Def. 1.6.1 (bounded set) A subset $X \subset \mathbb{R}^n$ is called bounded if $\exists\ R > 0$ s.t. $X \subset B_R(0)$.

Def 1.6.2: (compact set) A non-empty set $X \subset \mathbb{R}^n$ is called compact if X is bounded and closed.

e.g.: any closed ball in $\mathbb{R}^n$ is compact
any closed box or cube is compact.
closed bounded intervals are compact.

Theorem: If $C \subset \mathbb{R}^n$ is compact, then any sequence of points of C has a convergent subsequence (the limit of the subsequence is then also in C)

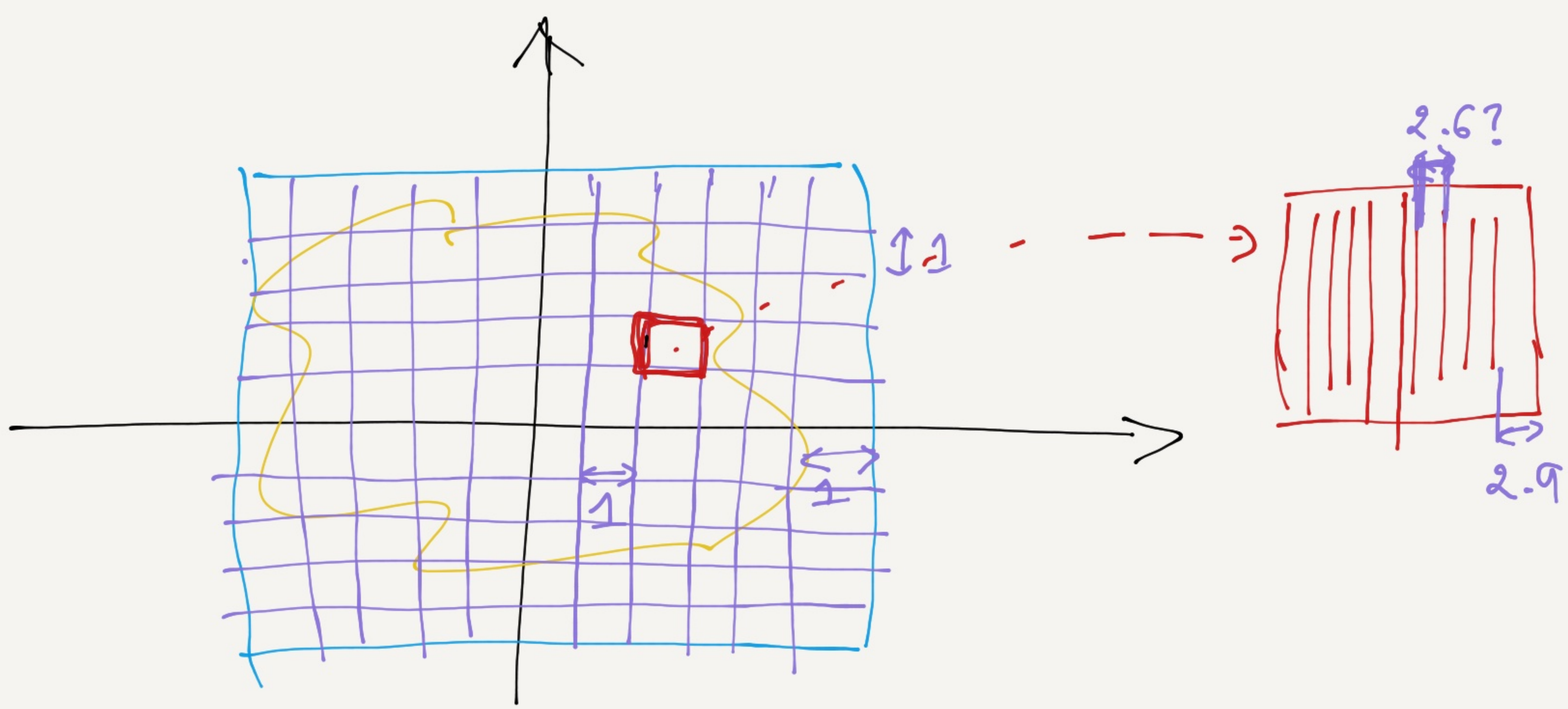
Proof: Let $i \longmapsto v_i \in C$ be a sequence of points of C.

There exists a large real number $R > 0$ s.t

$$C \subset B_R(0).$$

$\cap \quad C \subset$ box with side length M centered at the origin O.

Divide the box into a union of closed boxes of side length 1.



$\exists$ box which contains infinitely many of the v_i.

let $v_{i(1)}$ be a point in this first box

Now divide the first box into boxes of side length $\frac{1}{10}$: we get 10^n boxes.

One of the smaller boxes contains infinitely many points of the sequence. Choose a box of side length $\frac{1}{10}$ which contains ∞ly many v_i.

Let $\boxed{v_{i(2)}}$ be a point in this second box.

we can choose $i(2) > i(1)$ because we have ∞ly many v_i to choose from.

Continue: divide the last box into boxes of side length $\frac{1}{10^2}$ .... $\rightsquigarrow$ $v_{i(3)}$

So we obtain a subsequence $j \mapsto v_{i(j)}$

Let $a =$ point of $\mathbb{R}^n$ s.t. the decimal digits of the k-th coordinate of a are equal to those of the successive boxes we chose.

Then the distance between a and $v_{i(j)}$ is less than or equal to $\sqrt{\left(\frac{1}{10^{j-1}}\right)^2 + \cdots + \left(\frac{1}{10^{j-1}}\right)^2}$

$$= \sqrt{n}\,\frac{1}{10^{j-1}}$$ goes to 0 as $j \to \infty$

$\square$

Def 1.6.5 (supremum): Give a function $f : C \longrightarrow \mathbb{R}$ (where $C \subset \mathbb{R}^n$ is a subset), the supremum of f is the least upper bound of the values of f, it is denoted $\sup_{v \in C} f(v)$.

In other words, if we write $M := \sup_{v \in C} f(v)$, then $\forall\ v \in C,\ f(v) \leq M$

and if a number N satisfies $f(v) \leq N\ \forall v \in C$, then $M \leq N$.

Definition 1.6.6 (maximum) The maximum value of $f : C \to \mathbb{R}$, is a number M, denoted $\max_{v \in C} f(v)$,

s.t. $M = \sup_{v \in C} f(v)$ and $\exists\ b \in C$ s.t.
$f(b) = M$.

<u>Def 1.6.7</u> (infimum)

<u>Def. 1.6.8</u> (minimum)

<u>e.g.</u>: the function $f(x) = -e^{x}$ has the supremum 0 on $\mathbb{R}$ but never reaches 0.

<u>Theorem 1.6.9</u> Let $C \subset \mathbb{R}^n$ be a compact set, let $f : C \longrightarrow \mathbb{R}$ be a continuous function.
Then, $\exists\ a \in C$ s.t. $f(a) \geq f(v)\ \forall v \in C$
and $\exists\ b \in C$ s.t. $f(b) \leq f(v)\ \forall\ v \in C$.

In other words, continuous functions on compact sets have a maximum and a minimum.

Proof: read in text. (please bring questions to class or office hours)

Theorem 1.6.11 If $C \subset \mathbb{R}^n$ is compact, then any function continuous on C is uniformly continuous.

Proof: uniform continuity says:

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall v, w \in C$

$$|v-w| < \delta \implies |f(v)-f(w)| < \varepsilon$$

If f is not uniformly continuous, then

$\exists \varepsilon > 0$ s.t. $\forall \delta > 0, \exists v, w$ s.t. $|v-w| < \delta$ and $|f(v)-f(w)| \geq \varepsilon$

In particular for $\delta = \frac{1}{m}$ $\quad m \in \mathbb{N}$

$\exists$ v_m, w_m s.t. $|v_m - w_m| < \frac{1}{m}$ and $|f(v_m) - f(w_m)| \geq \varepsilon$

$\exists$ subsequence $j \longmapsto v_{n(j)}$ which converges to a point $b \in C$.

Because $|v_n - w_n| < \frac{1}{n} \longrightarrow 0$, we also have

$$\lim_{j \to \infty} w_{n(j)} = b$$

f is continuous, so $\lim\limits_{j \to \infty} f(v_{n(j)}) = f(b)$

$$= \lim_{j \to \infty} f(w_{n(j)})$$

So $\lim\limits_{j \to \infty} \left(f(v_{n(j)}) - f(w_{n(j)}) \right) = 0$ contradiction! because

$|f(v_{n(j)}) - f(w_{n(j)})| \geq \varepsilon > 0$

$\square$

<u>Prop. 1.6.12</u>: If $g:(a,b)\longrightarrow \mathbb{R}$ is differentiable and $c\in(a,b)$ is a min or a max, then $g'(c)=0$.

<u>Proof</u>: We do it for a max:

$$g'(c)=\lim_{h\to 0}\frac{g(c+h)-g(c)}{h}$$

if $h>0$, then $\frac{g(c+h)-g(c)}{h}\leq 0$

if $h<0$, then $\frac{g(c+h)-g(c)}{h}\geq 0$

the uniqueness of ~~the~~ limit implies that ~~the~~ limit is 0.

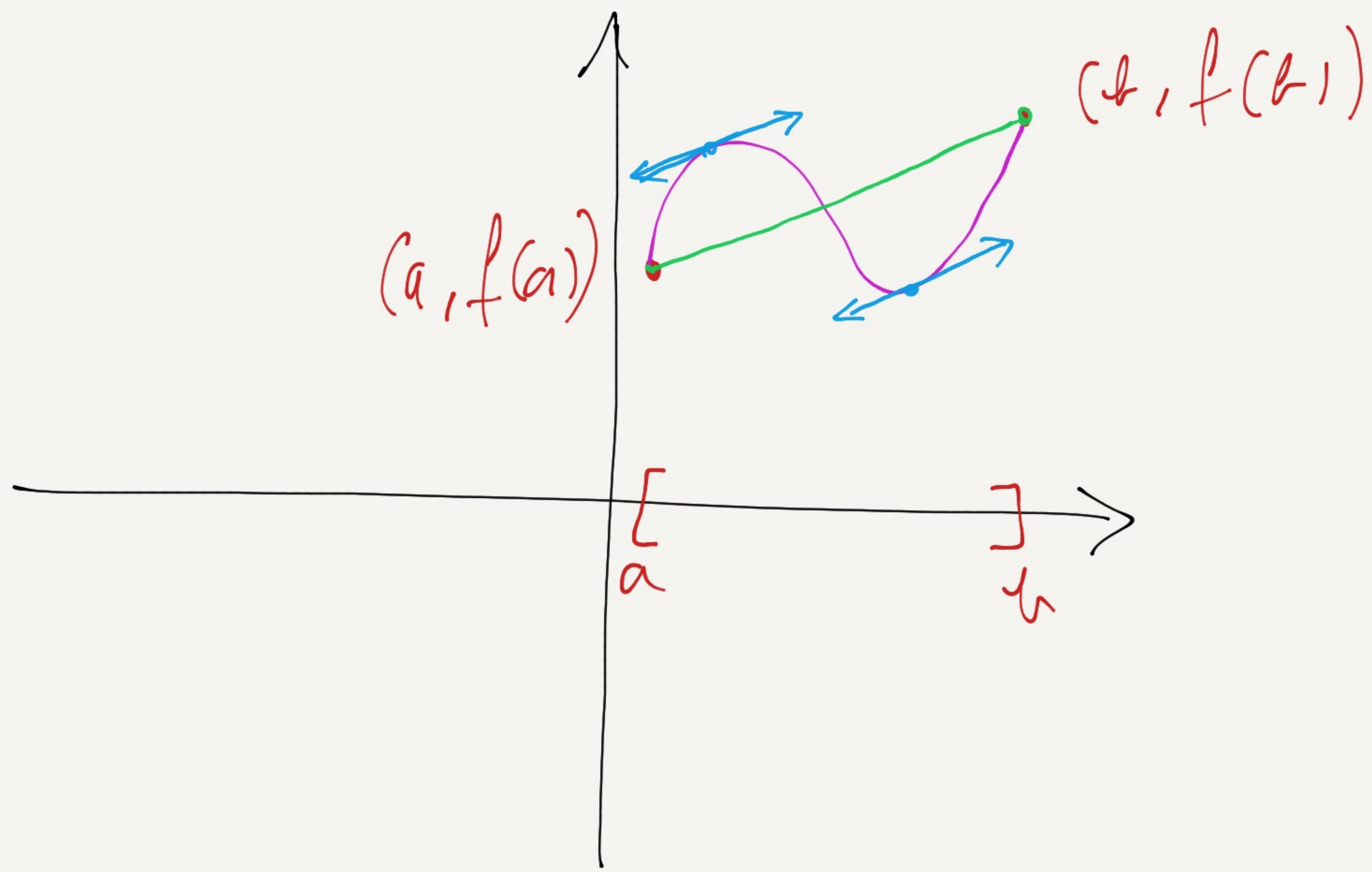
$\square$

Theorem 1.6.13 (mean value theorem)

If $f : [a,b] \to \mathbb{R}$ is continuous and differentiable on (a,b), then $\exists\, c \in (a,b)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

Proof:



We use a function which allows us to use the previous result. Put $m := \dfrac{f(b)-f(a)}{b-a}$

Then $$g(x) = f(x) - mx$$

In fact, if we put

$$h(x) = f(x) - f(a) - m(x-a)$$

then $h(a) = 0$ and $h(b) = 0$