Some pathological functions:

e.g.: $f\begin{pmatrix} x \\ y \end{pmatrix} = \dfrac{x^2 y}{x^2+y^2}$ if $(x,y) \neq (0,0)$

$$f\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$$

f is a continuous function $\mathbb{R}^2 \longrightarrow \mathbb{R}$

f is a rational function, hence it is differentiable everywhere except possibly at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Partial derivatives: $\dfrac{\partial f}{\partial x} = \dfrac{2xy(x^2+y^2) - x^2y(2x)}{(x^2+y^2)^2} = \dfrac{2xy^3}{(x^2+y^2)^2}$

$$\frac{\partial f}{\partial y} = \frac{x^2(x^2+y^2) - x^2y(2y)}{(x^2+y^2)^2} = \frac{x^2(x^2-y^2)}{(x^2+y^2)^2}$$

Look at $\displaystyle\lim_{\binom{x}{y}\to\binom{0}{0}} \frac{\partial f}{\partial x}$

On the axes: $x=0$ or $y=0$ $\quad \frac{\partial f}{\partial x} = 0$

So $\displaystyle\lim_{\binom{0}{h}\to\binom{0}{0}} \frac{\partial f}{\partial x}\binom{0}{h} = 0 = \lim_{\binom{h}{0}\to\binom{0}{0}} \frac{\partial f}{\partial x}\binom{h}{0}$

along the line $y=x$, we have $\displaystyle\frac{\partial f}{\partial x}\binom{x}{y} = \frac{2x^4}{(2x^2)^2} = \frac{1}{2}$

So $\displaystyle\lim_{\binom{h}{h}\to\binom{0}{0}} \frac{\partial f}{\partial x}\binom{h}{h} = \frac{1}{2}$

So $\frac{\partial f}{\partial x}$ does not have a limit when $\binom{x}{y}\to\binom{0}{0}$.

$\frac{\partial f}{\partial y}$ is similar.

To compute $\frac{\partial f}{\partial x}\begin{pmatrix}0\\0\end{pmatrix}$ we have to use the definition:

$$\lim_{h\to 0}\frac{1}{h}\left(f\begin{pmatrix}h\\0\end{pmatrix}-f\begin{pmatrix}0\\0\end{pmatrix}\right)=\lim_{h\to 0}\frac{0}{h}=\lim_{h\to 0}0=0$$

$$\frac{\partial f}{\partial y}\begin{pmatrix}0\\0\end{pmatrix}=\lim_{h\to 0}\frac{1}{h}\left(f\begin{pmatrix}0\\h\end{pmatrix}-f\begin{pmatrix}0\\0\end{pmatrix}\right)$$

$$=\lim_{h\to 0}\frac{0}{h}=\lim_{h\to 0}0=0$$

The partials exist everywhere but are not continuous at $\begin{pmatrix}0\\0\end{pmatrix}$.

If f is differentiable at $\begin{pmatrix}0\\0\end{pmatrix}$, its derivative is represented by the Jacobian matrix, which in this case is $\begin{pmatrix} 0 & 0 \end{pmatrix}$.

To see if f is differentiable, we compute

$$\lim_{\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}} \frac{1}{\sqrt{h_1^2+h_2^2}} \left(f\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} - f\begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \right)$$

$$= \lim_{\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}} \frac{1}{\sqrt{h_1^2+h_2^2}} \left(\frac{h_1^2 h_2}{h_1^2+h_2^2} \right) = \lim_{\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}} \frac{h_1^2 h_2}{(h_1^2+h_2^2)^{3/2}}$$

approach along $h_1 = h_2 = h$: $\displaystyle\lim_{h \to 0} \frac{h^3}{(2h^2)^{3/2}} = \lim_{h \to 0} \frac{1}{\sqrt{8}} = \frac{1}{\sqrt{8}}$

$\neq 0$

So f is <u>not</u> differentiable at 0

Also, we cannot use the Jacobian matrix to compute directional derivatives (we can when f is differentiable).

e.g.: In the direction of $\begin{pmatrix}1\\1\end{pmatrix}$:

$$\lim_{h\to 0} \frac{1}{h}\left(f\left(h\begin{pmatrix}1\\1\end{pmatrix}\right) - f\begin{pmatrix}0\\0\end{pmatrix}\right) = \lim_{h\to 0} \frac{1}{h}\,\frac{h^3}{2h^2} = \frac{1}{2}$$

The Jacobian matrix would have given us 0!

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e.g.: $f(x) = \frac{x}{2} + x^2 \sin\frac{1}{x}$ if $x \neq 0$

$f(0) = 0$

f is continuous. f is differentiable away from 0.

Let's look at 0.

$$f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{1}{h}\left(\frac{h}{2} + h^2 \sin\frac{1}{h}\right)$$

$$= \lim_{h\to 0} \left(\frac{1}{2} + h \sin\frac{1}{h}\right) = \frac{1}{2}$$

So f is differentiable everywhere!

for $x \neq 0$ $\quad f'(x) = \frac{1}{2} + 2x \sin\frac{1}{x} + x^2\left(-\frac{1}{x^2}\right)\cos\frac{1}{x}$

$$= \frac{1}{2} + 2x \sin\frac{1}{x} - \cos\frac{1}{x}$$

The limit of $f'(x)$ as $x \to 0$ does not exist:

$x = \frac{1}{n\pi}$ $\quad$ or $\quad$ $x = \frac{1}{n\frac{\pi}{2}}$

$\cos\frac{1}{x} = \cos(n\pi) = (-1)^n$ $\qquad$ $\cos\frac{1}{x} = \cos\frac{n\pi}{2} = 0$

$\lim_{n\to\infty} \cos\left(\frac{1}{\frac{1}{2n\pi}}\right) = 1$ $\qquad$ $\lim_{n\to\infty} \cos\left(\frac{1}{\frac{1}{n\frac{\pi}{2}}}\right) = 0$

__Definition 1.9.6-7__: A function $f: U \longrightarrow \mathbb{R}^m$ is continuously differentiable on U if all the partial derivatives of f exist and are continuous on U.

We also say f is C^1.

A C^p function is a function which is p times continuously differentiable ($p \geq 1$ integer).

__Theorem 1.9.8__: If $f: U \to \mathbb{R}^m$ is C^1, then f is differentiable on U.

Before proving this, we talk a bit about rules of differentiation.

Theorem 1.8.1 $\quad U \subset \mathbb{R}^n$ open

1. If $f: U \longrightarrow \mathbb{R}^m$ is constant, then f is differentiable and $Df(a) = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \\ 0 & \cdots & 0 \end{pmatrix}$ everywhere.

2. If $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is linear, then f is differentiable everywhere and $Df(a) = f$

i.e., $Df(a)(v) = f(v)$

3. If $f = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix} : U \longrightarrow \mathbb{R}^m$, then f is differentiable at $a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ iff f_i is differentiable at a $\forall\, i = 1, \dots, m$.

4. If $f, g : U \to \mathbb{R}^m$ are differentiable, then so is $f+g$ and $D(f+g)(a) = Df(a) + Dg(a)$ $\forall a$

5. If $f : U \to \mathbb{R}$ and $g : U \to \mathbb{R}^m$ are differentiable at a, then so is fg and $(\ldots\ldots)$

$$D(fg)(a)[v] = f(a)\, Dg(a)[v] + \underbrace{Df(a)[v]}_{\text{scalar}}\, g(a)$$

6. If $f : U \to \mathbb{R}$ and $g : U \to \mathbb{R}^m$ are differentiable and $f(a) \neq 0$, then so is $\frac{g}{f}$ and

$$D\left(\frac{g}{f}\right)(a)[v] = \frac{1}{f(a)}\, Dg(a)[v] - \frac{Df(a)[v]}{(f(a))^2}\, g(a)$$