Manifolds:

Def 3.1.1 (Graph) Given a function $f: \mathbb{R}^k \to \mathbb{R}^{n-k}$.
The graph of f is the set of points $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$.
s.t. $y = f(x)$.

e.g.: $f: \mathbb{R}^2 \to \mathbb{R}$, $\begin{pmatrix} u \\ v \end{pmatrix} \mapsto uv$. The graph of f is $\left\{ \begin{pmatrix} x \\ y \\ xy \end{pmatrix} \right\} \subset \mathbb{R}^3$

Def 3.1.2: (Smooth manifold in $\mathbb{R}^n$) A subset $M \subset \mathbb{R}^n$ is a smooth k-dimensional manifold if it is locally the graph of a C^1 function expressing $n-k$ variables in terms of the remaining k variables.

This means: $\forall\ x \in M$, $\exists$ open set $U \subset \mathbb{R}^n$

s.t. $x \in U$ and $\exists\ V \subset \mathbb{R}^k$, $f : V \longrightarrow \mathbb{R}^{n-k}$

s.t. $U \cap M =$ graph of $f : V \to \mathbb{R}^{n-k}$.

e.g.: unit circle: $M = \{x^2 + y^2 = 1\} \subset \mathbb{R}^2$

show M is a manifold: express one variable in terms of the other (locally).

$$y = \sqrt{1-x^2} \quad \text{works when } y > 0$$

$$f :]-1,1[\longrightarrow \mathbb{R}, \quad x \longmapsto \sqrt{1-x^2}$$

the upper half circle is locally the graph of f.

If we proceed this way, we need 4 open sets to cover M.

Theorem 3.1.10 (When is the zero locus of a function a manifold?)

1. Let $U \subset \mathbb{R}^n$ be open, $F : U \longrightarrow \mathbb{R}^{n-k}$ a C^1 function

Let M be a subset of $\mathbb{R}^n$ s.t.

$$M \cap U = \{ v \in U \mid F(v) = 0 \}.$$

If $DF(v)$ is surjective (onto) for every $v \in M \cap U$, then $M \cap U$ is a smooth k-dimensional manifold in $\mathbb{R}^n$.

2. Conversely, if M is a smooth k-dimensional manifold in $\mathbb{R}^n$, then every $v \in M$ has an open neighborhood $U \subset \mathbb{R}^n$ s.t. $\exists$ a C^1 function $F : U \longrightarrow \mathbb{R}^{n-k}$ with $DF(v)$ surjective and $M \cap U = \{ v \mid F(v) = 0 \}$

Theorem 2.10.11 (The implicit function theorem):

Let $U \subset \mathbb{R}^n$ be open, $b \in U$ a point.

$F: U \longrightarrow \mathbb{R}^{n-k}$ a C^1 function such that $F(b)=0$

and $DF(b)$ is surjective.

If necessary, permute the coordinates in $\mathbb{R}^n$, so that the last $n-k$ columns of a matrix representing $DF(b)$ are linearly independent.

Then there exists an open neighborhood V of b, an open set $W \subset \mathbb{R}^k$ and a function $g: W \longrightarrow \mathbb{R}^{n-k}$ $g := \begin{pmatrix} g_1 \\ \vdots \\ g_{n-k} \end{pmatrix}$

s.t. $V \cap \{F(v)=0\}$ is the graph of $g: W \to \mathbb{R}^{n-k}$:

For $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in Z(F) := \{v \mid F(v)=0\}$, we have

$$v_{i+k} = g_i \begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix} \text{ for } i = 1, \dots, n-k$$

<u>e.g.</u> : The unit sphere: $n=3$, $k=2$, $n-k=1$

$$F : \mathbb{R}^3 \longrightarrow \mathbb{R}$$
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto x^2+y^2+z^2-1$$

$$b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad DF(b) = \left(\frac{\partial F}{\partial x}(b) \quad \frac{\partial F}{\partial y}(b) \quad \frac{\partial F}{\partial z}(b) \right)$$

$$DF(b) = (0, 0, 2)$$

$DF(b) : \mathbb{R}^3 \longrightarrow \mathbb{R}$ has rank $1 = \dim(\mathbb{R})$
so it is surjective.

$DF(b)$ is surjective iff its image is everything.
Because $\mathbb{R}$ has dimension 1, it only has two linear subspaces: $\{0\}$ and $\mathbb{R}$. So $DF(b)$ is surjective iff

it is not 0.

In this case $DF(b) \neq 0$ is surjective.

In fact, because $\frac{\partial F}{\partial z}(b) \neq 0$ we can write

$$z = g(x, y)$$

In our example, we can write g explicitly,

$$z = \sqrt{1 - x^2 - y^2} \quad \text{locally:}$$

$$g : B_1(0,0) \longrightarrow \mathbb{R}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \sqrt{1 - x^2 - y^2}$$

graph of g: $\left\{ \begin{pmatrix} x \\ y \\ \sqrt{1 - x^2 - y^2} \end{pmatrix} \right\} \subset \text{sphere} \subset \mathbb{R}^3$

Tangent spaces:

Def 3.2.1 (tangent vector space to a manifold)

Let $M \subset \mathbb{R}^n$ be a k-dimensional manifold, locally at $z_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, M is the graph of $f : \underset{\mathbb{R}^k}{\overset{}{V}} \longrightarrow \mathbb{R}^{n-k}$.

$$y_0 = f(x_0)$$

The tangent vector space to M at z_0 is the graph of the linear transformation $Df(x_0) : \mathbb{R}^k \longrightarrow \mathbb{R}^{n-k}$

Note: the graph of $Df(x_0)$ is in $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$.

$$= \left\{ \begin{pmatrix} v \\ Df(x_0)(v) \end{pmatrix} \right\}.$$

<u>e.g.:</u> curve in $\mathbb{R}^2$: $f: U \longrightarrow \mathbb{R}$ C^1 function.

graph of f : $C = \left\{ \begin{pmatrix} x \\ f(x) \end{pmatrix} \right\} \subset \mathbb{R}^2$.

The derivative: $Df(x_0) : \mathbb{R} \longrightarrow \mathbb{R}$

$$v \longmapsto f'(x_0)v$$

graph: $\left\{ \begin{pmatrix} v \\ f'(x_0)v \end{pmatrix} \right\} \subset \mathbb{R}^2$ line through the origin of slope $f'(x_0)$.