A little bit of topology:

Open sets

First: open balls: n is a positive integer

Def 1.5.1: For any $v \in \mathbb{R}^n$ and $r \in \mathbb{R}, r > 0$, the open ball of radius r and center v is the set

$$B_r(v) := \{ w \in \mathbb{R}^n \mid |v - w| < r \}.$$

---

recall: $|v - w|$ is the length of the vector $v - w$ in terms of coordinates: if $v = (x_1, \ldots, x_n)$, then $|v| = \sqrt{x_1^2 + \cdots + x_n^2}$.

e.g.: $n=1$: the intervall $]-1,1[\,:=(-1,1)$ is the open ball of radius 1, centered at 0.

$n=2$: the disc $\{(x,y) : x^2+y^2<1\}$ is the open ball of radius 1, centered at $(0,0)$

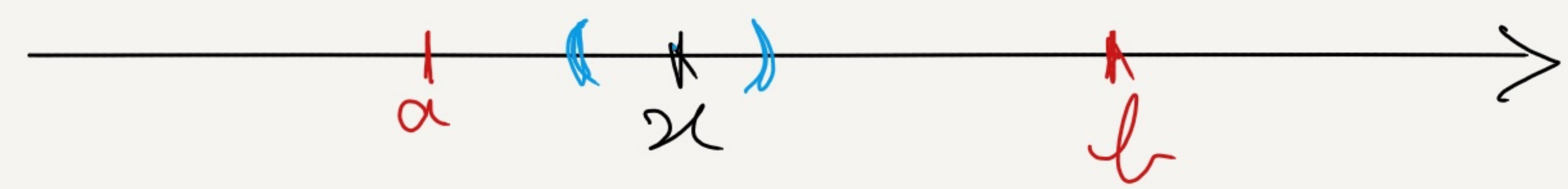
the disc $\{(x,y) \quad (x-2)^2+(y-3)^2<4\}$ is the open ball of radius 2, centered at $(2,3)$.

Def 1.5.2 : A subset $U\subset\mathbb{R}^n$ is open if $\forall\, v\in U$, $\exists\, r>0$ s.t. the open ball $B_r(v)\subset U$.

e.g.: $n=1$:

any open interval is open: $]a,b[= (a,b)$

$x \in (a,b)$

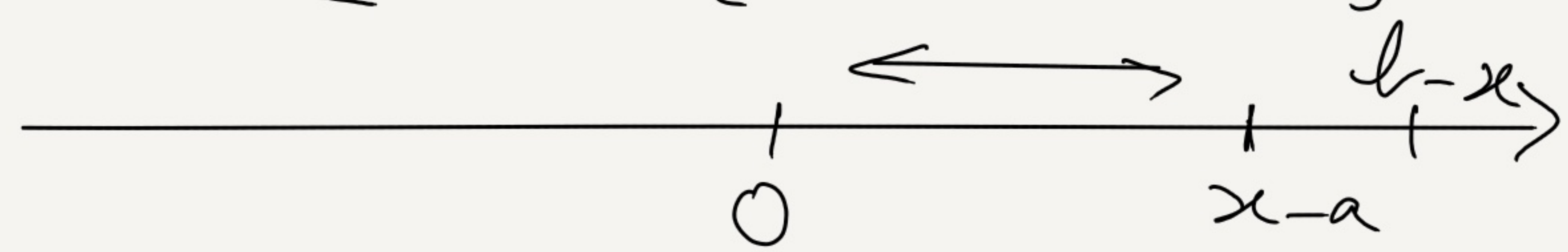


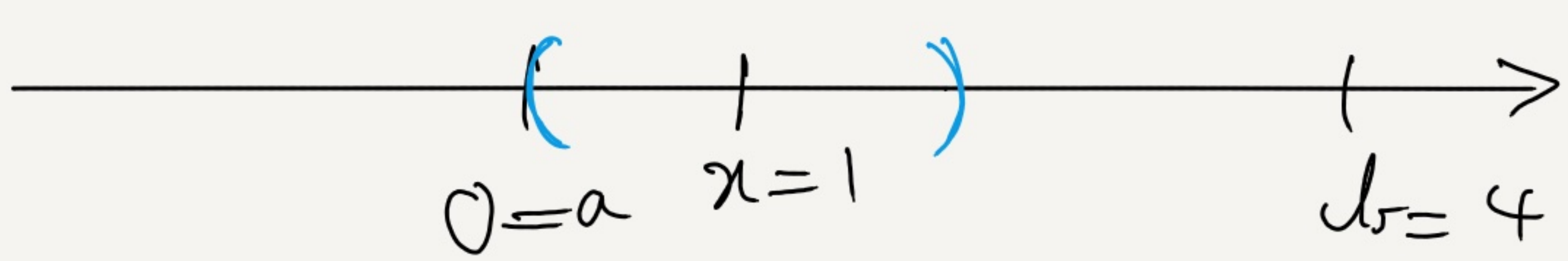
choose $r>0$ s.t. $r \le x-a$ and $r \le b-x$.

$x \in (a,b)$ so $x-a>0$ and $b-x>0$

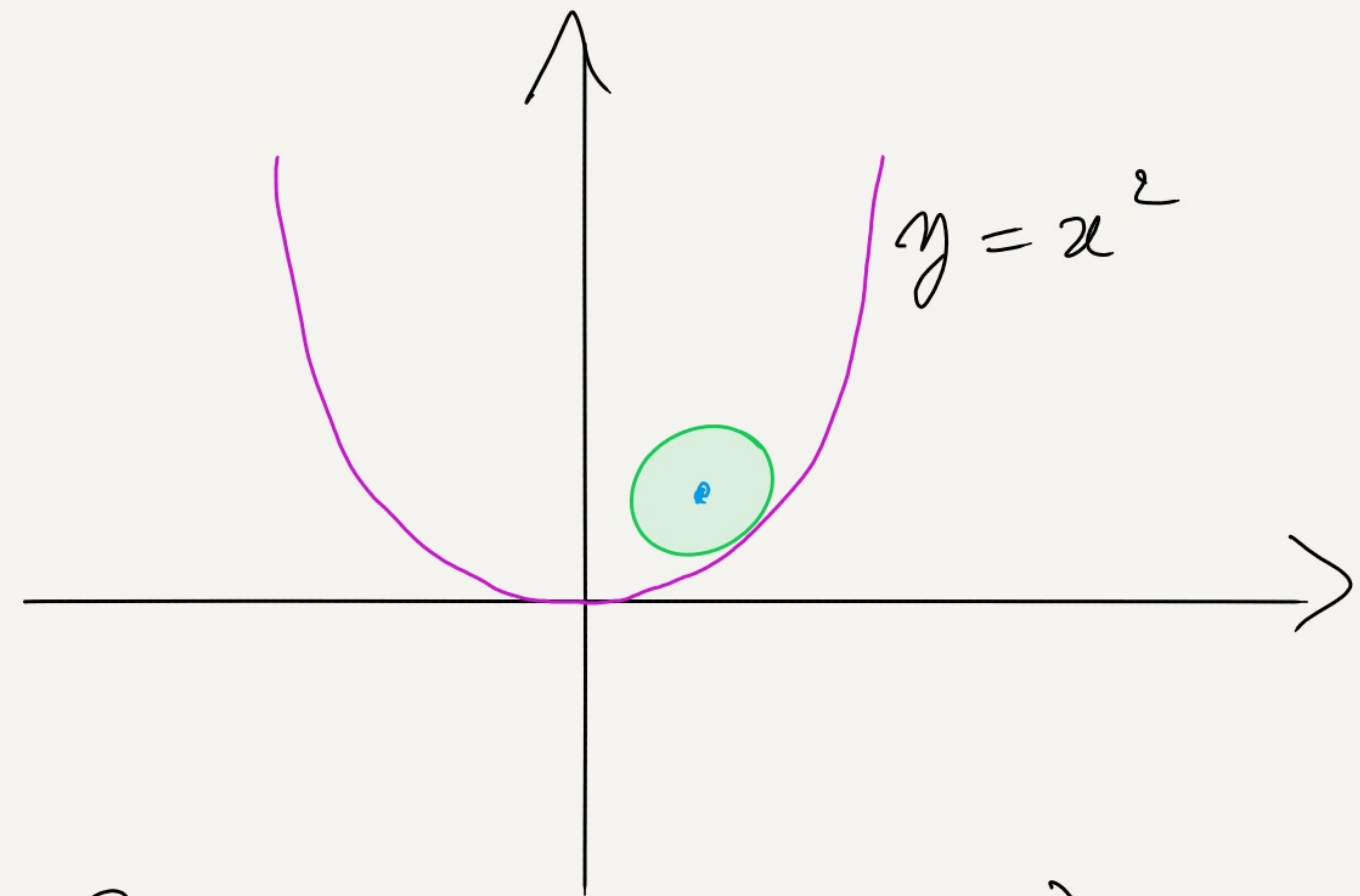
we can take $r = \min\{x-a, b-x\}$

or $r \le \min\{x-a, b-x\}$





$n = 2$, parabola: $y = x^2$



the set $\{(x,y) \mid y > x^2\}$ is open.

If (x,y) is above the parabola, we can take

$$r = \min\left\{\text{distance between } (x,y) \text{ and a point on the parabola}\right\}$$
$$= \min\left\{\sqrt{(x-a)^2 + (y-a^2)^2}\ ,\ a \in \mathbb{R}\right\}$$

Def 1.5.4: A subset $C \subset \mathbb{R}^n$ is closed if its complement $\mathbb{R}^n \setminus C$ is open.

e.g.: $U = \{ (x,y) \mid y > x^2 \}$.

$$C = \mathbb{R}^n \setminus U = \{ (x,y) \mid y \leq x^2 \}.$$

Def 1.5.7: Given a point $v \in \mathbb{R}^n$, a neighborhood V of v is a subset of $\mathbb{R}^n$ s.t. V contains an open ball of center v.

<u>Remark:</u> Given an open set U. For all $v \in U$, U is a neighborhood of v.

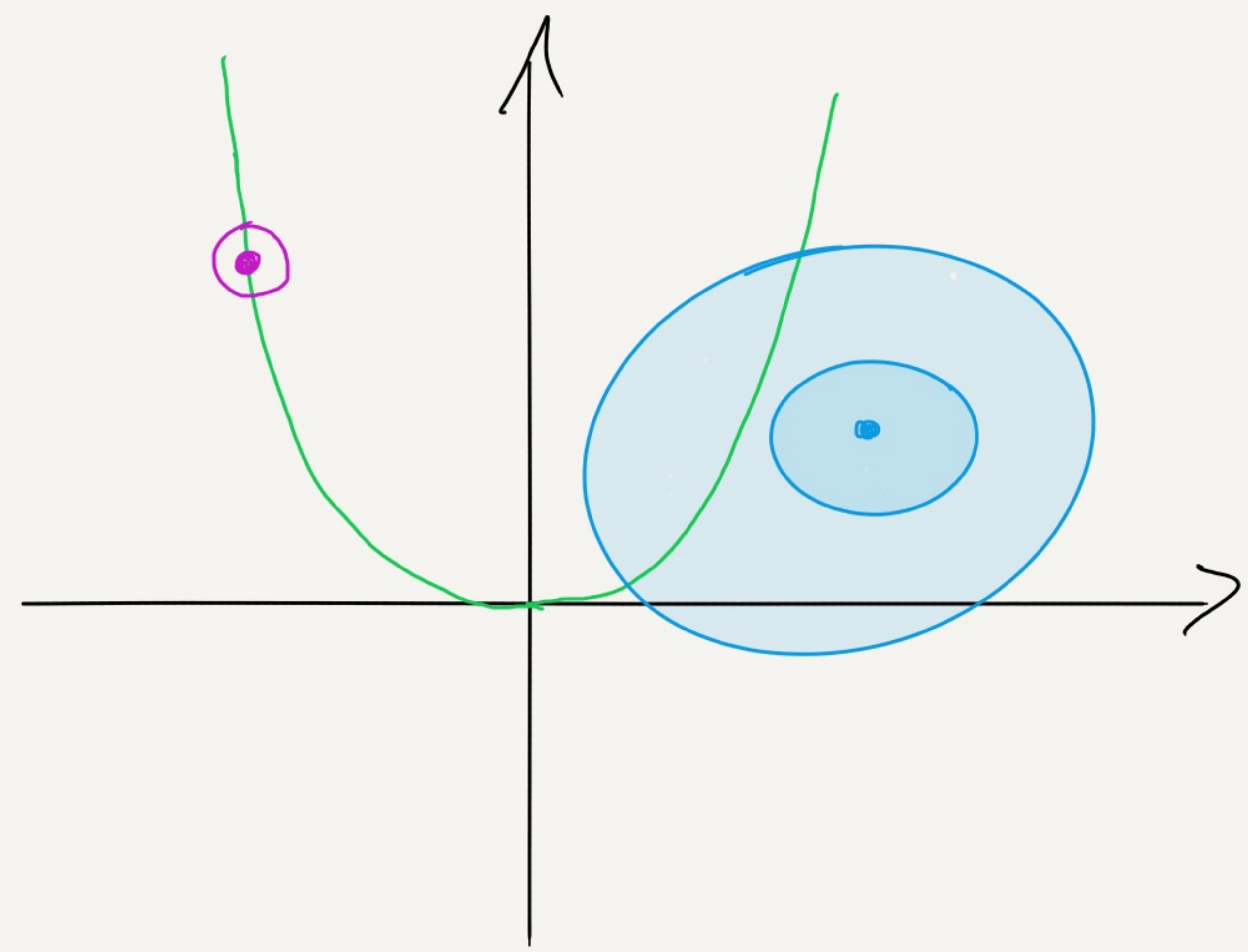
<u>Def: 1.5.8:</u> Given a subset $A \subset \mathbb{R}^n$, the closure $\bar{A}$ of A is the set of $v \in \mathbb{R}^n$ s.t. $\forall\ r > 0 \quad B_r(v) \cap A \neq \emptyset$

<u>Note:</u> If $v \in A$, then

$$v \in B_r(v) \cap A$$

hence $\quad B_r(v) \cap A \neq \emptyset$

e.g.: $A = \{(x,y) \mid y > x^2\}$



points on the parabola are in $\overline{A}$.

points strictly below the parabola are <u>not</u> in $\overline{A}$.

$$\overline{A} = \{(x,y) \mid y \geq x^2\}$$

the complement $\mathbb{R}^2 \setminus \overline{A} = \{(x,y) \mid y < x^2\}$ is open

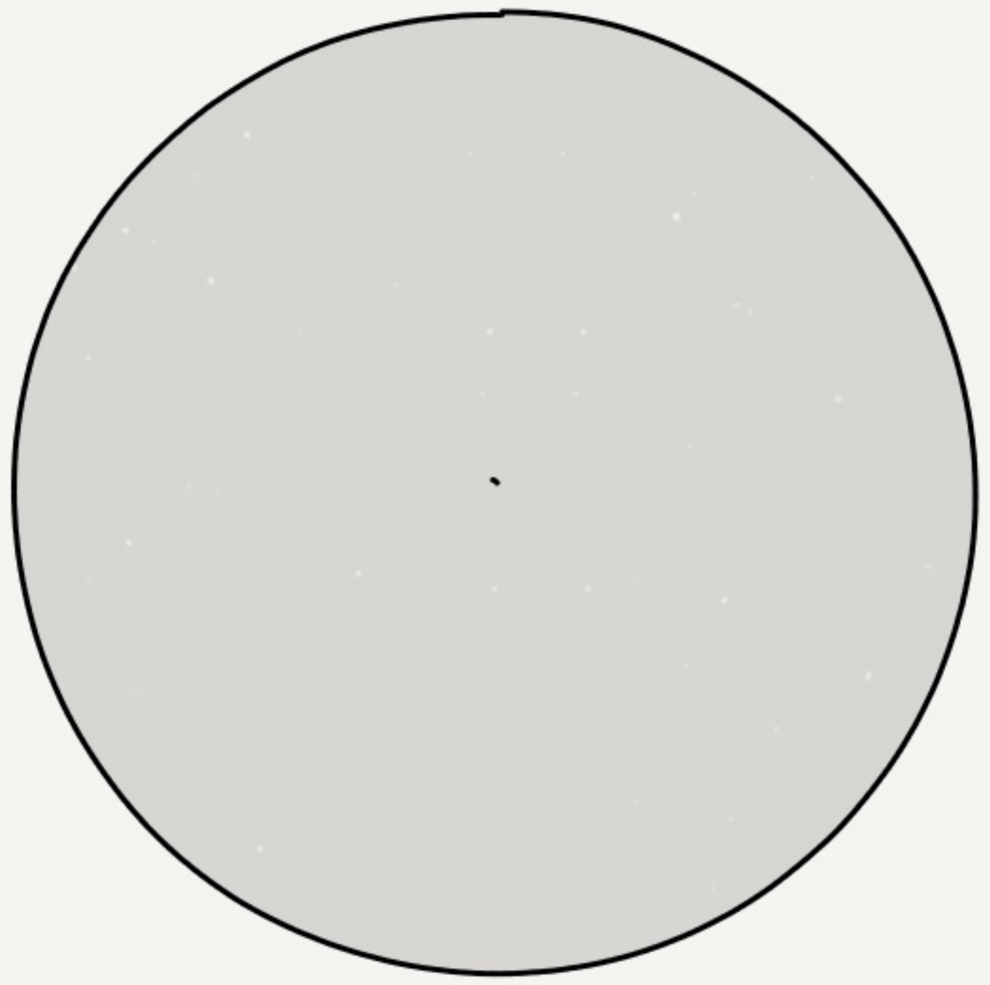
__Def 1.5.9__ (interior): If A is a subset of $\mathbb{R}^n$, the interior of A, denoted $\mathring{A}$ is the set of $v \in \mathbb{R}^n$ s.t. A is a neighborhood of v, i.e., $\exists\, r > 0$ s.t. $B_r(v) \subset A$.

__Note:__ Open balls are open.

So $\mathring{A}$ is open

If A is open, then $\mathring{A} = A$

If A is closed, then $\overline{A} = A$



Def 1.5.10: The boundary of a subset $A \subset \mathbb{R}^n$ is $\partial A := \bar{A} \setminus \mathring{A}$

(∂ del)