Notes: 1. Any open set is a manifold:

$U \subset \mathbb{R}^n \qquad f: U \longrightarrow \{0\} = \mathbb{R}^0$

graph of f: $\quad \Gamma(f) \subset \mathbb{R}^n \times \mathbb{R}^0 = \mathbb{R}^n$

$$\Gamma(f) = \{(v, 0) : f(v) = 0\} = \{(v,0)\}$$

$$= U \times \{0\} \subset \mathbb{R}^n \times \{0\}$$

$$U \times \{0\} = U \qquad \mathbb{R}^n \times \{0\} = \mathbb{R}^n$$

$g: U \times \{0\} \longrightarrow U$ bijection

$(v, 0) \longmapsto v$

$(v, 0) \longleftarrow\!\shortmid\; v$

2. Closed sets are NOT always manifolds.

Zeros of continuous functions are examples of closed sets:

e.g. ⓐ $f(x,y) = y - x^2$

$$Z(f) = \{ y = x^2 \}$$ parabola

smooth manifold because it is the graph of the C^1 function $g(x) = x^2$.

ⓑ $f(x,y) = |x| - y$, $Z(f) = \{ f(x,y) = 0 \}$ is a graph

but $g(x) = |x|$ is NOT C^1

$$Z(f) = \{ |x| - y = 0 \} = \Gamma(g)$$

(c) $F(x,y) = (x^2+y^2-1)^3$

F is differentiable because it is a polynomial

$$\frac{\partial F}{\partial x} = 3 \cdot 2x \cdot (x^2+y^2-1)^2$$

$$\frac{\partial F}{\partial y} = 3 \cdot 2y \cdot (x^2+y^2-1)^2$$

on $Z(F)$, we have $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0$

put $f(x,y) = x^2+y^2-1$, then

$Z(F) = Z(f)$

3. $F : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-k}$

$$F = \begin{pmatrix} F_1 \\ \vdots \\ F_{n-k} \end{pmatrix}$$

$$Z(F) = Z(F_1) \cap Z(F_2) \cap \cdots \cap Z(F_{n-k})$$

$$DF(a) = \begin{pmatrix} DF_1(a) \\ \vdots \\ DF_{n-k}(a) \end{pmatrix} : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-k}$$

$$DF_1(a) : \mathbb{R}^n \longrightarrow \mathbb{R}$$

$DF(a)$ surjective $\Leftrightarrow$ $\dim(\ker DF(a)) = k$

$DF_1(a)$ surjective $\Leftrightarrow$ $\dim(\ker DF_1(a)) = n-1$

$$\ker DF(a) = \ker DF_1(a) \cap \ker DF_2(a) \cap \dots \cap \ker DF_{n-k}(a).$$

---

(7) On preparation (practice) sheet:

$$\begin{cases} x^2+y^2+z^2-yw=2 \\ xyw-yz+z^2+w^2=0 \end{cases}$$

$(0,1,1,0)$ satisfies the system: plug in. ✓

$F : \mathbb{R}^4 \longrightarrow \mathbb{R}^2$ $\quad 2 = n-k \quad 4 = n \quad \Rightarrow k=2$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \longmapsto \begin{pmatrix} x^2+y^2+z^2-yw-2 \\ xyw-yz+z^2+w^2 \end{pmatrix}$$

$Z(F)$? $\quad$ F is C^1: the components are polynomials.

$$JF = \begin{pmatrix} 2x & 2y-w & 2z & -y \\ yw & xw-z & 2z-y & xy+2w \end{pmatrix}$$

$$JF\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 2 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix}$$

Any two of the last 3 columns are linearly independent. So, in a neighborhood of $\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$, we can write (y,z) as a function of (x,w)

or (y,w) as a function of (x,z)

or (z,w) as a function of (x,y)

We pick the last possibility: $\exists$ function $g(x,y)$ s.t. in a neighborhood of $\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$,

$$(z,w) = g(x,y) \iff F(x,y,z,w) = 0$$

$\Leftrightarrow$ in a neighborhood of $\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$: $\Gamma(g) = Z(F)$

$\Leftrightarrow$ $\forall$ (x,y) in a neighborhood of $(0,1)$

$$F(x, y, g_1(x,y), g_2(x,y)) = 0$$

where $g = \begin{pmatrix}g_1\\g_2\end{pmatrix}$

The implicit function theorem says g is C^1 in a small neighborhood of $(0,1)$.

We can ask where, a particular set of variables can be expressed as functions of the others.

e.g.: where can (y,z) be locally expressed as functions of (x,w)? This is the locus where the columns corresponding to y and z are linearly independent, i.e., the submatrix

$$\begin{pmatrix} 2y-w & 2z \\ xw-z & 2z-y \end{pmatrix}$$ has rank 2.

The determinant is $(2y-w)(2z-y) - 2z(xw-z)$

The matrix has rank 2 when the determinant is not 0.

So, if $(2y-w)(2z-y) - 2z(xw-z) \neq 0$, and

$F(x,y,z,w) = 0$, we can locally express (y,z) as

a function of (x,w).

If (e.g.) we just had $F(x,y,z,w) = x^2+y^2+z^2-yw-z$.

then, we could look at $DF(a_1,\dots,a_4)$

$$DF = \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} & \frac{\partial F}{\partial w} \end{pmatrix}$$

$F: \mathbb{R}^4 \to \mathbb{R}$

$DF(0,1,1,0): \mathbb{R}^4 \to \mathbb{R}$

$$DF(0,1,1,0) = \begin{pmatrix} 0 & 2 & 2 & -1 \end{pmatrix}$$

$\frac{\partial F}{\partial y}(0,1,1,0) \neq 0 \Rightarrow y$ is locally an implicit function of (x,z,w) near $(0,1,1,0)$

Similarly, z can be locally expressed as a function of x, y, w.

And w can be locally expressed as a function of x, y, z.

<u>derivative of g</u>: $\dfrac{\partial}{\partial x}\Big(F\,(x, y, g_1(x,y), g_2(x,y))\Big) = 0$.

$$\frac{\partial F}{\partial x}(a) + \frac{\partial y}{\partial x}(?)\frac{\partial F}{\partial y}(a) + \frac{\partial g_1}{\partial x}(?)\frac{\partial F}{\partial z}(a) + \frac{\partial g_2}{\partial x}(?)\frac{\partial F}{\partial w}(a) = 0$$

The tangent <u>vector</u> space to $Z(F)$ at $\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ is the kernel of $DF\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ represented by the matrix $\begin{pmatrix} 0 & 2 & 2 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix}$

The kernel of this is the set of (x, y, z, w) s.t.

$$\begin{pmatrix} 0 & 2 & 2 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = 0 \quad \text{or:} \quad \begin{cases} 2y + 2z - w = 0 \\ -y + z = 0 \end{cases}$$

The equation of the tangent <u>affine</u> space is the set of (x, y, z, w) s.t. $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ belongs to the tangent vector space:

$$\begin{cases} 2(y-1) + 2(z-1) - w = 0 \\ -(y-1) + (z-1) = 0 \end{cases}$$

Definition and Proposition (3.2.9 C^r map on a manifold and its derivative)

Let $M \subset \mathbb{R}^n$ be a k-dimensional smooth manifold.
A map $f : M \to \mathbb{R}^m$ is of class C^r if every $x \in M$ has an open neighborhood $U \subset \mathbb{R}^n$ s.t. $\exists$ a map $\tilde{f} : U \to \mathbb{R}^m$ of class C^r with $\tilde{f}|_{U \cap M} = f|_{U \cap M}$

(this means that, for all $y \in U \cap M$, $\tilde{f}(y) = f(y)$)

Furthermore, if $r \geq 1$, the derivative of f at x is defined to be $Df(x) := D\tilde{f}(x)|_{T_xM} : T_xM \to \mathbb{R}^m$

(this means that $Df(x)$ is only well-defined on the tangent vector space T_xM and $\forall\, v \in T_xM$, $(Df(x))(v) = (D\tilde{f}(x))(v)$)

$Df(x) : T_xM \to \mathbb{R}^m$ only depends on f and not on the choice of $\tilde{f}$.