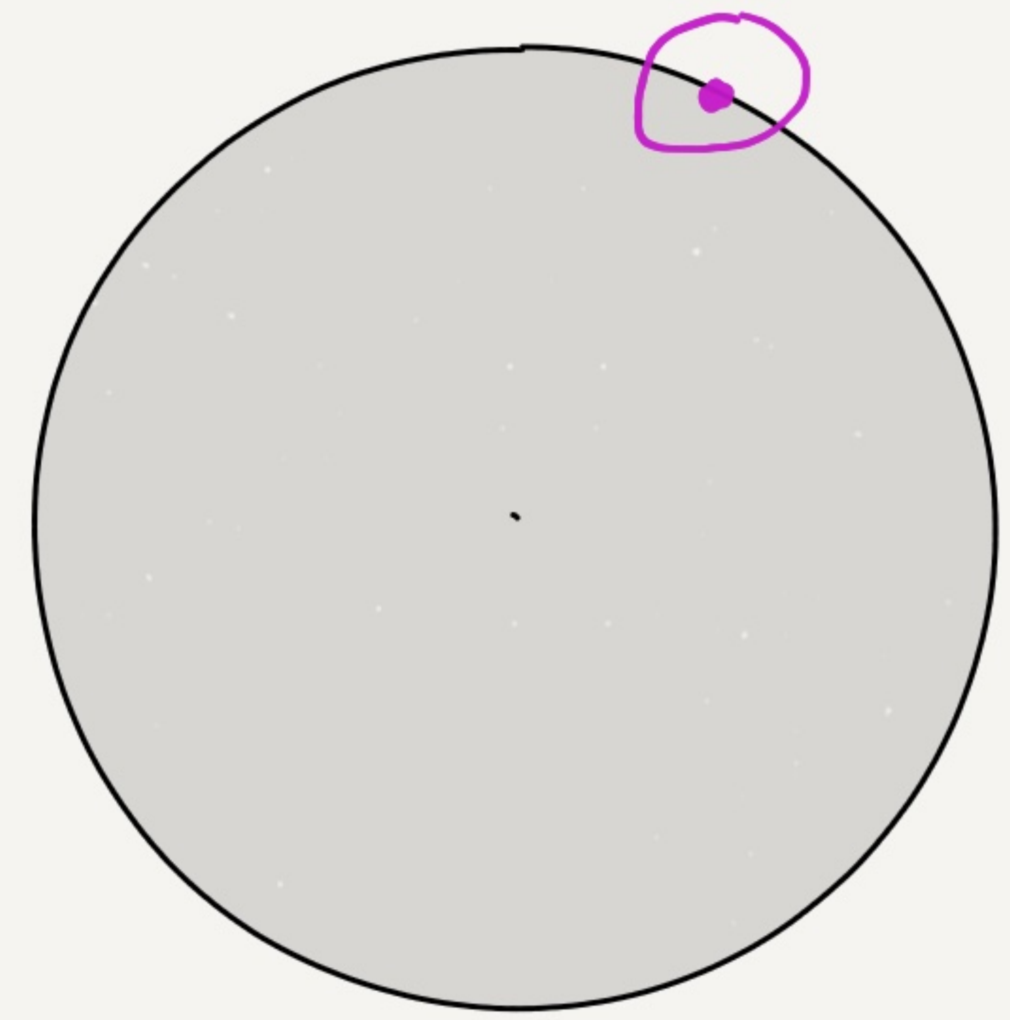
If A = open disc, then ∂A = circle



Def 1.5.10: The boundary of a subset $A \subset \mathbb{R}^n$ is $\partial A := \bar{A} \setminus \mathring{A}$

($\uparrow$ del)

This is equivalent to

$$\partial A := \{ v \in \mathbb{R}^n \mid \text{every neighborhood of } v \text{ intersects both } A \text{ and } \mathbb{R}^n \setminus A \}$$

we also have:

$$\partial A = \{ v \in \mathbb{R}^n \mid \forall\, r>0 \quad B_r(v) \cap A \neq \phi \text{ and } B_r(v) \cap (\mathbb{R}^n \setminus A) \neq \phi \}$$

(see homework for more details).

Convergence and limits:

Fix n. We can think of a sequence as a function

$$\mathbb{N} \longrightarrow \mathbb{R}^n \qquad \{v_0, v_1, v_2, \dots\}$$
$$i \longmapsto v_i$$

Def: 1.5.12: A sequence $i \mapsto v_i$ of elements of $\mathbb{R}^n$ converges to $v \in \mathbb{R}^n$ if

$\forall\, \varepsilon > 0$, $\exists\, M$, s.t. $\forall\, m > M$, $|v_m - v| < \varepsilon$.

Def. 1.5.20 (Limit of a function).

Let X be a subset of $\mathbb{R}^n$, v_0 a point of $\overline{X}$.

A function $f: X \longrightarrow \mathbb{R}^m$ (m and n are fixed) has limit a at v_0: $\lim\limits_{v \to v_0} f(v) = a$

if $\forall\ \varepsilon > 0,\ \exists\ \delta > 0$ s.t. $\forall\ v \in X$

$$|v - v_0| < \delta \implies |f(v) - a| < \varepsilon.$$

equivalently,

$\forall\ \varepsilon > 0,\ \exists\ \delta > 0$ s.t. for all $v \in B_\delta(v_0)$ we have $f(v) \in B_\varepsilon(a)$.

Proposition (1.5.15 and 1.5.21)

Limits are unique when they exist.

Proof: We will prove the case of functions, sequences are similar.

Suppose $X \subset \mathbb{R}^n$ is a subset, $v_0 \in \bar{X}$

$f : X \longrightarrow \mathbb{R}^m$ is a function.

Suppose $\lim_{v \to v_0} f(v) = a$ and $\lim_{v \to v_0} f(v) = b$

we will show $a = b$.

Consider $|a - b|$

$\forall \ \varepsilon > 0, \ \exists \ \delta_1$ s.t. $\forall \ v \in B_{\delta_1}(v_0)$

$$|f(v) - a| < \frac{\varepsilon}{2}$$

$\forall \ \varepsilon > 0, \ \exists \ \delta_2$ s.t. $\forall \ v \in B_{\delta_2}(v_0)$

$$|f(v) - b| < \frac{\varepsilon}{2}$$

So if we put $\delta := \text{Min}\{\delta_1, \delta_2\} > 0$

then $\forall \ v \in B_\delta(v_0)$

$$|f(v) - a| < \frac{\varepsilon}{2} \text{ and } |f(v) - b| < \frac{\varepsilon}{2}$$

$$\Rightarrow \ |a - b| = |a - f(v) + f(v) - b|$$

$$\leq |a - f(v)| + |f(v) - b| < \varepsilon$$

So, $\forall \varepsilon > 0 \quad |a-b| < \varepsilon.$

$$\Rightarrow |a-b| = 0$$

$\square$

or q.e.d.

We can take limits coordinately coordinate:

<u>Proposition 1.5.25</u> (the case of functions, sequences are similar)

Suppose $U \subset \mathbb{R}^n$ is a subset and $x_0 \in \overline{U}$

$f : U \longrightarrow \mathbb{R}^m$. We can write the coordinates

of f as:

$$f(v) = \begin{pmatrix} f_1(v) \\ \vdots \\ \vdots \\ f_m(v) \end{pmatrix}$$

Then if $a = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} \in \mathbb{R}^m$

$\lim\limits_{v \to v_0} f(v) = a$ iff (if and only if) $\forall\ i = 1, \dots, m,\ \lim\limits_{v \to v_0} f_i(v) = a_i$

<u>Proof:</u> Assume $\lim\limits_{v \to v_0} f(v) = a$, we want to prove $\lim\limits_{v \to v_0} f_i(v) = a_i \quad \forall i$

$\forall\ \varepsilon > 0,\ \exists\ \delta > 0$ s.t. $\forall\ v \in B_\delta(v_0)$

$|f_i(v) - a_i| < \varepsilon$?

we know $\exists\ \delta > 0$ s.t. $\forall\ v \in B_\delta(v_0)$

$|f(v) - a| < \varepsilon$

$= \sqrt{(f_1(v) - a_1)^2 + \cdots + (f_m(v) - a_m)^2}$

for $i = 1, \ldots, m$

so $|f_i(v) - a_i| = \sqrt{(f_i(v) - a_i)^2}$

$\leq \sqrt{(f_1(v) - a_1)^2 + \cdots + (f_m(v) - a_m)^2}$

$< \varepsilon$

Now assume $\forall i = 1, \dots, m$, $\lim\limits_{v \to v_0} f_i(v) = a_i$.

$\forall \varepsilon > 0 \quad \exists \ \delta_1$ s.t. for $v \in B_{\delta_1}(v_0)$

$$|f_1(v) - a_1| < \frac{\varepsilon}{\sqrt{m}}$$

$\vdots \qquad \vdots$

$\exists \ \delta_m$ s.t. for $v \in B_{\delta_m}(v_0)$

$$|f_m(v) - a_m| < \frac{\varepsilon}{\sqrt{m}}$$

Set $\delta := \mathrm{Min}\{\delta_1, \dots, \delta_m\}$, then

for $v \in B_\delta(v_0)$, $|f_i(v) - a_i| < \varepsilon \quad \forall i = 1, \dots, m$

$$|f(v)-a| = \sqrt{(f_1(v)-a_1)^2 + \cdots + (f_m(v)-a)^2}$$

$$< \sqrt{\varepsilon^2 + \varepsilon^2 + \cdots + \varepsilon^2} = \sqrt{m\,\varepsilon^2}$$

$$= \varepsilon \sqrt{m}$$

So we have

$\forall\ \varepsilon > 0,\ \exists\ \delta$ s.t. $\forall\ v \in B_\delta(v)$

$$|f(v) - a| < \varepsilon \cancel{\sqrt{m}} \qquad \square$$

Please read Prop. 1.5.14.