We use a function which allows us to use the previous result. Put $m := \dfrac{f(b)-f(a)}{b-a}$

Then $g(x) = f(x) - mx$

In fact, if we put

$$h(x) = f(x) - f(a) - m(x-a)$$

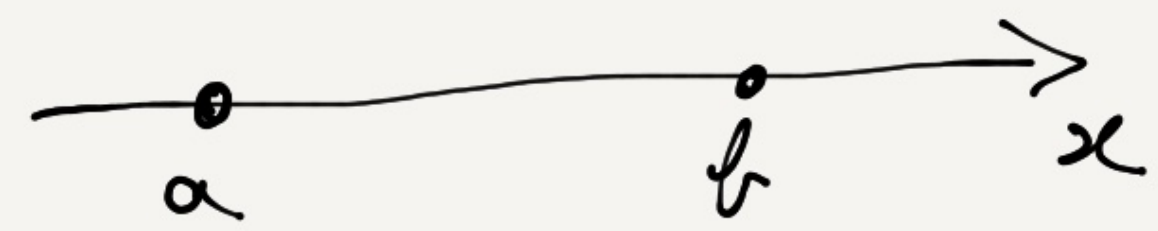
then $h(a) = 0$ and $h(b) = 0$

If h is constant then $h'(x) = 0$ for all x and $f'(x) = m$ for every $x \in (a, b)$, and we are done.

If h is not constant, since $h(a) = h(b) = 0$,

h takes either positive values

or negative values



$h(a) = h(b) = 0$

$[a, b]$ is compact, so h has a minimum or a maximum on $[a, b]$.

So h has either a positive maximum

or a negative minimum

$\Rightarrow \exists\, c \in [a, b]$ s.t. $h(c)$ is

either a positive maximum

or a negative minimum

$c \neq a, b$ because $h(a) = h(b) = 0$

By Theorem 1.6.12, $h'(c) = 0 = f'(c) - m$ $\square$

Proof of Theorem 1.6.9:

We first show that f has a supremum, i.e., the set of values of f has an upper bound.

Suppose that f does not have a supremum.

Then $\forall \quad i \in \mathbb{N}, \ \exists \ v_i \in C$, s.t. $f(v_i) \geq i$.

This gives us a sequence $i \mapsto v_i$ of points of C.

By Theorem 1.6.3, $i \mapsto v_i$ has a convergent subsequence, say $j \mapsto v_{i(j)}$. Put $b := \lim_{j \to \infty} v_{i(j)} \in C$

Since f is continuous, we have $\lim_{j \to \infty} f(v_{i(j)}) = f(b)$

$\Rightarrow \ \forall \ \varepsilon > 0, \ \exists \delta$, s.t. $|v - b| < \delta \Longrightarrow |f(v) - f(b)| < \varepsilon$.

$$\Rightarrow f(b) - \varepsilon < f(v) < f(b) + \varepsilon$$

given $\delta > 0$, $\exists$ M s.t. $\forall$ $j \geq M$

$$|v_{i(j)} - b| < \delta$$

$$\Rightarrow f(b) - \varepsilon < f(v_{i(j)}) < f(b) + \varepsilon$$

for j very large, $i(j) \geq f(b) + 2\varepsilon$

and $f(v_{i(j)}) \geq i(j)$

we have a contradiction.

So far we proved that f has a supremum M.

Next, we show $\exists$ $v \in C$ s.t. $f(v) = M$:

There is a sequence $i \mapsto v_i \in C$ s.t.

$$\lim_{i \to \infty} f(v_i) = M$$

$$\forall \varepsilon > 0 \quad \exists\, v_\varepsilon \in C \text{ s.t. } M - \varepsilon \le f(v_\varepsilon) \le M$$

$$\forall n \quad \exists\, v_n \in C \text{ s.t. } M - \frac{1}{n} \le f(v_n) \le M$$

$$\Rightarrow \lim_{n \to \infty} f(v_n) = M.$$

$\exists$ convergent subsequence $j \mapsto v_{n(j)}$

say $\displaystyle\lim_{j \to \infty} v_{n(j)} =: b \in C$

$$\text{Then } \left.\begin{aligned} \lim_{j \to \infty} f(v_{n(j)}) &= f(b) \\ \text{and } &= M \end{aligned}\right\} \Rightarrow f(b) = M$$

$\square$

Derivatives in several variables:

Recall

Def 1.7.1: $U \subset \mathbb{R}$ open, $f : U \to \mathbb{R}$ a function.

f is differentiable at $a \in U$ with derivative $f'(a)$ if

$$\lim_{h \to 0} \frac{1}{h}\big(f(a+h) - f(a)\big) \text{ exists and is equal to } f'(a).$$

To generalize this, we first do:

Def 1.7.3: $U \subset \mathbb{R}^n$ open $f : U \to \mathbb{R}$ function.

The partial derivative of f with respect to the i-th variable, at $a = (a_1, \dots, a_n)$ (if it exists) is the limit

$$\lim_{h\to 0} \frac{1}{h}\left(f\begin{pmatrix} a_1 \\ \vdots \\ a_i+h \\ \vdots \\ a_n\end{pmatrix} - f\begin{pmatrix} a_1 \\ \vdots \\ a_n\end{pmatrix}\right)$$

this is denoted $\dfrac{\partial f}{\partial x_i}(a)$ or $D_i f(a)$

Note! $\begin{pmatrix} a_1 \\ \vdots \\ a_i+h \\ \vdots \\ a_n\end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_n\end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ h \\ \vdots \\ 0\end{pmatrix} = a + he_i$

where $e_1, \dots, e_n$ are the vectors $\begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \dots$

Note: If $f: U \longrightarrow \mathbb{R}^m$, then f has coordinate functions: $f(a) = (f_1(a), \dots, f_m(a))$ and, for $1 \leq i \leq n$ and $1 \leq j \leq m$ we can compute $\dfrac{\partial f_j}{\partial x_i}(a)$

Def 1.7.7 : The Jacobian matrix of f at a is:
$(f : U \longrightarrow \mathbb{R}^n)$

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \frac{\partial f_1}{\partial x_2}(a) & \cdots & \frac{\partial f_1}{\partial x_n}(a) \\ \frac{\partial f_2}{\partial x_1}(a) & & & \vdots \\ \vdots & & & \vdots \\ \frac{\partial f_m}{\partial x_1}(a) & \cdots & \cdots & \frac{\partial f_m}{\partial x_n}(a) \end{pmatrix}$$

Example: $f(x, y) = \begin{pmatrix} x^2 \\ \sin(xy) \end{pmatrix}$

$$Jf(x,y) = \begin{pmatrix} 2x & 0 \\ y\cos(xy) & x\cos(xy) \end{pmatrix}$$