Example: $f(x,y) = \begin{pmatrix} x^2 \\ \sin(xy) \end{pmatrix}$ $\quad f: \mathbb{R}^2 \to \mathbb{R}^2$

$$Jf(x,y) = \begin{pmatrix} 2x & 0 \\ y\cos(xy) & x\cos(xy) \end{pmatrix}$$

The total derivative,

Let's briefly go back to the one variable case:

$f: \mathbb{R} \to \mathbb{R}$ differentiable:

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$$

rewrite: $\displaystyle \lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h} - f'(a) \right) = 0$

or $$\lim_{h\to 0} \frac{1}{h}\left(f(a+h)-f(a)-f'(a)h\right)=0$$

We think of this as approximating f near a with an affine transformation.

(An affine transformation is the sum of a linear transformation and a constant (vector))

So, in higher dimensions, we also try to approximate $f:\mathbb{R}^n \longrightarrow \mathbb{R}^m$ with an affine transformation.

Proposition and definition 1.7.9:

(total Derivative)

Let $U \subset \mathbb{R}^n$ be an open subset and $f: U \to \mathbb{R}^m$ a mapping (or function). Let $a \in U$.

We say f is differentiable at a if there exists a linear transformation $L: \mathbb{R}^n \to \mathbb{R}^m$

s.t.

$$\lim_{\vec{h} \to \vec{0}} \frac{1}{|h|}\Big(f(a+h) - f(a) - L(h)\Big) = 0.$$

The transformation L is unique and L is the derivative of f at a, denoted $Df(a)$.

Theorem 1.7.10 If f is differentiable at a, then all partial derivatives of f at a exist, and the matrix representing $Df(a)$ is the Jacobian matrix $Jf(a)$.

Proof of 1.7.9 and 1.7.10:

$$L := Df(a)$$

Let $e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}$, $\ldots$, $e_n \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

be the standard basis of $\mathbb{R}^n$.

Then the matrix of L in this basis is:

$$\begin{array}{cc} \begin{pmatrix} L(e_1), \ldots, L(e_n) \end{pmatrix} & \updownarrow \ m \text{ rows} \end{array}$$

Note that $\dfrac{\partial f_i}{\partial x_1}(a) = \lim\limits_{h_1 \to 0} \dfrac{f_i\left(a + \begin{pmatrix} h_1 \\ 0 \\ \vdots \end{pmatrix}\right) - f_i(a)}{h}$

$f = (f_1, \dots, f_m)$ $\qquad = \lim\limits_{h_1 \to 0} \dfrac{f_i(a + h_1 e_1) - f_i(a)}{h_1}$

We know f is differentiable, so

$$\lim_{h_1 e_1 \to 0} \frac{1}{|h_1 e_1|} \Big(f(a + h_1 e_1) - f(a) - L(h_1 e_1) \Big) = 0$$

or $\displaystyle\lim_{h_1 \to 0} \frac{1}{|h_1|} \Big(f(a + h_1 e_1) - f(a) - h_1 L(e_1) \Big) = 0$

Since the limit is 0, we also have:

$$\lim_{h_1 \to 0} \frac{1}{h_1} \Big(f(a + h_1 e_1) - f(a) - h_1 L(e_1) \Big) = 0$$

Or $\lim_{h_1 \to 0} \left[\frac{1}{h_1} \left(f(a+h_1 e_1) - f(a) \right) - L(e_1) \right] = 0$

or $\lim_{h_1 \to 0} \frac{1}{h_1} \left(f(a+h_1 e_1) - f_1(a) \right) = L(e_1)$

or $\lim_{h_1 \to 0} \frac{1}{h_1} \left(f\begin{pmatrix} a_1 + h_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} - f\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \right) = L(e_1)$

or $\lim_{h_1 \to 0} \frac{1}{h_1} \left[\begin{pmatrix} f_1(a+h_1 e_1) \\ \vdots \\ f_m(a+h_1 e_1) \end{pmatrix} - \begin{pmatrix} f_1(a) \\ \vdots \\ f_m(a) \end{pmatrix} \right] = L(e_1)$

<u>first row</u>: $\lim_{h_1 \to 0} \frac{1}{h_1} \left(f_1(a+h_1 e_1) - f_1(a) \right) =$ first coordinate of $L(e_1)$

So $\dfrac{\partial f_1}{\partial x_1}(a)$ = first coordinate of $L(e_1)$.

(And similarly for the other coordinates)

$\square$

<u>e.g.</u>: $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2+1+y \\ \sin(xy) \\ e^x - e^y \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

then the Jacobian matrix is $\begin{pmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \\ \dfrac{\partial f_3}{\partial x} & \dfrac{\partial f_3}{\partial y} \end{pmatrix}$

we compute : $Jf(x,y) = \begin{pmatrix} 2x & 1 \\ y\cos(xy) & x\cos(xy) \\ e^x & -e^y \end{pmatrix} = M(x,y)$

To show f is differentiable, compute the limit :

$$\lim_{h\to 0} \frac{1}{|h|} \left(f(a+h) - f(a) - M(h) \right)$$

$$h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \qquad a = \begin{pmatrix} x \\ y \end{pmatrix} \qquad a+h = \begin{pmatrix} x+h_1 \\ y+h_2 \end{pmatrix}$$

$$f(a+h) = \begin{pmatrix} (x+h_1)^2 + 1 + (y+h_2) \\ \sin((x+h_1)(y+h_2)) \\ e^{x+h_1} - e^{y+h_2} \end{pmatrix} \qquad f(a) = \begin{pmatrix} x^2+1+y \\ \sin(xy) \\ e^x - e^y \end{pmatrix}$$

$$\frac{1}{|h|}\left[\begin{pmatrix} (x+h_1)^2+1+(y+h_2) & - & (x^2+1+y) \\ \sin((x+h_1)(y+h_2)) & - & \sin(xy) \\ e^{x_1+h_1}-e^{y+h_2} & - & e^{x}+e^{y} \end{pmatrix}\right]$$

$$M(h) = \begin{pmatrix} 2x & 1 \\ y\cos(xy) & x\cos(xy) \\ e^{x} & -e^{y} \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

$$= \begin{pmatrix} 2xh_1+h_2 \\ y\cos(xy)h_1+x\cos(xy)h_2 \\ e^{x}h_1-e^{y}h_2 \end{pmatrix}$$

$$\frac{1}{|h|}\Big(f(a+h)-f(a)-M(h)\Big)=$$

$$\frac{1}{|h|}\begin{pmatrix} (x+h_1)^2+1+(y+h_2)-x^2-1-y-2xh_1-h_2 \\ \sin\big((x+h_1)(y+h_2)\big)-\sin(xy)-y\cos(xy)h_1-x\cos(xy)h_2 \\ e^{x+h_1}-e^{y+h_2}-e^x+e^y-e^xh_1+e^yh_2 \end{pmatrix}$$