Our guess for the derivative at A is

$$H \longmapsto AH + HA \quad \text{linear map } M_n(\mathbb{R}) \to M_n(\mathbb{R})$$

Now we show that this is the total derivative of $A \mapsto A^2$.

We show $\lim\limits_{H \to 0} \frac{1}{|H|}\left((A+H)^2 - A^2 - (AH + HA)\right) = 0$

($H \longmapsto AH + HA$ is the derivative of $A \mapsto A^2$ at A applied to H.)

we compute: $(A+H)^2 - A^2 - AH - HA = H^2$

$$\lim_{H \to 0} \frac{1}{|H|} H^2 = 0 \quad \text{because.}$$

$$\lim_{H \to 0} \frac{1}{|H|} |H^2| = 0 \quad \text{because}$$

$$0 \leq \frac{1}{|H|} |H^2| \leq \frac{1}{|H|} |H|^2$$ result from last quarter: for any matrices A, B, $|AB| \leq |A| \cdot |B|$ (1.4.11)

$$\lim_{H \to 0} \frac{|H|^2}{|H|} = \lim_{H \to 0} |H| = 0$$

So we are done. $\square$

<u>Example:</u> The derivative of $A \longmapsto A^{-1}$
$M_n(\mathbb{R}) \longrightarrow M_n(\mathbb{R})$.

First we guess the answer: analyse $f(A+H) - f(A)$:

$$(A+H)^{-1} - A^{-1} = \left(A(I + A^{-1}H)\right)^{-1} - A^{-1}$$

$$= (I + A^{-1}H)^{-1} A^{-1} - A^{-1} = \left[(I + A^{-1}H)^{-1} - I\right] A^{-1}$$

$$H \to 0 \implies A^{-1}H \to 0$$

So for $|H|$ small enough, we can assume $|A^{-1}H| < 1$

so $$\left(I - (-A^{-1}H)\right)^{-1} = I + (-A^{-1}H) + (-A^{-1}H)^2 + \cdots$$
$$= \sum_{m=0}^{\infty} (-A^{-1}H)^m$$

and $$\left[(I + A^{-1}H)^{-1} - I\right]A^{-1} = \left[\sum_{m=0}^{\infty} (-A^{-1}H)^m - I\right]A^{-1}$$
$$= \left[-A^{-1}H + (-A^{-1}H)^2 + \cdots\right]A^{-1}$$

We keep the linear term: our guess for the derivative at A is $$H \longmapsto -A^{-1}HA^{-1} = [Df(A)](H)$$

Now we prove that this the total derivative.

We show $\lim_{H \to 0} \frac{1}{|H|}\left((A+H)^{-1} - A^{-1} - (-A^{-1}HA^{-1})\right) = 0$

We already computed: $(A+H)^{-1} - A^{-1} = \left[-A^{-1}H + (-A^{-1}H)^2 + \cdots\right]A^{-1}$

So we compute

$$\lim_{H \to 0} \frac{1}{|H|}\left(\left[-A^{-1}H + (-A^{-1}H)^2 + \cdots\right]A^{-1} + A^{-1}HA^{-1}\right)$$

$$= \lim_{H \to 0} \frac{1}{|H|}\left((-A^{-1}H)^2A^{-1} + (-A^{-1}H)^3A^{-1} + \cdots\right)$$

$$= \lim_{H \to 0} \frac{1}{|H|}(A^{-1}H)^2\left(I + (-A^{-1}H) + (-A^{-1}H)^2 + \cdots\right)A^{-1}$$

We compute $\lim_{H \to 0} \frac{1}{|H|}\left|(A^{-1}H)^2\left(I + (-A^{-1}H) + \cdots\right)A^{-1}\right|$

$$\left|(A^{-1}H)^2\left(I + (-A^{-1}H) + \cdots\right)A^{-1}\right| \leq |A^{-1}H|^2 \cdot \left|I + (-A^{-1}H) + \cdots\right| \cdot |A^{-1}|$$

$$\leq |A^{-1}|^3 |H|^2 \left|I + (-A^{-1}H) + \cdots\right|$$

$$\left|I + (-A^{-1}H) + \cdots + (-AH)^m\right| \leq |I| + |A^{-1}H| + \cdots + |A^{-1}H|^m$$

take the limit as $m \to \infty$ to obtain

$$\left|I + (-A^{-1}H) + \cdots\right| \leq |I| + |A^{-1}H| + \cdots$$

$$= |I| - 1 + 1 + |A^{-1}H| + \cdots$$

$$= |I| - 1 + \frac{1}{1 - |A^{-1}H|}$$

$H \to 0$, so for $|H|$ small enough, we can assume

$$|A^{-1}H| \leq \frac{1}{2} \quad \text{so} \quad \frac{1}{1 - |A^{-1}H|} \leq 2$$

So $\left| I + |A^{-1}H| + \cdots \right| \leq |I| - 1 + 2 = \sqrt{n} + 1$

and $\frac{1}{|H|} |A^{-1}H|^2 \left| I + (A^{-1}H) + \cdots \right| |A^{-1}| \leq \frac{1}{|H|} |A^{-1}|^3 |H|^2 (\sqrt{n}+1)$

$$= |A^{-1}|^3 (\sqrt{n}+1) |H|$$

So the limit is 0 as $H \to 0$. $\square$

The mean value theorem in several variables:

In one variable, the mean value theorem is the following:

$f : [a,b] \to \mathbb{R}$ continuous, differentiable on (a,b),
then $\exists\, c \in (a,b)$ s.t. $f(b) - f(a) = f'(c)(b-a)$

In several variables we have:

Theorem 1.9.1: $U \subset \mathbb{R}^n$, $f : U \to \mathbb{R}$ differentiable. Suppose $a, b \in U$ satisfy $[a,b] \subset U$ where $[a,b]$ is the segment joining a to b in $\mathbb{R}^n$.

Then $\exists$ $c_0 \in (a,b) \subset U$ s.t.

$$f(b) - f(a) = [Df(c_0)](b-a)$$

Proof: Put $g(t) := f((1-t)a + tb)$ $t \in \mathbb{R}$

note $g(0) = f(a)$ $g(1) = f(b)$

$$g'(t) = \lim_{s \to 0} \frac{g(t+s) - g(t)}{s}$$

$$= \lim_{s \to 0} \frac{f((1-t-s)a + (t+s)b) - f((1-t)a + tb)}{s}$$

note $f((1-t-s)a+(t+s)b) = f((1-t)a+tb-s(b-a))$

So $$\lim_{s\to 0} \frac{g(t+s)-g(t)}{s}$$

$$= \lim_{s\to 0} \frac{f((1-t)a+tb+s(b-a))-f((1-t)a+tb)}{s}$$

this is the directional derivative of f in the direction of $b-a$ so it is equal to

$$[Df((1-t)a+tb)](b-a)$$

The one variable mean value theorem says $\exists\, s_0 \in (0,1)$

s.t. $$g(1)-g(0) = g'(s_0)(1-0)$$

Substitute to get:

$$f(b) - f(a) = [Df((1-s_0)a + s_0 b)](b-a)$$

we can take $c_0 = (1-s_0)a + s_0 b$

$\square$