<u>Prop. 3.4.4</u> : (Chain rule for Taylor polynomials)

Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}$ be open.

$f : U \longrightarrow V$ and $g : V \longrightarrow \mathbb{R}$ of class C^k.

Then (we already know) $g \circ f : U \longrightarrow \mathbb{R}$ is of class C^k

and if $f(a) = b$, we have

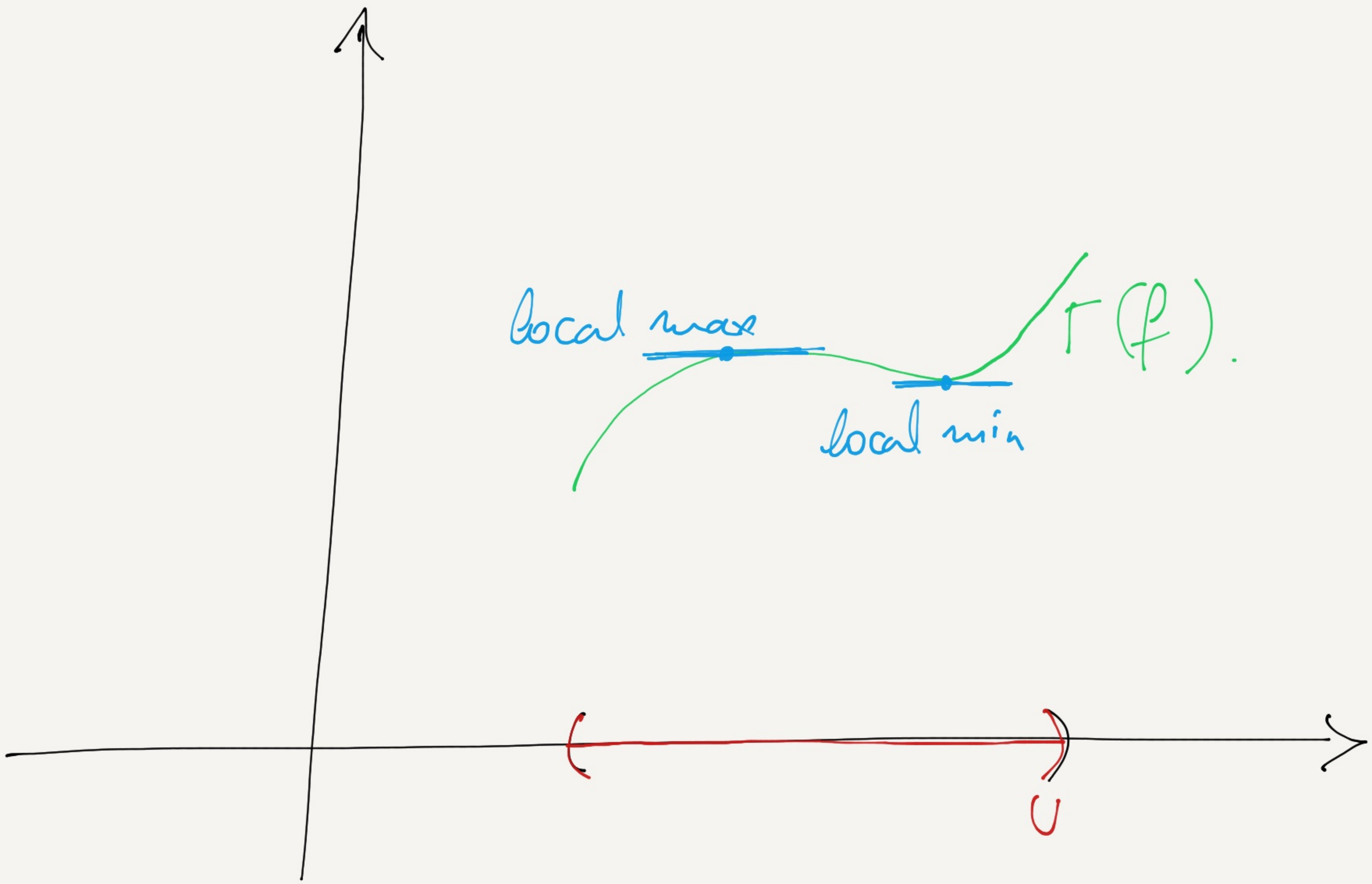
$$P^k_{g\circ f, a}(a+h) = P^k_{g,b}\left(P^k_{f,a}(a+h)\right)$$ minus the terms of degree $> k$.

Understanding (local) extrema and critical points:

Recall:

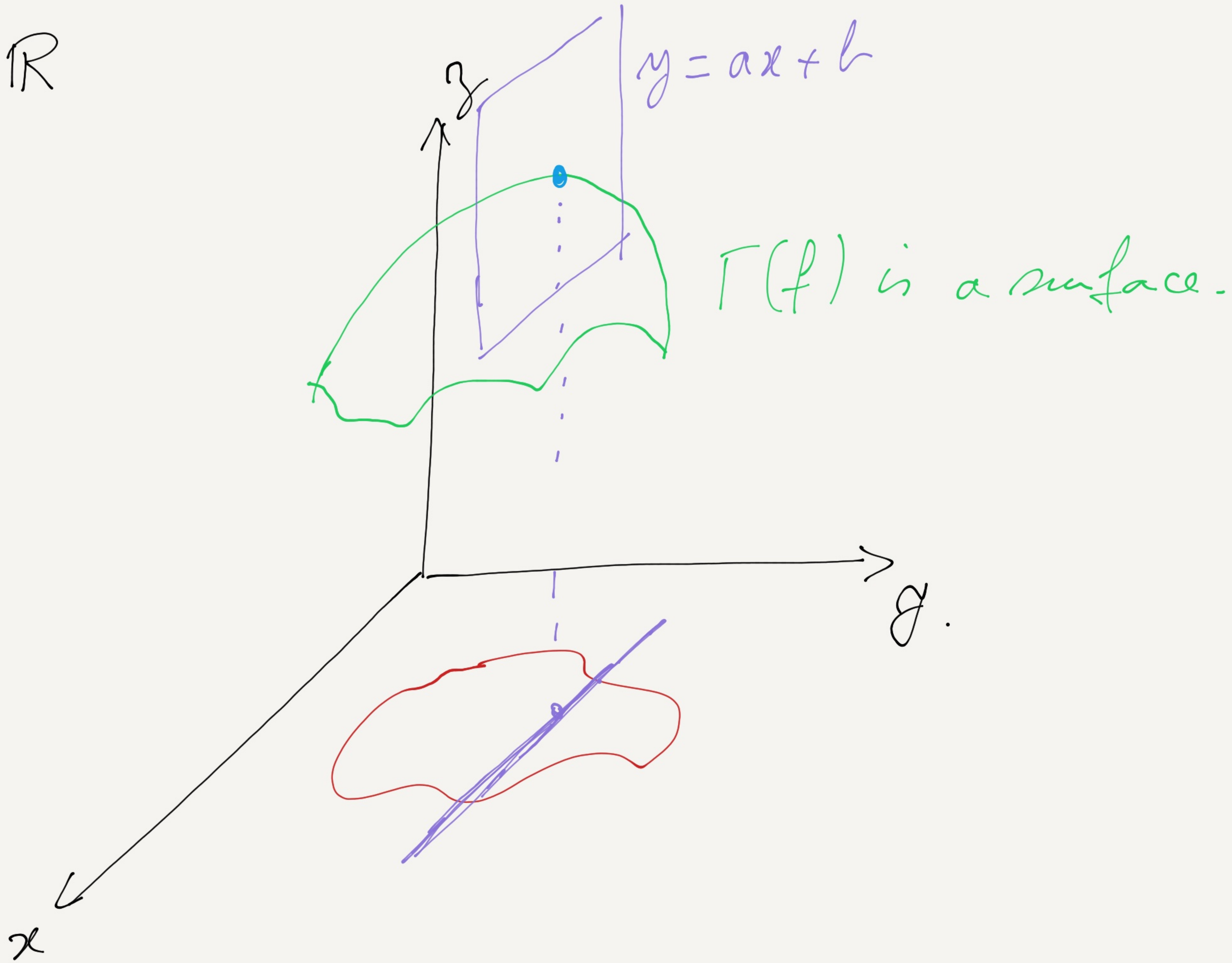
Theorem 3.6.1 Let $U \subset \mathbb{R}$ be an open interval, $f : U \longrightarrow \mathbb{R}$ a differentiable function. Then

1. If $x_0 \in U$ is a local extremum for f, then $f'(x_0) = 0$
2. If f is twice differentiable and $f'(x_0) = 0$ and
   (a) $f''(x_0) > 0$, then f has a local minimum at x_0.
   (b) $f''(x_0) < 0$, then f has a local maximum at x_0.


local max
local min
Γ(f).
U

$f : U \longrightarrow \mathbb{R}$

$U \subset \mathbb{R}^2$



Theorem 3.6.3 (Derivative is zero at extrema)

$U \subset \mathbb{R}^n$ open $\quad f : U \longrightarrow \mathbb{R}$ differentiable.

If $x_0 \in U$ is a local extremum for f, then $Df(x_0) = 0$.

Proof: $Df(x_0)$ is represented by the Jacobian matrix, whose entries are the partial derivatives of f.

Now, if we take $g_i(x) := f(a_1, \dots, a_i + x, a_{i+1}, \dots, a_n)$

where $a = (a_1, \dots, a_n)$

If f has a local maximum or minimum at a,

then g has a local maximum or minimum at 0.

So $g'(0) = 0$, but $g'(x) = \frac{\partial f}{\partial x_i}(a_1, \dots, a_i + x, \dots, a_n)$

Definition 3.6.2 $U \subset \mathbb{R}^n$ open, $f : U \to \mathbb{R}$ differentiable. A critical point of f is a point $a \in U$ s.t. $Df(a) = 0$. The value of f at a critical point is a critical value.

---

e.g. all local maximum (or minimum) values of f are critical values.

We want to generalize the second derivative test to several variables.

We assume $f : U \to \mathbb{R}$ is C^2.

Then f has a Taylor polynomial of degree 2 at any $a \in U$.

The terms of degree 2 of the Taylor polynomial are

$$\frac{1}{2}\frac{\partial^2 f}{(\partial x_1)^2}(a)\,h_1^2 + \frac{\partial^2 f}{\partial x_1 \partial x_2}(a)\,h_1 h_2 + \cdots + \frac{1}{2}\frac{\partial^2 f}{(\partial x_n)^2}(a)\,h_n^2$$

This is a homogeneous quadratic polynomial, i.e., all the monomials have degree 2.

Such polynomials can always be written into a sum of $\pm l_i^2$ where l_i are homogeneous linear polynomials and are linearly independent.

e.g.: $x^2 + xy \quad = \left(x + \frac{1}{2}y\right)^2 - \frac{1}{4}y^2$

xz

<u>note</u>: $(x-z)(x+z) = x^2 - z^2$

$x = u - v$ and $z = u + v$?

solve $u = \frac{1}{2}(x+z)$ $\quad v = \frac{1}{2}(z-x)$

so $xz = (u-v)(u+v) = u^2 - v^2$

$= \frac{1}{4}(x+z)^2 - \frac{1}{4}(x-z)$

<u>Note</u>: x, z are linearly independent (as functions)

If $\lambda x + \mu z = 0$ for all values of x, z, then $\lambda = \mu = 0$.

plug in $x = 1, z = 0$ to get $\lambda = 0$

$x = 0, z = 1$ to get $\mu = 0$

x, y, z are functions $\mathbb{R}^3 \longrightarrow \mathbb{R}$

$x : (a, b, c) \longmapsto a$

$y : (a, b, c) \longmapsto b$

$z : (a, b, c) \longmapsto c$

<u>Theorem 3.5.3</u>

For any homogeneous quadratic polynomial $Q : \mathbb{R}^n \to \mathbb{R}$,

(1) there exist $m = k + l$ linearly independent linear functions $\alpha_1, \dots, \alpha_m : \mathbb{R}^n \longrightarrow \mathbb{R}$ s.t.

$$Q(x) = \alpha_1(x)^2 + \dots + \alpha_k(x)^2 - \alpha_{k+1}(x)^2 - \dots - \alpha_m(x)^2$$

where m is a nonnegative integer $\leq n$.

(2) The numbers k and l depend only on Q

Def 3.5.4 The pair (k, l) is called the signature of Q.

Def 3.6.6 $U \subset \mathbb{R}^n$ open, $f: U \to \mathbb{R}$ of class C^2. $a \in U$ a critical point of f. The signature of a is the signature of the quadratic polynomial

$$Q_{f,a}(h) := \sum_{I \in \mathcal{I}_n^2} \frac{1}{I!} \frac{\partial^I f}{(\partial x)^I}(a)\, h^I$$

$$= \frac{1}{2} \frac{\partial^2 f}{(\partial x_1)^2}(a)\, h_1^2 + \frac{\partial^2 f}{\partial x_1 \partial x_2}(a)\, h_1 h_2 + \cdots$$

Def 3.5.9: If Q has signature $(n, 0)$, we say Q is positive definite. If Q has signature $(0, n)$, we say Q is negative definite.

Theorem 3.6.8: $U \subset \mathbb{R}^n$ open, $f: U \longrightarrow \mathbb{R}$ of class C^2.

$a \in U$ critical point of f

1. If $Q_{f,a}$ is positive definite, then a is a strict local minimum.
2. If $Q_{f,a}$ is negative definite, then a is a strict local maximum.
3. If the signature of a is (k, l) with $l > 0$, then a is NOT a local minimum.
4. If the signature of a is (k, l) with $k > 0$, then a is NOT a local maximum.

example of positive definite Q:

$$Q(x) = x_1^2 + x_2^2 + \dots + x_n^2.$$

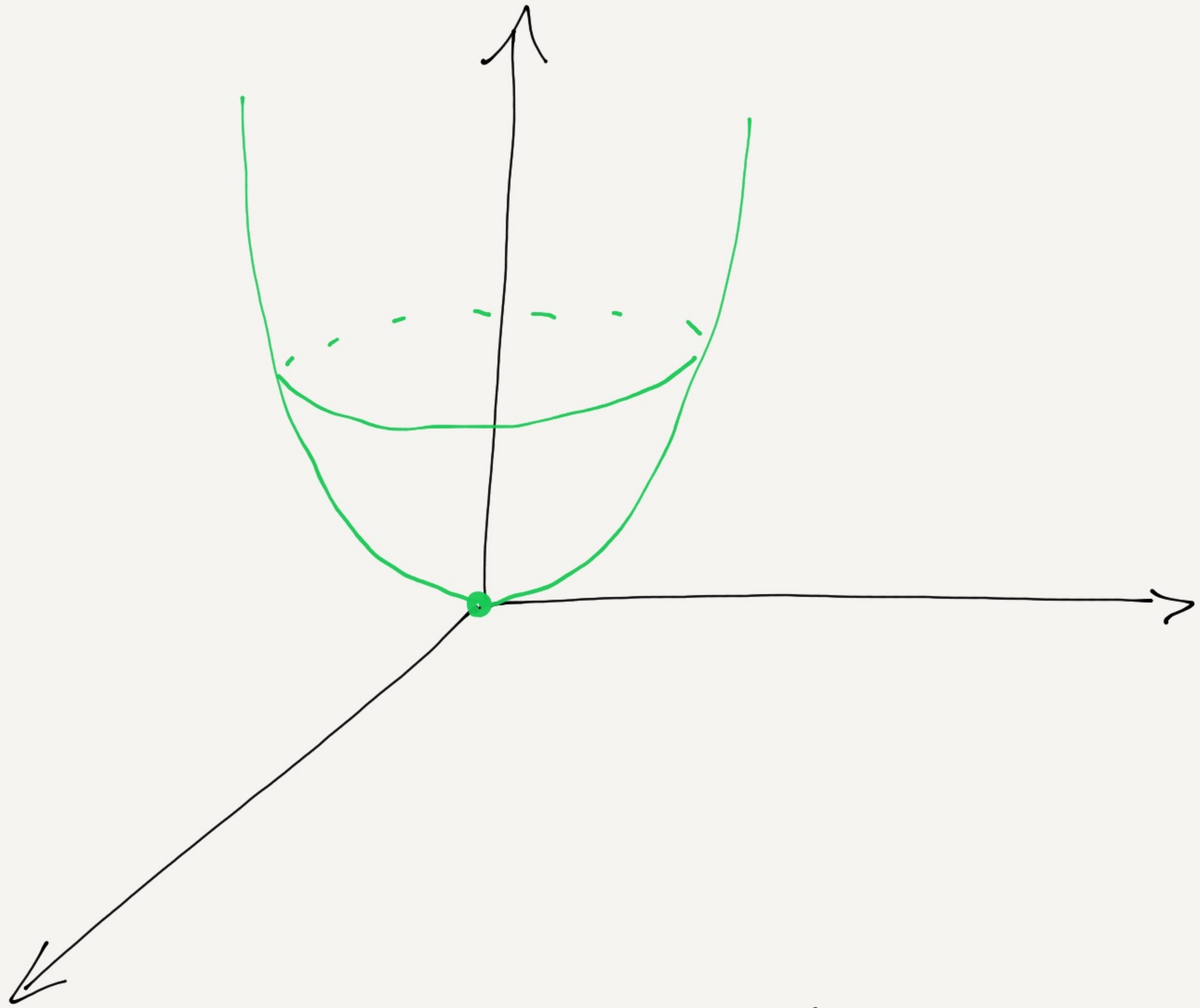
negative definite $Q = -x_1^2 - \dots - x_n^2$

<u>Def</u>: <u>Saddle</u>: If p is a critical point with $Q_{f,p}$ of signature (k, l) and $k > 0$ and $l > 0$, then we call p a saddle

---

<u>Examples</u>: $f(x, y) = x^2 + y^2$

$$T(f) \subset \mathbb{R}^3 \qquad T(f) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; z = x^2 + y^2 \right\}.$$

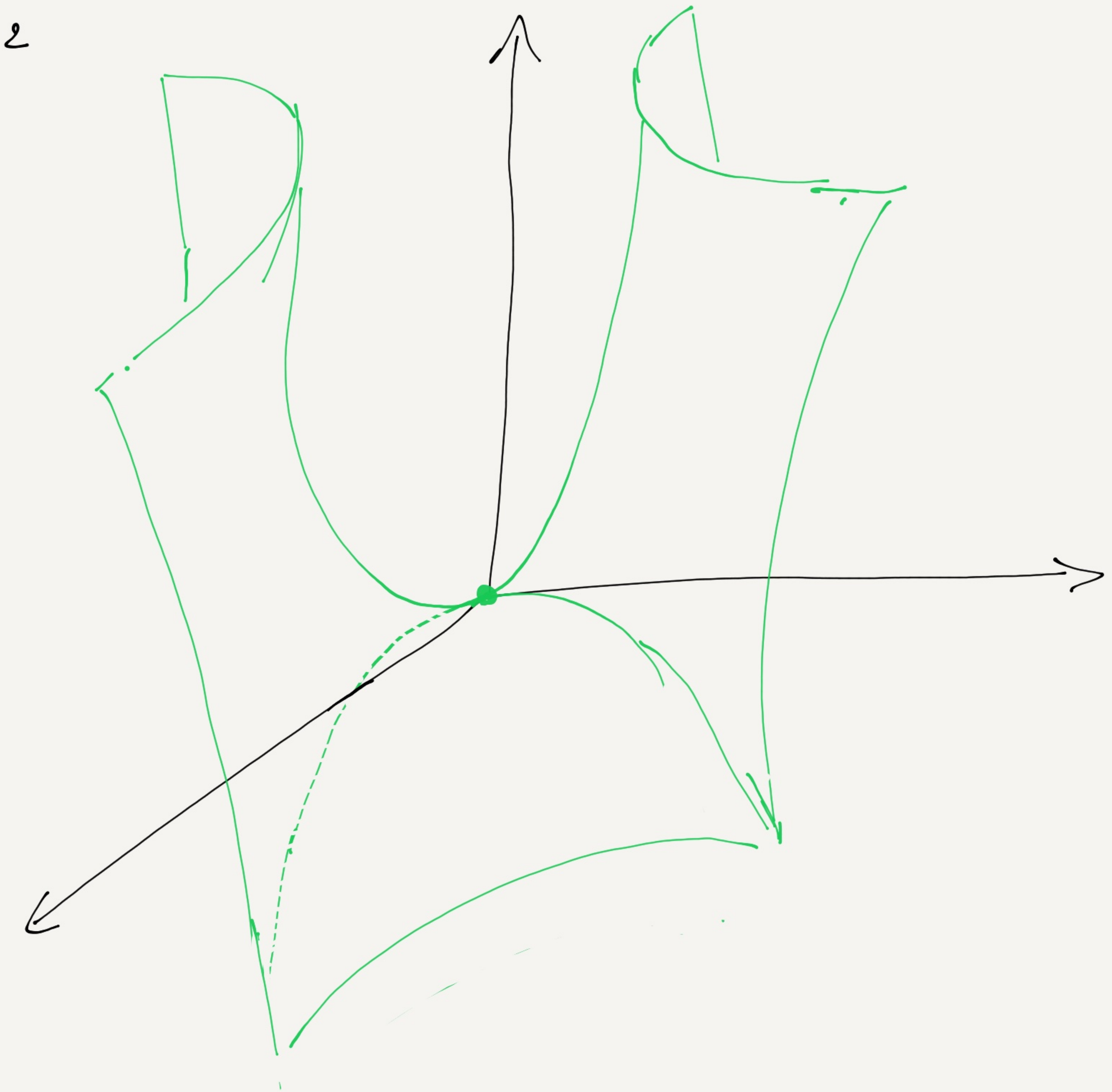


critical point : $(0,0)$ $\quad f(0,0)=0$

minimum : $\quad Q_{f,0} = x^2+y^2$ positive definite.

$f(x,y) = -x^2-y^2$ has a max at 0.

$f(x,y) = x^2 - y^2$



Degenerate critical point: $(k+l<n)$

$$f(x,y) = x^2 + y^3$$

picture of graph in book page 348