So $\dfrac{\partial f_1}{\partial x_1}(a)$ = first coordinate of $L(e_1)$.

(And similarly for the other coordinates)

$\square$

<u>e.g.</u>: $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2+1+y \\ \sin(xy) \\ e^x - e^y \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

then the Jacobian matrix is

$$\begin{pmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \\ \dfrac{\partial f_3}{\partial x} & \dfrac{\partial f_3}{\partial y} \end{pmatrix}$$

we compute : $Jf(x,y) = \begin{pmatrix} 2x & 1 \\ y\cos(xy) & x\cos(xy) \\ e^x & -e^y \end{pmatrix} = M(x,y)$

To show f is differentiable, compute the limit :

$$\lim_{h \to 0} \frac{1}{|h|} \Big(f(a+h) - f(a) - M(h) \Big)$$

$$h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \qquad a = \begin{pmatrix} x \\ y \end{pmatrix} \qquad a+h = \begin{pmatrix} x+h_1 \\ y+h_2 \end{pmatrix}$$

$$f(a+h) = \begin{pmatrix} (x+h_1)^2 + 1 + (y+h_2) \\ \sin((x+h_1)(y+h_2)) \\ e^{x+h_1} - e^{y+h_2} \end{pmatrix} \qquad f(a) = \begin{pmatrix} x^2+1+y \\ \sin(xy) \\ e^x - e^y \end{pmatrix}$$

$$\frac{1}{|h|}\left[\begin{pmatrix} (x+h_1)^2+1+(y+h_2) - (x^2+1+y) \\ \sin((x+h_1)(y+h_2)) - \sin(xy) \\ e^{x_1+h_1} - e^{y+h_2} - e^{x} + e^{y} \end{pmatrix}\right]$$

$$M(h) = \begin{pmatrix} 2x & 1 \\ y\cos(xy) & x\cos(xy) \\ e^{x} & -e^{y} \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

$$= \begin{pmatrix} 2x h_1 + h_2 \\ y\cos(xy)h_1 + x\cos(xy)h_2 \\ e^{x}h_1 - e^{y}h_2 \end{pmatrix}$$

$$\frac{1}{|h|}\Big(f(a+h)-f(a)-M(h)\Big)=$$

$$\frac{1}{|h|}\begin{pmatrix} (x+h_1)^2+1+(y+h_2)-x^2-1-y-2xh_1-h_2 \\ \sin\big((x+h_1)(y+h_2)\big)-\sin(xy)-y\cos(xy)h_1-x\cos(xy)h_2 \\ e^{x+h_1}-e^{y+h_2}-e^x+e^y-e^xh_1+e^yh_2 \end{pmatrix}$$

We show that all three entries are 0:

$$l_1 := \frac{1}{|h|}\Big((x+h_1)^2+1+(y+h_2)-x^2-1-y-2xh_1-h_2\Big)$$

$$= \frac{1}{|h|}\left(h_1^2\right) \qquad = \frac{h_1^2}{\sqrt{h_1^2+h_2^2}}$$

$$0 \leq \frac{h_1^2}{\sqrt{h_1^2 + h_2^2}} \leq \frac{h_1^2}{\sqrt{h_1^2}} = \frac{h_1^2}{|h_1|} = |h_1|$$

goes to 0.

Assume f is bounded on $X \subset \mathbb{R}^n$ compact.
Then f has a supremum and f has an infimum.

$$M := \sup_{x \in X} f(x) \qquad \exists \text{ sequence } i \mapsto y_i$$

$$\exists x_i : y_i = f(x_i) \text{ s.t. } \lim_{i \to \infty} y_i = M$$

<u>Because:</u> M is the least upper bound of the image of f.

$\Rightarrow \forall \varepsilon > 0 \quad \exists \ x \in X$ s.t. $f(x) > M - \varepsilon$

Choose $\varepsilon = \frac{1}{i} \quad i \in \mathbb{N}$

$$\forall i \quad \exists \quad x_i \in X \text{ s.t. } \underset{y_i}{\underset{\shortparallel}{f(x_i)}} > M - \frac{1}{i}$$

<u>Note:</u> $\lim_{i \to \infty} y_i = M$

$\exists \quad j \longmapsto i(j)$ s.t. $j \longmapsto x_{i(j)}$ is convergent because X is compact. Put $b := \lim_{j\to\infty} x_{i(j)}$

If f is continuous, then

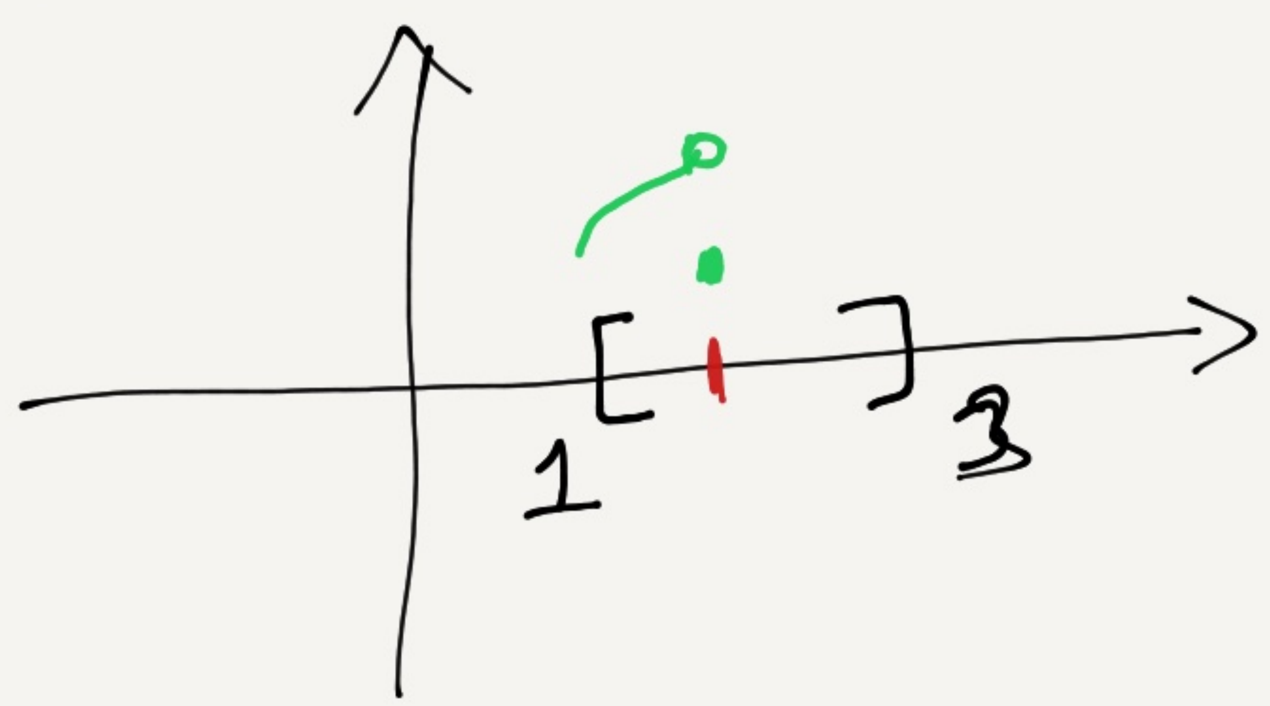
$$\lim_{j\to\infty} f(x_{i(j)}) = f\left(\lim_{j\to\infty} x_{i(j)}\right) = f(b)$$

We also have

$$\lim_{j\to\infty} f(x_{i(j)}) = \lim_{j\to\infty} y_{i(j)} = M$$

$$\Longrightarrow \quad f(b) = M$$

If f is not continuous, we can make examples where this fails:



Differentiability implies continuity: (1.7.11)

Suppose $f: U \to \mathbb{R}^m$ is differentiable at $a \in U$, where $U \subset \mathbb{R}^n$ is open. Then f is continuous at a.

Proof: Since f is differentiable at a, we have

$$\lim_{h \to 0} \frac{1}{|h|}\Big(f(a+h) - f(a) - L(h) \Big) = 0$$

where $L : \mathbb{R}^n \to \mathbb{R}^m$ is linear $\big(L = Df(a)\big)$.

Since $\lim_{h \to 0} |h| = 0$, we also have

$$\lim_{h \to 0} \frac{1}{|h|}\Big(f(a+h) - f(a) - L(h) \Big) |h| = 0$$

Or: $\lim_{h \to 0} \Big(f(a+h) - f(a) - L(h) \Big) = 0$

Since L is linear (think multiplication by a constant matrix), $\lim_{h \to 0} L(h) = 0$. So $\lim_{h \to 0} \big(f(a+h) - f(a) \big) = 0$ and f is continuous at a.

$\square$

(3) from Practice:

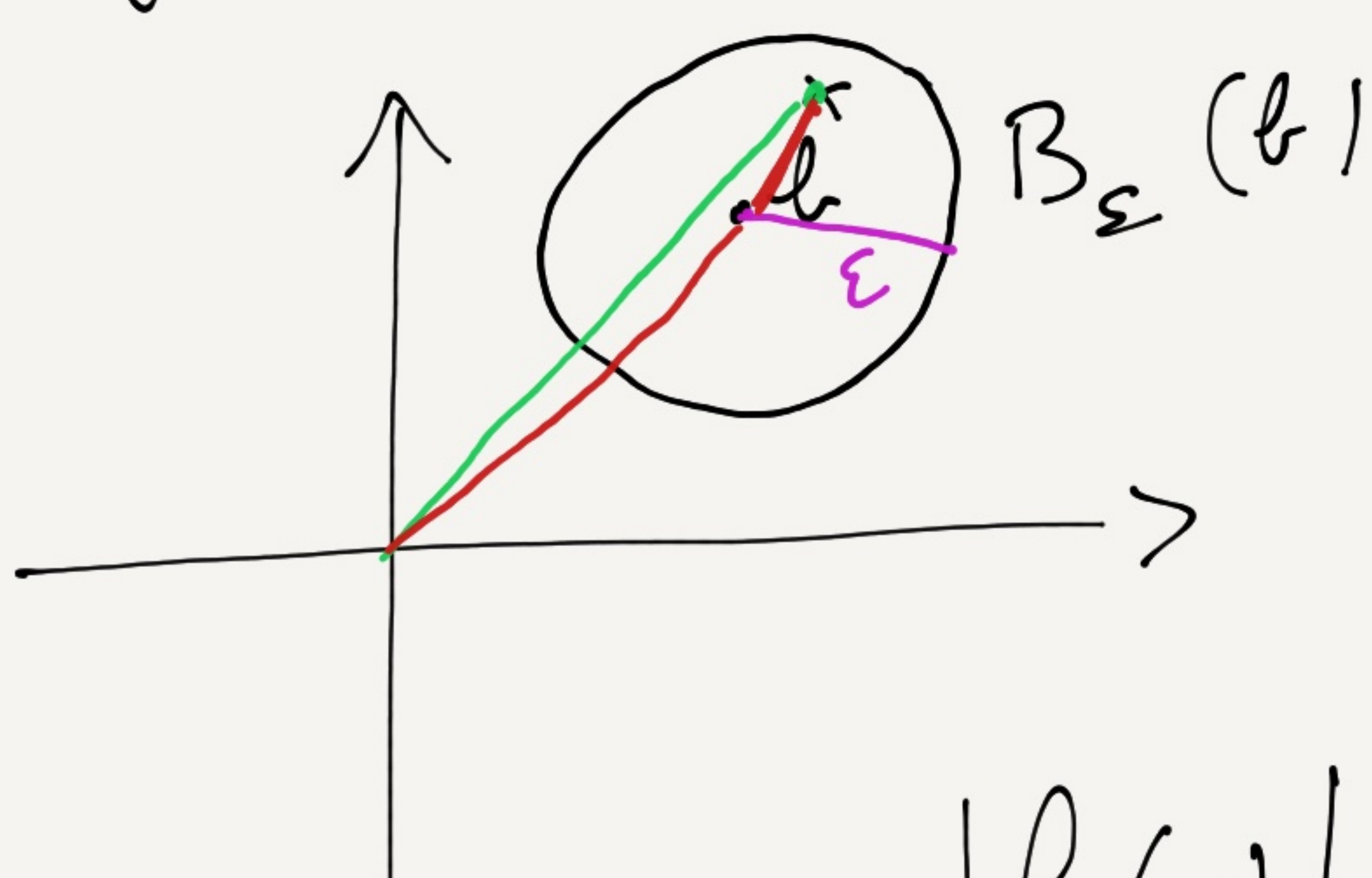
If $f: X \longrightarrow \mathbb{R}^m$ has a limit when $v \longrightarrow v_0 \in X$, then $\exists$ neighborhood of v_0 on which f is bounded.

Put $b := \lim f(v)$ as $v \longrightarrow v_0$.

Proof: $\forall \ \varepsilon > 0 \ \exists \ \delta > 0$ s.t.

$$|v - v_0| < \delta \Longrightarrow |f(v) - b| < \varepsilon$$

So for $v \in B_\delta(v_0)$, we have $f(v) \in B_\varepsilon(b)$



$$\Rightarrow |f(v)| < |b| + \varepsilon$$

$$|f(v)| \leq |b| + |f(v) - b| < |b| + \varepsilon$$

$$\Rightarrow f(v) \in B_{|b|+\varepsilon}(0)$$

$\square$

f is bounded on a neighborhood of v_0.

means:

$\exists$ neighborhood U of v_0 and a constant M

s.t. $\forall \; v \in U$, $|f(v)| \leq M$

(5) <u>From practice</u>: $g: \mathbb{N} \to \mathbb{N}$ strictly increasing.

<u>By induction</u>: Initial step: $g(1) \geq 1$ because $g(1) \in \mathbb{N}$.

Induction hypothesis: $g(k) \geq k$

Induction step: prove $g(k+1) \geq k+1$

g is strictly increasing, so $g(k+1) > g(k) \geq k$

(induction hypothesis)

so $g(k+1) > k \Rightarrow g(k+1) \geq k+1$

because $g(k+1)$ is an integer

$\square$