Taylor polynomials:

A useful notion:

Definition 3.4.1 (Little o) Let $U \subset \mathbb{R}^n$ be an open set containing 0. Let $f, g : U \setminus \{0\} \longrightarrow \mathbb{R}$ be two functions with $g > 0$. Then f is in / a $o(g)$ or we write $f = o(g)$ if

$$\lim_{h \to 0} \frac{f(h)}{g(h)} = 0.$$

e.g.: If f is differentiable $f : \mathbb{R} \to \mathbb{R}$, then

$$f(h) - f(0) - f'(0)h = o(|h|)$$

because $$\lim_{h \to 0} \frac{1}{h} \left(f(h) - f(0) - f'(0)h \right) = 0.$$

We start by reviewing Taylor polynomials in one variable.

<u>Definition and Theorem 3.3.1</u> $U \subset \mathbb{R}$ open

$f : U \longrightarrow \mathbb{R}$ of class C^k, $k \in \mathbb{N}$.

The k-th Taylor polynomial of f at $a \in U$ is

$$P^k_{f,a}(a+h) := f(a) + f'(a)h + \frac{1}{2!} f''(a)h^2 + \dots$$
$$\dots + \frac{1}{k!} f^{(k)}(a)\, h^k$$

The polynomial $P^k_{f,a}$ is the unique polynomial of degree $\leq k$ s.t. $f(a+h) - P^k_{f,a}(a+h) = o(|h|^k)$

which means: $\displaystyle \lim_{h \to 0} \frac{1}{h^k} \left(f(a+h) - P^k_{f,a}(a+h) \right) = 0$

Note: (1) Saying $f = o(g)$ means

$$\lim_{h\to 0} \frac{f(h)}{g(h)} = 0 \quad \text{or} \quad f(h) = g(h)\cdot \text{something that goes to } 0 \text{ as } h \to 0$$

$$\left(f = \frac{f}{g} \cdot g \right)$$

(2) $$P^k_{f,a}(x) = f(a) + f'(a)(x-a) + \frac{1}{2!} f''(a)(x-a)^2 + \dots + \frac{1}{k!} f^{(k)}(a)(x-a)^k$$

We can similarly define Taylor polynomials for scalar valued functions of several variables.

<u>Theorem 3.3.16</u> (Taylor's theorem in several variables)

$U \subset \mathbb{R}^n$ open, $a \in U$, $f : U \to \mathbb{R}$ a function of class C^k.

We define a polynomial $P^k_{f,a}(a+h)$ of degree k, using the partial derivatives of f.

The polynomial $P^k_{f,a}(a+h)$ is the unique polynomial of degree $\leq k$ s.t. $f(a+h) - P^k_{f,a}(a+h) = o(|h|^k)$.

This means $\lim_{h \to 0} \frac{1}{|h|^k} \left(f(a+h) - P^k_{f,a}(a+h) \right) = 0$

Some notation: $\mathcal{I}_n :=$ set of all multi-indices or multi-exponents with n entries

$$\mathcal{I}_n = \{ (i_1, \dots, i_n) \mid i_j \in \mathbb{Z}_+ \text{ for } j = 1, \dots, n \}$$

$\mathcal{I}_n^k :=$ set of multi-indices or multi-exponents of total degree k.

$$\subset \mathcal{I}_n$$

$$\mathcal{I}_n^k := \{ (i_1, \dots, i_n) \mid i_1 + \dots + i_n = k \}$$

For $x := (x_1, \dots, x_n) \in \mathbb{R}^n$, $I \in \mathcal{I}_n^k$

$x^I := x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ is a monomial of total degree k.

$f : U \longrightarrow \mathbb{R}$ $\quad U \subset \mathbb{R}^n$ open

C^k function. $\quad I \in \mathcal{I}_n^l \quad l \leq k$

$$\frac{\partial^I f}{(\partial x)^I} := \frac{\partial^{i_1}}{(\partial x_1)^{i_1}}\left(\frac{\partial^{i_2}}{(\partial x_2)^{i_2}}\left(\cdots\left(\frac{\partial^{i_n}}{(\partial x_n)^{i_n}} f\right)\right)\cdots\right)$$

where $I = (i_1, \dots, i_n)$ $\quad i_1 + \cdots + i_n = l \leq k$

e.g. $\quad I = (1, 1, 0, \dots, 0)$

$$\frac{\partial^I f}{(\partial x)^I} = \frac{\partial}{\partial x_1}\left(\frac{\partial}{\partial x_2} f\right) = \frac{\partial^2 f}{\partial x_1 \partial x_2}$$

<u>Question:</u> Does the order matter?

<u>Answer:</u> Not for "nice" functions.

Theorem 3.3.8 (Equality of mixed partials):

$U \subset \mathbb{R}^n$ open $f: U \to \mathbb{R}$ function s.t. all the first partial derivatives of f are differentiable at $a \in U$. Then, $\forall\, i, j$

$$\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j} f\right)(a) = \frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_i} f\right)(a).$$

More useful is the following:

Corollary 3.3.10: If $f: U \to \mathbb{R}$ is C^k, then the partials of order up to k do not depend on the order of differentiation.

Factorials of multi-exponents:

$$I = (i_1, \dots, i_n) \in \mathcal{I}_n$$

then $\quad I! := i_1!\, i_2! \cdots\cdots i_n!$

Definition 3.3.13 $\quad U \subset \mathbb{R}^n$ open, $f : U \to \mathbb{R}$ C^k.

$a \in U$. The Taylor polynomial of degree k of f at a is

$$P^k_{f,a}(a+h) := \sum_{l=0}^{k} \sum_{I \in \mathcal{I}_n^l} \frac{1}{I!} \frac{\partial^I f}{(\partial x)^I}(a)\, h^I$$

Note: $\quad h^I = h_1^{i_1} h_2^{i_2} \cdots h_n^{i_n}$

We can also write $\quad P^k_{f,a}(x) = \sum_{l=0}^{k} \sum_{I \in \mathcal{I}_n^l} \frac{1}{I!} \frac{\partial^I f}{(\partial x)^I}(a)\,(x-a)^I$

<u>Examples:</u> (1) $f(x,y,z) = x^2y - z^3 + y^4$

$f: \mathbb{R}^3 \to \mathbb{R}$ $\quad C^\infty$

$$P^1_{f,0}(h) = f(0) + \frac{\partial f}{\partial x}(0)h_1 + \frac{\partial f}{\partial y}(0)h_2 + \frac{\partial f}{\partial z}(0)h_3 = 0$$

$h = (h_1, h_2, h_3)$

$$P^2_{f,0}(h) = P^1_{f,0}(h) + \frac{1}{2}\frac{\partial^2 f}{(\partial x)^2}(0)h_1^2 + \frac{1}{2}\frac{\partial^2 f}{(\partial y)^2}(0)h_2^2$$

$$+ \frac{1}{2}\frac{\partial^2 f}{(\partial z)^2}(0)h_3^2 + \frac{1}{1!1!}\frac{\partial^2 f}{\partial x \partial y}(0)h_1h_2 + \frac{\partial^2 f}{\partial x \partial z}(0)h_1h_3$$

$$+ \frac{\partial^2 f}{\partial y \partial z}(0)h_2h_3 = 0$$

More generally, every polynomial is its own Taylor expansion at 0.

To get the Taylor expansion at $a = (a_1, a_2, a_3)$, compute $f(a_1 + h_1, a_2 + h_2, a_3 + h_3)$ and expand as a polynomial in $h = (h_1, h_2, h_3)$.

(2) $f(x,y) = \sin(x^2 + y)$ $\quad f : \mathbb{R}^2 \to \mathbb{R}$ C^∞

At $0 = (0,0)$, $\quad P^0_{f,0} = f(0) = 0$

$$P^1_{f,0} = P^0_{f,0} + \frac{\partial f}{\partial x}(0) h_1 + \frac{\partial f}{\partial y}(0) h_2$$

$$\frac{\partial f}{\partial x} = 2x \cos(x^2 + y) \qquad \frac{\partial f}{\partial y} = \cos(x^2 + y)$$

$$P^1_{f,0} = 0 + h_2 = h_2$$

$$P^2_{f,0}(h) = P^1_{f,0}(h) + \frac{1}{2}\frac{\partial^2 f}{(\partial x)^2}(0)h_1^2 + \frac{1}{1!\,1!}\frac{\partial^2 f}{\partial x \partial y}h_1 h_2 + \frac{1}{2}\frac{\partial^2 f}{(\partial y)^2}(0)h_2^2$$

$$\frac{\partial^2 f}{\partial x^2} = 2\cos(x^2+y) + 2x(-2x)\sin(x^2+y)$$

$$\frac{\partial^2 f}{\partial x \partial y} = -2x\sin(x^2+y)$$

$$\frac{\partial^2 f}{(\partial y)^2} = -\sin(x^2+y)$$

$$P^2_{f,0}(h) = h_2 + h_1^2$$

There are rules which make it a lot faster to compute Taylor polynomials.

<u>Proposition 3.4.3</u>: Sums and products of Taylor polynomials: $U \subset \mathbb{R}^n$ open $a \in U$

$f, g : U \longrightarrow \mathbb{R}$ C^k, then $f+g$ and fg are also C^k (we already know this) and

(1) $$P^k_{f+g,a}(a+h) = P^k_{f,a}(a+h) + P^k_{g,a}(a+h)$$

$$P^k_{fg,a}(a+h) = P^k_{f,a}(a+h) \cdot P^k_{g,a}(a+h)$$ minus the terms of degree $> k$.