Definition (Affine tangent space or tangent affine space).

M a k-dimensional smooth manifold, locally at $z_0 \in \mathbb{R}^n$, M is the graph of $f : V \longrightarrow \mathbb{R}^{n-k}$

open $V \subset \mathbb{R}^k$

write $z_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ $x_0 \in V$, $y_0 = f(x_0) \in \mathbb{R}^{n-k}$

then the tangent affine space to M at z_0 is the affine space through z_0, parallel to the tangent vector space.

i.e., if $T_{z_0}^{v} M := \left\{ \begin{pmatrix} v \\ Df(x_0)(v) \end{pmatrix} \right\} \subset \mathbb{R}^n$

then $T_{z_0}^{a} M := \left\{ \begin{pmatrix} v + x_0 \\ Df(x_0)(v) + y_0 \end{pmatrix} \right\} \subset \mathbb{R}^n$

Tangent spaces via equations:

Theorem 3.2.4: $F : U \longrightarrow \mathbb{R}^{n-k}$ $\quad U \underset{\text{open}}{\subset} \mathbb{R}^n$

F C^1 $\quad Z(F) := \{ v \mid F(v) = 0 \}$ $\quad z_0 \in Z(F)$

If $Z(F)$ is a smooth manifold, and $DF(z_0)$ is onto (a surjective), then $T^v_{z_0} Z(F) = \ker(DF(z_0)) \subset \mathbb{R}^n$

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e.g.: curve in $\mathbb{R}^2$: $\quad F(x,y) = x^7 + 2x^3 - y + y^2$

$$Z(F) = \{ (x,y) \mid x^7 + 2x^3 - y + y^2 = 0 \}$$

F is C^1 everywhere and $DF\begin{pmatrix} x \\ y \end{pmatrix} = (7x^6 + 6x^2, -1 + 2y)$

$(0,0) \in Z(F)$ $\quad DF\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \end{pmatrix} : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$\ker DF\begin{pmatrix} 0 \\ 0 \end{pmatrix} = Z(y) = x\text{-axis}$

More generally, at $\begin{pmatrix} a \\ b \end{pmatrix} \in Z(F)$, then

$$DF\begin{pmatrix} a \\ b \end{pmatrix} = \left(7a^6 + 6a^2, -1+2b\right) = \left(\frac{\partial F}{\partial x}\begin{pmatrix} a \\ b \end{pmatrix} \quad \frac{\partial F}{\partial y}\begin{pmatrix} a \\ b \end{pmatrix}\right)$$

$$\ker DF\begin{pmatrix} a \\ b \end{pmatrix} = \left\{ \begin{pmatrix} v \\ w \end{pmatrix} \text{ s.t. } DF\begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix} = 0 \right\}$$

$$= \left\{ \begin{pmatrix} v \\ w \end{pmatrix} \;\middle|\; \left(\frac{\partial F}{\partial x}\begin{pmatrix} a \\ b \end{pmatrix} \quad \frac{\partial F}{\partial y}\begin{pmatrix} a \\ b \end{pmatrix}\right)\begin{pmatrix} v \\ w \end{pmatrix} = 0 \right\}$$

$$= \left\{ \begin{pmatrix} v \\ w \end{pmatrix} \;\middle|\; \frac{\partial F}{\partial x}\begin{pmatrix} a \\ b \end{pmatrix} v + \frac{\partial F}{\partial y}\begin{pmatrix} a \\ b \end{pmatrix} w = 0 \right\}$$

$\begin{pmatrix} x \\ y \end{pmatrix} \in T^{a}_{\begin{pmatrix} a \\ b \end{pmatrix}} Z(F)$ means $\begin{pmatrix} x-a \\ y-b \end{pmatrix} \in T^{v}_{\begin{pmatrix} a \\ b \end{pmatrix}} Z(F)$

so the equation of the affine tangent line is

$$\frac{\partial F}{\partial x}\begin{pmatrix} a \\ b \end{pmatrix}(x-a) + \frac{\partial F}{\partial y}\begin{pmatrix} a \\ b \end{pmatrix}(y-b) = 0$$

e.g. A smooth surface in $\mathbb{R}^3$:

$$F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \sin(xy+z) \qquad Z(F) \subset \mathbb{R}^3$$

$$F : \mathbb{R}^3 \longrightarrow \mathbb{R} \qquad n=3, \quad n-k=1, \quad k=2$$

F is C^1 because polynomials are C^1 and sine is C^1 and compositions of C^1 functions are C^1.

$$DF\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b\cos(ab+c) & a\cos(ab+c) & \cos(ab+c) \end{pmatrix}$$

$$DF\begin{pmatrix} a \\ b \\ c \end{pmatrix} : \mathbb{R}^3 \longrightarrow \mathbb{R}$$

On $Z(F)$ $\quad \sin(ab+c)=0$, so $\cos(ab+c) \neq 0$

We can apply the implicit function theorem: we can express z as an implicit function of x, y

in a small neighborhood of $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$: $z = g(x,y)$

and $F(x, y, g(x,y)) = 0$

$$G : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x \\ y \\ g(x,y) \end{pmatrix}$$

$$F \circ G = 0$$

$$F \circ G : \mathbb{R}^2 \xrightarrow[G]{} \mathbb{R}^3 \xrightarrow[F]{} \mathbb{R}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} \overset{G}{\longmapsto} \begin{pmatrix} a \\ b \\ g(a,b) \end{pmatrix} \overset{F}{\longmapsto} 0$$

$$D(F \circ G) = 0$$

$$DF\begin{pmatrix} a \\ b \\ g(a,b) \end{pmatrix} \circ DG\begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$$\left(\frac{\partial F}{\partial x}\begin{pmatrix} a \\ b \\ g(a,b) \end{pmatrix} \; \frac{\partial F}{\partial y}\begin{pmatrix} a \\ b \\ g(a,b) \end{pmatrix} \; \frac{\partial F}{\partial z}\begin{pmatrix} a \\ b \\ g(a,b) \end{pmatrix} \right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial g}{\partial x}\begin{pmatrix} a \\ b \end{pmatrix} & \frac{\partial g}{\partial y}\begin{pmatrix} a \\ b \end{pmatrix} \end{pmatrix} = 0$$

So

$$\frac{\partial F}{\partial x}\begin{pmatrix} a \\ b \\ g(a,b) \end{pmatrix} + \frac{\partial g}{\partial x}\begin{pmatrix} a \\ b \end{pmatrix} \frac{\partial F}{\partial z}\begin{pmatrix} a \\ b \\ g(a,b) \end{pmatrix} = 0$$

$$\frac{\partial F}{\partial y}\begin{pmatrix} a \\ b \\ g(a,b) \end{pmatrix} + \frac{\partial g}{\partial y}\begin{pmatrix} a \\ b \end{pmatrix} \frac{\partial F}{\partial z}\begin{pmatrix} a \\ b \\ g(a,b) \end{pmatrix} = 0$$

$$\Rightarrow \quad \frac{\partial g}{\partial x}\begin{pmatrix} a \\ b \end{pmatrix} = -\frac{1}{\frac{\partial F}{\partial z}\begin{pmatrix} a \\ b \\ g(a,b) \end{pmatrix}} \frac{\partial F}{\partial x}\begin{pmatrix} a \\ b \\ g(a,b) \end{pmatrix}$$

$$\frac{\partial g}{\partial y}\begin{pmatrix} a \\ b \end{pmatrix} = -\frac{1}{\frac{\partial F}{\partial z}\begin{pmatrix} a \\ b \\ g(a,b) \end{pmatrix}} \frac{\partial F}{\partial y}\begin{pmatrix} a \\ b \\ g(a,b) \end{pmatrix}$$

So we can compute the derivatives of implicitly defined functions (g in the example) even though we may not be able to compute the implicit function itself (g).

The inverse function theorem is useful in a similar way: a function may not be invertible, but it could be locally invertible. It is usually very hard to compute a local inverse, but with the inverse function theorem, we can compute the derivative of the local inverse.

Theorem 2.10.4: Suppose $U \subset \mathbb{R}^n$ open, $f: U \longrightarrow \mathbb{R}^n$ C^1, suppose for some $x_0 \in U$, $Df(x_0)$ is invertible. Then f is locally invertible with differentiable inverse and if $f(x_0) = y_0$ and f^{-1} is the local inverse at y_0, then

$$D(f^{-1})(y_0) = ((Df)(x_0))^{-1}$$

e.g.:

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} \sin(x+y) \\ x^2 - 2y^2 \end{pmatrix}$$

f is C^1 because polynomials, sine and compositions of C^1 functions are C^1.

$$DF\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos(x+y) & \cos(x+y) \\ 2x & -4y \end{pmatrix}$$

The matrix is invertible when its determinant is not zero.

$$\det\left(Df\begin{pmatrix}x\\y\end{pmatrix}\right) = -4y\cos(x+y) - 2x\cos(x+y)$$
$$= -2\cos(x+y)\left(x+2y\right)$$

So f is locally invertible if $x \neq -2y$ and

$\cos(x+y) \neq 0$ meaning $x+y \neq \frac{\pi}{2} + k\pi$ for all $k \in \mathbb{Z}$

When f is locally invertible the jacobian matrix of f^{-1}

is $$D(f^{-1})\begin{pmatrix}\sin(a+b)\\ a^2-2b^2\end{pmatrix} = \begin{pmatrix}\cos(a+b) & \cos(a+b)\\ 2a & -4b\end{pmatrix}^{-1}$$

Parametrizations:

Def 3.1.18 A parametrization of a k-dimensional manifold $M \subset \mathbb{R}^n$ is a mapping $\gamma : \underset{\text{open} \subset \mathbb{R}^n}{U} \longrightarrow M$ s.t. :

1. U is open
2. γ is C^1, injective, onto M
3. $D\gamma(u)$ is injective for all $u \in U$.