3 Immersions and Embeddings

A prototypical theme in geometry is the study of "spaces with structure", i.e. a set X equipped with some sort of additional geometric structure, such as a topology in the case of topological spaces, or in our case an atlas (to turn X into a topological manifold) along with differentiable transition functions (making X a differentiable manifold). From this perspective, we should investigate how functions between our spaces with structure interact with the structure on those spaces, and concern ourselves only with those functions which "preserve" the structure (e.g. in this class we are concerned with differentiable maps between differentiable manifolds, which are maps respecting the differentiable structure).

With this theme in mind, though set-theoretically injective maps give some notion of embedding sets into other sets, set-theoretic injectivity alone (even of differentiable maps) should be inadequate to describe the vague notion of "embedding" a differentiable manifold into another, since something should additionally be prescribed for how the map acts on the differentiable structure.

Definition (*Immersions*) A differentiable mapping

$$\varphi : M^m \to N^n$$

of differentiable manifolds is said to be an **immersion** if $d\varphi_p : T_pM \to T_{\varphi(p)}N$ is injective for all points $p \in M$.

This is at least a weak version of what one should expect to happen if one has an embedding (whatever that means) of a differentiable manifold. Indeed, since T_pM parametrizes the different directions in which one can move at a point $p \in M$, moving in distinct directions about p should then map to moving in distinct directions about $\varphi(p)$ in N if φ is an embedding at least in the vicinity of p. This is simply then the statement that $d\varphi_p : T_pM \to T_{\varphi(p)}N$ is injective.

Note: Since $d\varphi_p : T_pM \to T_{\varphi(p)}N$ is a linear map between vector spaces and $\dim T_pM = \dim M$ for all points p of a differentiable manifold M, it follows by linear algebra that if φ is an immersion then $\dim M \leq \dim N$. We call $\dim N - \dim M$ the *codimension* of φ.

Examples

(i) The curve $\alpha : \mathbb{R} \to \mathbb{R}^2$ given by

$$\alpha(t) = (t, |t|^2)$$

is not differentiable at $t = 0$, hence it is not a differentiable map since, much less an immersion, since $|t|$ is not differentiable at $t = 0$. Nevertheless, α is still injective.

(ii) The curve $\alpha : \mathbb{R} \to \mathbb{R}^2$ given by

$$\alpha(t) = (t^3, t^2)$$

is not an immersion, since $d_t\alpha$ is the zero map for $t = 0$.

(iii) The curve $\alpha : \mathbb{R} \to \mathbb{R}^2$ given by

$$\alpha(t) = (t^3 - 4t, t^2 - 4)$$

is an immersion, since $\alpha'(t)$ is never zero (as $3t^2 - 4 = 2t = 0$ has no solution in t). However, it is not an injective map, as $\alpha(2) = \alpha(-2)$, so this is a curve with self-intersection at $\alpha(2) = (0, 0)$.

As seen in the last example, immersions aren't necessarily injective on points, so they don't fully capture the notion of injectively "embedding" a space into another (though as alluded to by our discussion of immersions, we will see how immersions can be seen as yielding local embeddings). To properly capture this notion in the setting of differentiable manifolds, we have the following definition.

Definition: (*Embeddings*) An immersion $\varphi : M \to N$ of differentiable manifolds is an **embedding** if φ is a homeomorphism of M onto its image $\varphi(M) \subset N$, where $\varphi(M)$ inherits the subspace topology from N. If $M \subset N$ and the inclusion map $M \hookrightarrow N$ is an embedding, then we say that M is a **submanifold** of N.

Thus, an embedding is a map which a homeomorphism onto its image (which is what one should expect at least as topological spaces), along with the extra condition of being injective on tangent spaces.

Examples

(i) Define the curve $\alpha(t)$ as

$$\alpha(t) = (0, -(t + 2))$$

for $t \in (-3, -1)$, as

$$\alpha(t) = (-t, -\sin(\frac{1}{t}))$$

for $t \in (-\frac{1}{\pi}, 0)$, and then for $t \in [-1, -\frac{1}{\pi}]$ as any regular curve connecting the endpoint $(0, -1)$ of the first segment to the startpoint $(-\frac{1}{\pi}, 0)$ of the third segment which doesn't intersect the other two segments (see page 13 of do Carmo for a picture). This is an injective immersion from $(-3, 0)$ to $\mathbb{R}^2$ without self-intersection. However, it is not an embedding. To see this, let p be a point along the interior of the first segment (e.g. $p = (0, 0)$). Any open neighborhood of p in the subspace topology in $\mathbb{R}^2$ will consist of infinitely many disconnected open intervals (coming from the oscillating sine wave near $x = 0$). However, any open neighborhood of the corresponding point in the source manifold $(-3, 0)$ will simply be an open interval. Thus, α is not a homeomorphism onto its image and hence not an embedding.

(ii) As mentioned previously, the map $\alpha(t) = (t^3, t^2)$ from $\mathbb{R}$ to $\mathbb{R}^2$ is not an immersion for $t = 0$. However, it is both a differentiable map and a topological embedding (homeomorphism onto its image). This example shows the importance of the immersion condition as part of the definition of a smooth embedding - the image of α in $\mathbb{R}^2$ is a curve with a cusp at the origin, which does not give a differentiable manifold.

(iii) It follows from the definition of a regular surface S that the coordinate chart mappings $U_\alpha \to S \subset \mathbb{R}^3$ are smooth embeddings (they are differentiable homeomorphisms onto their images by definition of S and are immersions since we required the differential to be injective at each point).

In fact, we can prove the stronger condition that $S \subset \mathbb{R}^3$ is actually a submanifold. To show that the inclusion map i is a smooth embedding, let $p \in S$ be arbitrary. Then there exists a parametrization $\varphi : U \subset \mathbb{R}^2 \to S$ at p and another parametrization $id : V \subset \mathbb{R}^3 \to \mathbb{R}^3$ at $i(p)$ of the smooth manifold $\mathbb{R}^3$, where id is just the identity map. As $id^{-1} \circ i \circ \varphi = \varphi$ is a differentiable map by definition, we have that i is a differentiable map from the abstract regular surface S to its image in $\mathbb{R}^3$. It is immediate that i is an immersion as its differential is the identity, and the topology on S is already defined to agree with that of its image in $\mathbb{R}^3$ in the subspace topology, so that S is indeed a submanifold of $\mathbb{R}^3$.

Having seen several examples and non-examples of immersions and embeddings, we now prove a theorem showing that immersions are in a sense "the next best thing" to embeddings, in that they are locally embeddings.

Theorem: Let $\varphi : M_1^n \to M_2^m, n \leq m$ be an immersion of differentiable manifolds. Then for every point $p \in M_1$ there exists an open neighborhood $V \subset M_1$ of p such that the restriction $\varphi|_V : V \to \varphi(V)$ is an embedding.

Proof: Let $\phi_1 : U_1 \subset \mathbb{R}^n \to M_1$ and $\phi_2 : U_2 \subset \mathbb{R}^m \to M_2$ respectively be local charts about $p, \varphi(p)$, and let $(x_1, ..., x_n), (y_1, ..., y_m)$ respectively denote the coordinates in $\mathbb{R}^n$ and $\mathbb{R}^m$. In coordinates, we may write

$$\tilde{\varphi} := \phi_2^{-1} \circ \varphi \circ \phi_1 : \mathbb{R}^n \to \mathbb{R}^m = (y_1(x_1, ..., x_n), ..., y_m(x_1, ..., x_n)).$$

Set $q = \phi_1^{-1}(p)$. Since φ is an immersion, the Jacobian matrix of $\tilde{\varphi}$ at q has maximal rank n, so that after relabeling coordinates if necessary we have that the Jacobian determinant of the first n by n block

$$|\frac{\partial(y_1, ..., y_n)}{\partial(x_1, ..., x_n)}(q)| \neq 0.$$

In order to apply the inverse function theorem we need the dimensions of the source and target space to match up, so define

$$\phi : U_1 \times \mathbb{R}^{m-n=k} \to \mathbb{R}^m$$

via

$$\phi(x_1, ..., x_n, t_1, ..., t_k) = (y_1(x_1, ..., x_n), ..., y_n(x_1, ..., x_n), \quad (1)$$
$$y_{n+1}(x_1, ..., x_n) + t_1, ..., y_{n+k=m}(x_1, ..., x_n) + t_k), \quad (2)$$

where $t_1, ..., t_k$ are the coordinates on $\mathbb{R}^k$. By definition we have that the restriction of ϕ to $U_1 \times \{0\}$ is just $\tilde{\varphi}$. Moreover, if $(q, 0)$ denotes the point in $U_1 \times \mathbb{R}^k$ with q in the first factor and the zero vector in the second, then we can easily compute the Jacobian matrix

$$d\phi_{(q,0)} = \left[\begin{array}{c|c} \frac{\partial(y_1, ..., y_n)}{\partial(x_1, ..., x_n)} & 0 \\ \hline M & I_k \end{array} \right]$$

where M is some k by n matrix of partials, I_k is the k by k identity matrix, and 0 denotes the n by k zero matrix. This is a block lower triangular matrix and hence has determinant

$$\det(d\phi_{(q,0)}) = |\frac{\partial(y_1, ..., y_n)}{\partial(x_1, ..., x_n)}| \cdot \det(I_k) \quad (3)$$
$$= |\frac{\partial(y_1, ..., y_n)}{\partial(x_1, ..., x_n)}| \quad (4)$$
$$\neq 0. \quad (5)$$

The inverse function theorem now applies to ϕ, so that there exist open neighborhoods $W_1 \subset U_1 \times \mathbb{R}^k$ of $(q, 0)$ and $W_2 \subset \mathbb{R}^m$ of $\phi(q, 0)$ such that the restriction $\phi|_{W_1}$ is a diffeomorphism onto W_2. Set $\tilde{V} = W_1 \cap U_1$, so that $\phi|_{\tilde{V}} = \tilde{\varphi}|_{\tilde{V}}$. Note that ϕ_1, ϕ_2 are diffeomorphisms and that the composition and inverses of diffeomorphisms are diffeomorphisms. Moreover, we have that $\varphi = \phi_2 \circ \tilde{\varphi} \circ \phi_1^{-1}$ by definition of $\tilde{\varphi}$, so that the restriction of φ to $\phi_1(\tilde{V}) = V$ is a diffeomorphism onto $\varphi(V) \subset M_2$, and hence an embedding.