We finish the computation of the total derivative for the example $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2+1+y \\ \sin(xy) \\ e^x - e^y \end{pmatrix}$

$$b_2 := \frac{1}{|h|}\Big(\sin\big((x+h_1)(y+h_2)\big) - \sin(xy) - y\cos(xy)\,h_1 - x\cos(xy)\,h_2\Big)$$

One way to compute this limit is to use the McLaurin series for sine and cosine.

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \cdots$$

$$\Rightarrow \cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$$

$$b_2 = \frac{1}{|h|}\left((x+h_1)(y+h_2) - \frac{(x+h_1)^3(y+h_2)^3}{3!} + \cdots - xy + \frac{x^3y^3}{3!} - \cdots \right.$$
$$\left. - y\left(1 - \frac{(xy)^2}{2!} + \frac{(xy)^4}{4!} - \cdots\right)h_1 - x\left(1 - \frac{(xy)^2}{2!} + \frac{(xy)^4}{4!}\right)h_2 \right)$$

First: $(x+h_1)(y+h_2) - xy - yh_1 - xh_2 = h_1h_2$

degree 3: $-\frac{(x^3+3x^2h_1+\cdots)(y^3+3y^2h_2+\cdots)}{3!} - \frac{x^3y^3}{3!}$

$$+ y\frac{x^2y^2h_1}{2} + \cdots + x\frac{x^2y^2}{2}h_2 + \cdots$$

In all degrees the terms that have degree ≤ 1 in h_1 or h_2 cancel each other, so we only have terms of degree ≥ 2 in h_1 h_2.

So $\quad r_2 = \dfrac{1}{|h|}\Big(h_1^2 A(h_1,h_2,x,y) + h_1 h_2 B(h_1,h_2,x,y)$
$\qquad\qquad + h_2^2 C(h_1,h_2,x,y)\Big)$

A, B, C are continuous functions for all h_1, h_2, x, y because they are obtained from the everywhere absolutely convergent power series for sine and cosine.

and $\displaystyle \lim_{h\to 0} \frac{h_1^2}{|h|} = \lim_{h\to 0} \frac{h_1 h_2}{|h|} = \lim_{h\to 0} \frac{h_2^2}{|h|} = 0$

So $\displaystyle \lim_{h\to 0} r_2 = 0$

$$r_3 := \frac{1}{|h|}\left(e^{x+h_1} - e^{y+h_2} - e^{x} + e^{y} - e^{x} h_1 + e^{y} h_2 \right)$$

$$b_3 = \frac{1}{|h|}\left(e^x\left(e^{h_1} - 1 - h_1\right) - e^y\left(e^{h_2} - 1 - h_2\right)\right)$$

$$e^{h_1} = 1 + h_1 + \frac{h_1^2}{2!} + \cdots$$

$$\Rightarrow \quad e^{h_1} - 1 - h_1 = h_1^2 A(h_1)$$

$$e^{h_2} - 1 - h_2 = h_2^2 B(h_2)$$

$$b_3 = \frac{1}{|h|}\left(e^x h_1^2 A(h_1) - e^y h_2^2 B(h_2)\right)$$

As before, $\lim\limits_{h \to 0} b_3 = 0$

Directional derivatives:

$U \subset \mathbb{R}^n$ open $\quad f : U \longrightarrow \mathbb{R}^m$ function.

$$f = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}$$

recall $\dfrac{\partial f_i}{\partial x_j} = \lim\limits_{h_j \to 0} \dfrac{f_i\begin{pmatrix} x_1 \\ \vdots \\ x_j + h_j \\ \vdots \end{pmatrix} - f_i\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}{h_j}$

If $a = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and $e_1, \dots, e_n$ is the standard basis for $\mathbb{R}^n$

then $\begin{pmatrix} x_1 \\ \vdots \\ x_j + h_j \\ \vdots \end{pmatrix} = a + h_j e_j$ and $|h_j| = |h_j e_j|$

The partial derivative

$$\frac{\partial f}{\partial x_j} := \begin{pmatrix} \frac{\partial f_1}{\partial x_j} \\ \vdots \\ \frac{\partial f_m}{\partial x_j} \end{pmatrix}$$

is a special case of a directional derivative.

It is the directional derivative in the direction of e_j. More generally, we have

Def 1.7.13: $U \subset \mathbb{R}^n$ open $f : U \to \mathbb{R}^m$ a function. $0 \neq v \in \mathbb{R}^n$ a vector. The directional derivative of f in the direction of v, if it exists, is

$$\lim_{h \to 0} \frac{f(a+hv) - f(a)}{h}$$

Proposition 1.7.14: $U \subset \mathbb{R}^n$ open $f: U \to \mathbb{R}^m$ a function.

If f is differentiable at $a \in \mathbb{R}^n$, then all directional derivatives at a exist and, for $\vec{v} \in \mathbb{R}^n$, the directional derivative in the direction of v is

$$\lim_{h \to 0} \frac{f(a+hv) - f(a)}{h} = (Df(a))(v)$$

Remark: We can the directional derivative, when it exists, as $D_v(f)(a)$ or $\frac{\partial f}{\partial v}(a)$ or $\partial_v f(a)$

Proof of the Proposition: We know:

$$\lim_{h \to 0} \frac{1}{|h|} \left(f(a+h) - f(a) - (Df(a))h \right) = 0$$

So $\lim_{h \to 0} \frac{1}{|hv|} \Big(f(a+hv) - f(a) - \big(Df(a)\big)(hv) \Big) = 0$

$$\parallel$$

$\lim_{h \to 0} \frac{1}{|h|\,|v|} \Big(f(a+hv) - f(a) - h\big(Df(a)\big)(v) \Big) = 0$

$|v|$ is a nonzero constant, so:

$$\lim_{h \to 0} \frac{1}{|h|} \Big(f(a+hv) - f(a) - h\, Df(a)(v) \Big) = 0$$

and $\lim_{h \to 0} \frac{1}{h} \Big(f(a+hv) - f(a) - h\, Df(a)(v) \Big) = 0$

$\Rightarrow$ $\lim_{h \to 0} \frac{1}{h} \Big(f(a+hv) - f(a) \Big)$ exists and is equal to $Df(a)(v)$. $\square$

Sometimes the Jacobian matrix is too complicated.

We can sometimes compute total derivatives in different ways.

e.g.:

$$f : M_n(\mathbb{R}) \longrightarrow M_n(\mathbb{R})$$

$$\begin{array}{ccc} \mathbb{R}^{n^2} \cong M_n(\mathbb{R}) & & M_n(\mathbb{R}) \cong \mathbb{R}^{n^2} \\ A & \longmapsto & A^2 \end{array}$$

How do we guess the answer:

$$\lim_{H \to 0} \frac{1}{|H|}\Big(f(A+H) - f(A) - L(H)\Big) = 0 \quad ?$$

$$f(A+H) - f(A) = (A+H)^2 - A^2 = A^2 + AH + HA + H^2 - A^2$$

$$= AH + HA + H^2$$