Prop. 1.5.17 A set $C \subset \mathbb{R}^n$ is closed iff $\forall$ convergent sequence $i \mapsto v_i \in C$,

$$\lim_{i \to \infty} v_i \in C$$

Proof: read text and future homework.

Def 1.5.27 (continuous functions).

Let $X \subset \mathbb{R}^n$ be a subset of $\mathbb{R}^n$. Let $f: X \longrightarrow \mathbb{R}^m$ be a function.

f is continuous at $v_0 \in X$ if

$$\lim_{v \to v_0} f(v) = f(v_0).$$

equivalently, $\forall\ \varepsilon > 0,\ \exists\ \delta > 0$

s.t. $\forall\ v \in X,\ |v - v_0| < \delta \Rightarrow |f(v) - f(v_0)| < \varepsilon$

We say f is continuous on X if f is continuous at every point of X.

<u>Proposition:</u> <u>1.5.28</u>

f is continuous at $v_0 \in X$ iff

$\forall$ sequence $i \mapsto v_i \in X$ with limit v_0, $\left(= \lim_{i \to \infty} v_i\right)$

we have $\lim_{i \to \infty} f(v_i) = f(v_0)$

Proof: (1) Assume f is continuous at v_0.

Suppose given a sequence $i \longmapsto v_i \in X$ s.t. $\lim_{i\to\infty} v_i = v_0$

we need to prove $\lim_{i\to\infty} f(v_i) = f(v_0)$

Given $\varepsilon > 0$, we need to find $M \in \mathbb{N}$ s.t. for $i > M$, we have $|f(v_i) - f(v_0)| < \varepsilon$.

f <u>continuous</u>: $\exists\, \delta > 0$ s.t. $|v_i - v_0| < \delta$
$\Rightarrow |f(v_i) - f(v_0)| < \varepsilon$

$\lim_{i\to\infty} v_i = v_0$: $\exists\, M > 0$ s.t. $i > M$
$\Rightarrow |v_i - v_0| < \delta$.

(2) We prove that if f is not continuous at v_0, then $\exists$ sequence $i \mapsto v_i \in X$ with $\lim_{i \to 0} v_i = v_0$ s.t. $\lim_{i \to \infty} f(v_i)$ either does not exist or is not $f(v_0)$.

$\exists\, \varepsilon > 0$ s.t. $\forall\, \delta > 0$, $\exists\, v \in X$ with $|v - v_0| < \delta$ and $|f(v) - f(v_0)| \geq \varepsilon > 0$

For instance for $\delta = \frac{1}{i}$ $\quad i \in \mathbb{N}$

$\exists\, v_i \in X$ with $|v_i - v_0| < \frac{1}{i}$ and $|f(v_i) - f(v_0)| \geq \varepsilon > 0$

we have a sequence $i \mapsto v_i \in X$

We have $\lim_{i \to \infty} v_i = v_0$

and $\forall i \qquad |f(v_i) - f(v_0)| \geq \varepsilon > 0$

So $\lim_{i \to \infty} f(v_i)$ can't be $f(v_0)$.

$\square$

<u>Examples of continuous functions:</u>

polynomials $\mathbb{R}^n \longrightarrow \mathbb{R}$

e.g. $f: \mathbb{R}^3 \longrightarrow \mathbb{R} \qquad f(x, y, z) = x^2 y z^5 - 2y^2 + 3z$

rational functions where the denominator does not vanish: e.g. $g(x, y, z) = \dfrac{x^2 y - 4z + 1}{3yz + 5}$

Def. 1.5.32 (uniform continuity)

$X \subset \mathbb{R}^n$ subset, $f : X \longrightarrow \mathbb{R}^m$ a function.

We say f is uniformly continuous on X if

$\forall\ \varepsilon > 0,\ \exists\ \delta > 0$ s.t. $\forall\ v, w \in X$

$$|v - w| < \delta \Rightarrow |f(v) - f(w)| < \varepsilon$$

Example: Linear functions are uniformly continuous.

A linear function $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ can be given by a matrix A

$$\begin{array}{c} \mathbb{R}^n \\ \rotatebox{90}{$\in$} \\ v \\ \rotatebox{90}{$=$} \\ \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \end{array} \longmapsto Av$$

to show $v \mapsto Av$ is uniformly continuous,

note: $$Av - Aw = A(v-w)$$

$$|Av - Aw| = |A(v-w)| \leq |A| \cdot |v-w|$$

Prop 1.4.11, consequence of Schwarz's inequality Thm 1.4.5.

given $\varepsilon > 0$, if we take $\delta = \dfrac{\varepsilon}{|A|}$, then

we have $|v-w| < \delta \Rightarrow$

$$|Av - Aw| \leq |A| \cdot |v-w| < |A| \cdot \delta = \varepsilon$$

This works if $|A| \neq 0$. To avoid worrying about this, we can take $\delta = \dfrac{\varepsilon}{1+|A|}$.

Thm: $f: X \longrightarrow \mathbb{R}^m$, $X \subset \mathbb{R}^n$ subset.

f is continuous iff

$\forall$ open set $V \subset \mathbb{R}^m$, $f^{-1}(V)$ is open in X

iff $\forall$ closed set $C \subset \mathbb{R}^m$, $f^{-1}(C)$ is closed in X

Notation: If $Y \subset \mathbb{R}^m$ is a subset, then

$$f^{-1}(Y) \subset \mathbb{R}^n$$

and $f^{-1}(Y) := \{ v \in X \mid f(v) \in Y \}$

The theorem can be used to produce open sets and closed sets as inverse images:

<u>e.g.</u> $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$

$$(x,y) \longmapsto y - x^2$$

the parabola $y - x^2 = 0$ is $f^{-1}(\{0\})$

$\mathbb{R}^2 \supset \{(x,y) \mid y - x^2 = 0\}$ is closed because $\{0\}$ is closed in $\mathbb{R}$

$\mathbb{R}^2 \supset \{(x,y) \mid y > x^2\} = f^{-1}((0,\infty))$ is open because $(0,\infty) \subset \mathbb{R}$ is open

$\{(x,y) \mid y \geq x^2\} = f^{-1}([0,\infty))$ is closed.

Beginning of proof of theorem:

Suppose f is continuous and $V \subset \mathbb{R}^{box}$ is open.

Want to show $f^{-1}(V) \subset \mathbb{R}^{\infty}$ is open.

Let $v_0 \in f^{-1}(V)$ be any point. By definition

$$f(v_0) \in V$$

V is open so $\exists\ r > 0$ s.t. $B_r(f(v_0)) \subset V$

f continuous so, $\forall\ \varepsilon > 0$, $\exists\ \delta > 0$ s.t.

$$|v - v_0| < \delta \Rightarrow |f(v) - f(v_0)| < \varepsilon$$

Extra-credit homework: finish the proof of the theorem. $\square$