Topics in Classical Algebraic Geometry

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September 27, 2010

Preface

The main purpose of the present treatise is to give an account of some of the topics in algebraic geometry which while having occupied the minds of many mathematicians in previous generations have fallen out of fashion in modern times. Often in the history of mathematics new ideas and techniques make the work of previous generations of researchers obsolete, especially this applies to the foundations of the subject and the fundamental general theoretical facts used heavily in research. Even the greatest achievements of the past generations which can be found for example in the work of F. Severi on algebraic cycles or in the work of O. Zariski's in the theory of algebraic surfaces have been greatly generalized and clarified so that they now remain only of historical interest. In contrast, the fact that a nonsingular cubic surface has 27 lines or that a plane quartic has 28 bitangents is something that cannot be improved upon and continues to fascinate modern geometers. One of the goals of this present work is then to save from oblivion the work of many mathematicians who discovered these classic tenets and many so many beautiful results.

In writing this book the greatest challenge the author has faced was distilling the material down to what should be covered. The number of concrete facts, examples of special varieties and beautiful geometric constructions that have accumulated during the classical period of development of algebraic geometry is enormous and what the reader is going to find in the book is really only a tip of the iceberg; a work that is sort of a taste sampler of classical algebraic geometry. It avoids most of the material found in other modern books on the subject, such as, for example, [9] where one can find many of the classical results on algebraic curves. Instead, it tries to assemble or, in other words, to create a compendium of material that either cannot be found, is too dispersed to be found easily, or is simply not treated adequately by contemporary research papers. On the other hand, while most of the material treated in the book exists in classical treatises in algebraic geometry, their somewhat archaic terminology and what is by now completely forgotten background knowledge makes these books useful to but a handful of experts in the classical literature. Lastly, one must admit that the personal taste of the author also has much sway in the choice of material.

The reader should be warned that the book is by no means an introduction to algebraic geometry. Although some of the exposition can be followed with only a minimum background in algebraic geometry, for example, based on Shafarevich's book [386], it often relies on current cohomological techniques, such as those found in Hartshorne's book [206]. The idea was to reconstruct a result by using modern techniques but not necessarily its original proof. For one, the ingenious geometric constructions in those proofs were often beyond the authors abilities to follow them completely. Understandably, the price of this was often to replace a beautiful geometric argument with a dull cohomological one. For those looking for a less demanding sample of some of the topics covered in the book the recent beautiful book [24] maybe of great use.

No attempt has been made to give a complete bibliography. To give an idea of such an enormous task one could mention that the report on the status of topics in algebraic geometry submitted to the National Research Council in Washington in 1928 [389] contains more than 500 items of bibliography by 130 different authors only in the subject of planar Cremona transformations (covered in one of the chapters of the present book.) Another example is the bibliography on cubic surfaces compiled by J. E. Hill [215] in 1896 which alone contains 205 titles. Meyer's article [280] cites around 130 papers published 1896-1928. The title search in MathSciNet reveals more than 200 papers refereed since 1940, many of them published only in the last twenty years. How sad it is when one considers the impossibility of saving from oblivion so many names of researchers of the past years who have contributed so much to our subject.

A word about exercises: some of them are easy and follow from the definitions, some of them are hard and are meant to provide additional facts not covered in the main text. In this case we indicate the sources for the statements and solutions.

It is impossible to list all of my colleagues who helped me to improve the exposition by contributing their comments and corrections. For all the errors still found in the book the author bears sole responsibility.

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Chapter 1

Polarity

1.1 Polar hypersurfaces

1.1.1 The polar pairing

We will take $\mathbb{C}$ as the base field although many constructions in this book work over an arbitrary algebraically closed field. Let E be a finite-dimensional vector space. We denote by S^kE its symmetric k-th power and let $E^\vee$ denote its dual space of linear functions. We have a canonical bilinear pairing

$$\langle\ ,\ \rangle : E \otimes E^\vee \to \mathbb{C} \tag{1.1}$$

that can be extended, using the universal properties of symmetric products, to a bilinear pairing

$$S^kE \otimes S^dE^\vee \to S^{d-k}E^\vee, \quad d \geq k. \tag{1.2}$$

In coordinates, it can be described as follows. Pick up a basis $(\xi_0, \ldots, \xi_n)$ of E and let $(t_0, \ldots, t_n)$ be the dual basis in $E^\vee$. We can identify an element of $S^dE^\vee$ with a homogeneous polynomial f of degree d in the variables $t_0, \ldots, t_n$ and an element of S^kE with a homogeneous polynomial ψ of degree k in variables ξ_i. Since $\langle \xi_i, t_j \rangle = \delta_{ij}$, we view each ξ_i as the partial derivative operator $\partial_i = \frac{\partial}{\partial t_i}$. Hence any element $\psi \in S^kE$ can be viewed as a differential operator

$$D_\psi = \psi(\partial_0, \ldots, \partial_n).$$

The pairing (1.2) becomes

$$\langle \psi(\xi_0, \ldots, \xi_n), f(t_0, \ldots, t_n) \rangle = D_\psi(f). \tag{1.3}$$

For any monomial $\boldsymbol{\partial}^{\mathbf{i}} = \partial_0^{i_0} \cdots \partial_n^{i_n}$ and any monomial $\mathbf{t}^{\mathbf{j}} = t_0^{j_0} \cdots t_n^{j_n}$, we have

$$\boldsymbol{\partial}^{\mathbf{i}}(\mathbf{t}^{\mathbf{j}}) = \begin{cases} \frac{\mathbf{j}!}{(\mathbf{j}-\mathbf{i})!}\mathbf{t}^{\mathbf{j}-\mathbf{i}} & \text{if } \mathbf{j}-\mathbf{i} \geq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{1.4}$$

Here and later we use the vector notation:

$$\mathbf{i}! = i_0!\cdots i_n!, \quad \mathbf{i} = (i_0,\ldots,i_n) \geq 0 \Leftrightarrow i_0,\ldots,i_n \geq 0, \quad , |\mathbf{i}| = i_0+\cdots+i_n.$$

This gives an explicit expression for the pairing (1.2). Consider a special case when

$$\psi = (a_0\partial_0+\cdots+a_n\partial_n)^k = k!\sum_{|\mathbf{i}|=k}(\mathbf{i}!)^{-1}\mathbf{a}^{\mathbf{i}}\boldsymbol{\partial}^{\mathbf{i}}.$$

Then

$$D_\psi(f) = k!\sum_{|\mathbf{i}|=k}(\mathbf{i}!)^{-1}\mathbf{a}^{\mathbf{i}}\boldsymbol{\partial}^{\mathbf{i}}(f). \tag{1.5}$$

It follows from (1.4) that the pairing (1.2) is a perfect pairing, in particular there is a canonical isomorphisms of linear spaces

$$S^kE^\vee \cong (S^kE)^\vee, \quad S^kE \cong (S^kE^\vee)^\vee. \tag{1.6}$$

Let $|E|$ (or $\mathbb{P}_{\mathrm{sub}}(E)$) denote the projective space of one-dimensional subspaces of E. A basis $\xi_0,\ldots,\xi_n$ in E defines an isomorphism $E\cong\mathbb{C}^{n+1}$ and identifies $|E|$ with the projective space $\mathbb{P}^n = \mathbb{C}^{n+1}\setminus\{0\}/\mathbb{C}^*$. For any non-zero vector $v\in E$ we denote by $[v]$ the corresponding point in $|E|$. If $E=\mathbb{C}^{n+1}$ and $v=(a_0,\ldots,a_n)\in\mathbb{C}^{n+1}$ we set $[v]=[a_0,\ldots,a_n]$. We call $[a_0,\ldots,a_n]$ the *projective coordinates* of a point $[a]\in\mathbb{P}^n$. Other common notations are $(a_0:a_1:\ldots:a_n)$ or simply $(a_0,\ldots,a_n)$ if no confusion arises.

The projective space comes with the tautological invertible sheaf $\mathcal{O}_{|E|}(1)$ whose space of global sections is identified with the dual space $E^\vee$. Its d-th tensor power is denoted by $\mathcal{O}_{|E|}(d)$ and its sections are identified with the symmetric d-th power $S^dE^\vee$. For any $f\in S^dE^\vee$ we denote by $V(f)$ the corresponding closed subscheme of zeros of f, we call it a *hypersurface* of degree d in $|E|$ defined by equation $f=0$. A hypersurface of degree 1 is a *hyperplane*. A hypersurface could be also considered as an effective divisor in $\mathbb{P}^n$, not necessary reduced. By definition, $V(0)=\mathbb{P}^n$ (the zero divisor). Clearly, the set of hypersurfaces can be identified with the projective space $|S^dE^\vee|\cong\mathbb{P}^{N(d,n)}$, where $N(d,n)=\binom{n+d}{d}-1$.

The projective space $|E^\vee|$ is called the *dual projective space*. We will often denote it by $|E|^\vee$. Its points are hyperplanes in $|E|$. Using the isomorphisms (1.6), we can also view $|E^\vee|$ as the projective space $\mathbb{P}(E)$ of one-dimensional quotients of E. Also we may identify $|S^kE|$ with the projective space of hypersurfaces of degree k in the dual projective space. They are classically known as *envelopes* of *class* k.

We view $a_0\partial_0+\cdots+a_n\partial_n\neq 0$ as a point $a\in|E|$ with projective coordinates $[a_0,\ldots,a_n]$.

Definition 1.1. *Let $X=V(f)$ be a hypersurface of degree d in $|E|$. Let $a=[v]\in|E|$ for some $v\in E$. The hypersurface*

$$P_{a^k}(X) := V(D_{v^k}(f))$$

of degree $d-k$ is called the k-th polar hypersurface *of the point x with respect to the hypersurface $V(f)$ (or of the hypersurface with respect to the point).*

Example 1.1.1. Let $d = 2$, i.e.

$$f(t_0, \ldots, t_n) = \sum_{i=0}^{n} \alpha_{ii} t_i^2 + 2 \sum_{0 \le i < j \le n} \alpha_{ij} t_i t_j$$

is a quadratic form. Then $P_a(V(f)) = V(g)$, where

$$D_a(f) = \sum_{i=0}^{n} a_i \frac{\partial f}{\partial t_i} = 2 \sum_{0 \le i,j \le n} a_i \alpha_{ij} t_j, \quad \alpha_{ji} = \alpha_{ij}.$$

The linear map $a \mapsto \frac{1}{2} D_a(f)$ is a map from E to $E^\vee$ which can be identified with an element of $E^\vee \otimes E^\vee = (E \otimes E)^\vee$ which is the polar bilinear form associated to f with matrix (α_{ij}).

Example 1.1.2. Let $M_n(K)$ be the vector space of square matrices of size n with coordinates t_{ij}. We view the determinant function $\Delta : M_n(K) \to K$ as an element of $S^n(M_n(K)^\vee)$, i.e. a polynomial of degree n in the variables t_{ij}. Let $C_{ij} = \frac{\partial \Delta}{\partial t_{ij}}$. For any point $A = (a_{ij})$ in $M_n(K)$ the value of C_{ij} at A is equal to the ij-th cofactor of A. Then

$$D_{A^{n-1}}(\Delta) = (n-1)! \sum_{i,j=1}^{n} C_{ij}(a) t_{ij}$$

is a linear function on M_n identified with the cofactor matrix adj(A) of A (called in the classical literature the *adjugate matrix*, not the adjoint matrix as is customary to call it now).

Let us give another definition of the polar hypersurfaces $P_{a^k}(X)$. Choose two different points $a = [a_0, \ldots, a_n]$ and $b = [b_0, \ldots, b_n]$ in $\mathbb{P}^n$ and consider the line $\ell = \overline{a, b}$ spanned by the two points as the image of the map

$$\varphi : \mathbb{P}^1 \to \mathbb{P}^n, \quad [t_0, t_1] \mapsto t_0 a + t_1 b := [a_0 t_0 + b_0 t_1, \ldots, a_n t_0 + b_n t_1]$$

(a parametric equation of ℓ). The intersection $\ell \cap X$ is isomorphic to the positive divisor on $\mathbb{P}^1$ defined by the degree d homogeneous form

$$\varphi^*(f) = f(t_0 a + t_1 b) = f(a_0 t_0 + b_0 t_1, \ldots, a_n t_0 + b_n t_1).$$

Using the Taylor formula at $(0, 0)$, we can write

$$\varphi^*(f) = \sum_{k+m=d} \frac{d!}{k! m!} t_0^k t_1^m A_{km}(a, b), \tag{1.7}$$

where

$$A_{km}(p, q) = \frac{\partial^d \varphi^*(f)}{\partial t_0^k \partial t_1^m}(0, 0).$$

Using the Chain Rule, we get

$$\begin{aligned} A_{km}(a,b) &= k!m!\sum_{|\mathbf{i}|=k}(\mathbf{i}!)^{-1}\mathbf{a}^{\mathbf{i}}\boldsymbol{\partial}^{\mathbf{i}}(f)(b) = D_{a^k}(f)(b) = D_{b^m a^k}(f) \quad (1.8)\\ &= m!\sum_{|\mathbf{j}|=m}(\mathbf{j}!)^{-1}\mathbf{b}^{\mathbf{j}}\boldsymbol{\partial}^{\mathbf{j}}(f)(a) = m!D_{b^m}(f)(a) = D_{a^k b^m}(f). \end{aligned}$$

Observe the symmetry

$$A_{km}(a,b) = A_{mk}(b,a). \tag{1.9}$$

When we fix a and let b vary in $\mathbb{P}^n$ we obtain a hypersurface $V(A(a,x))$ of degree $d-k$ which is the k-th polar hypersurface of $X = V(f)$ with respect to the point a. When we fix b and vary a, we obtain the m-th polar hypersurface $V(A(x,b))$ of X with respect to the point b.

Since we are in characteristic 0, $D_{a^m}(f) \neq 0$ for $m \leq d$. To see this we use the *Euler formula*:

$$d\cdot f = \sum_{i=0}^n t_i\frac{\partial f}{\partial t_i}. \tag{1.10}$$

Applying this formula to the partial derivatives we obtain

$$d(d-1)\dots(d-k+1)f = k!\sum_{|\mathbf{i}|=k}(\mathbf{i}!)^{-1}\mathbf{t}^{\mathbf{i}}\boldsymbol{\partial}^{\mathbf{i}}(f) \tag{1.11}$$

(also called the Euler formula). It follows from this formula that for every k

$$a \in P_{a^k}(X) \Leftrightarrow a \in X. \tag{1.12}$$

In view of (1.8) and (1.9), we have

$$b \in P_{a^k}(X) \Leftrightarrow a \in P_{b^{d-k}}(X). \tag{1.13}$$

1.1.2 The first polars

Let us consider some special cases. Let $X = V(f)$ be a hypersurface of degree d. Obviously, any 0-th polar of X is equal to X, and, by (1.13), the d-th polar $P_{a^d}(X)$ is empty if $a \notin X$ and $\mathbb{P}^n$ if $a \in X$. Now take $k = 1, d-1$. Using (1.5), we obtain

$$D_a(f) = \sum_{i=0}^n a_i\frac{\partial f}{\partial t_i},$$

$$\frac{1}{(d-1)!}D_{a^{d-1}}(f) = \sum_{i=0}^n \frac{\partial f}{\partial t_i}(a)t_i.$$

Together with (1.13) this implies the following.

Theorem 1.1.1. *For any smooth point $x \in X$, we have*

$$P_{x^{d-1}}(X) = \mathbb{T}_x(X).$$

If x is a singular point $P_{x^{d-1}}(X) = \mathbb{P}^n$. Moreover, for any $x \in \mathbb{P}^n$,

$$X \cap P_x(X) = \{y \in X : x \in \mathbb{T}_y(X)\}.$$

Here and later on we denote by $\mathbb{T}_x(X)$ the *embedded tangent space* of a projective subvariety $X \subset \mathbb{P}^n$ at its nonsingular point x. It is a linear subspace of $\mathbb{P}^n$ equal to the projective closure of the affine tangent space $T_x(X)$ of X at x (see [203], p. 181).

In classical terminology, the intersection $X \cap P_a(X)$ is called the *apparent boundary* of X from the point a. If one projects X to $\mathbb{P}^{n-1}$ from the point a, then the apparent boundary is the ramification divisor of the projection map.

The following picture makes an attempt to show what happens in the case when X is a conic.

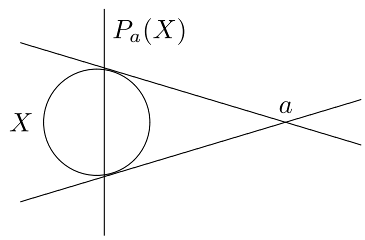


Figure 1.1: Polar line of a conic

The set of first polars $P_a(X)$ defines a linear system contained in the complete linear system $\left|\mathcal{O}_{\mathbb{P}^n}(d-1)\right|$. The dimension of this linear system $\leq n$. We will be freely using the language of linear systems and divisors on algebraic varieties (see [206]).

Proposition 1.1.2. *The dimension of the linear system of first polars $\leq r$ if and only if, after a linear change of variables, the polynomial f becomes a polynomial in $r+1$ variables.*

Proof. Induction on n and $n-r$. The assertion is obvious if $r = n$. Assume $r = n-1$. Let $\sum c_i \partial_i f = 0$ be a nontrivial linear relation between the first partial derivatives. Consider an invertible linear change of variables

$$t_i = \sum_{j=0}^{n} a_{ij} u_j, \quad i = 0, \ldots, n,$$

where $a_{i0} = c_i, i = 0, \ldots, n$. By the Chain Rule,

$$\frac{\partial f}{\partial u_0} = \sum_{i=0}^{n} c_i \frac{\partial f}{\partial t_i} = 0.$$

This proves the assertion in this case. Assume $r < n-1$. By induction on $n-r$, we may assume that, after a linear change of variables, f depends only on the variables $u_0, \ldots, u_{r+2}$. By induction on n, after a further change of variables, we may assume that f depends only on the variables $v_0, \ldots, v_{r+1}$. □

It follows from Theorem 1.1.1 that the first polar $P_a(X)$ of a point a with respect to a hypersurface X passes through all singular points of X. One can say more.

Proposition 1.1.3. *Let a be a singular point of X of multiplicity m. For each $r \leq \deg X - m$, $P_{a^r}(X)$ has a singular point at a of multiplicity m and the tangent cone of $P_{a^r}(X)$ at a coincides with the tangent cone $\mathrm{TC}_a(X)$ of X at a. For any point $b \neq a$, the r-th polar $P_{b^r}(X)$ has multiplicity $\geq m-r$ at a and its tangent cone at b is equal to the r-th polar of $\mathrm{TC}_a(X)$ with respect to b.*

Proof. Let us prove the first assertion. Without loss of generality, we may assume that $a = [1, 0, \ldots, 0]$. Then $X = V(f)$, where

$$f = t_0^{d-m} f_m(t_1, \ldots, t_n) + t_0^{d-m-1} f_{m+1}(t_1, \ldots, t_n) + \cdots + f_d(t_1, \ldots, t_n).$$

The equation $f_m(t_1, \ldots, t_n) = 0$ defines the tangent cone of X at b. The equation of $P_{a^r}(X)$ is

$$\frac{\partial^r f}{\partial t_0^r} = (d-m)\cdots(d-m-r) t_0^{d-m-r} f_m(t_1, \ldots, t_n) + \cdots + r! f_{d-r}(t_1, \ldots, t_n) = 0.$$

It is clear that $[1, 0, \ldots, 0]$ is a singular point of P_{a^r} of multiplicity m with the tangent cone $V(f_m(t_1, \ldots, t_n))$.

Now we prove the second assertion. Without loss of generality, we may assume that $a = [1, 0, \ldots, 0]$ and $b = [0, 1, 0, \ldots, 0]$. Then the equation of $P_{a^r}(X)$ is

$$\frac{\partial^r f}{\partial t_1^r} = t_0^{d-m} \frac{\partial^r f_m}{\partial t_1^r} + \cdots + \frac{\partial^r f_d}{\partial t_1^r} = 0.$$

The point a is a singular point of multiplicity $\geq (d-r)-(d-m) = m-r$. The tangent cone at b is equal to $V(\frac{\partial^r f_m}{\partial t_1^r})$ and this coincides with the r-th polar of $\mathrm{TC}_b(X) = V(f_m)$ with respect to a. □

For any nonsingular quadric Q, the map $x \mapsto P_x(Q)$ defines a projective isomorphism from the projective space to the dual projective space. This is a special case of a correlation.

An invertible projective map (a *collineation*) $\mathfrak{k}$ from a projective space $|V|$ to the dual $\mathbb{P}(W)$ of a projective space $|W|$ is called a *correlation*. It is given by an invertible linear map $\phi : V \to W^\vee$ defined uniquely up to proportinality. A correlation transforms points in $|V|$ to hyperplanes in $|W|$. A point $x \in |V|$ is called *conjugate* to a point $y \in |W|$ with respect to polarity $\mathfrak{k}$ if $y \in \mathfrak{k}(x)$. The map ${}^t\phi^{-1} : V^\vee \to W$ transforms hyperplanes in $|V|$ to points in $|W|$. It can be considered as as a correlation between the dual spaces $\mathbb{P}(V)$ and $\mathbb{P}(W)$. It is denoted by $\mathfrak{k}^\vee$ and is called the *dual correlation*. It is clear that $(\mathfrak{k}^\vee)^\vee = \mathfrak{k}$. If H is a hyperplane in $|V|$ and x is a point in

H, then point $y \in |W|$ conjugate to x under $\mathfrak{k}$ belongs to any hyperplane H' in $|W|$ conjugate to H under $\mathfrak{k}^\vee$.

A correlation can be considered as a line in $(V \otimes W)^\vee = V^\vee \otimes W^\vee$ spanned by a non-degenerate bilinear form, or, in other words as a nonsingular correspondence of type $(1,1)$ in $|V| \times |W|$. The dual correlation is the image of the divisor under the switch of the factors. A pair $(x,y) \in |V| \times |W|$ of conjugate points is just a point on this divisor.

In the case when $V = W$, we can define the *composition of correlations*. It is a collineation $\mathfrak{k}' \circ \mathfrak{k} := \mathfrak{k}' \circ \mathfrak{k}^\vee$. Collineations and correlations form a group $\Sigma\mathrm{PGL}(V)$ isomorphic to the group of outer automorphisms of $\mathrm{PGL}(V)$. The subgroup of collineations is of index 2.

A correlation $\mathfrak{k}$ of order 2 in the group $\Sigma\mathrm{PGL}(V)$ is called a *polarity*. In linear representative, this means that ${}^t\phi = \lambda\phi$ for some nonzero scalar λ. After transposing, we obtain $\lambda = \pm 1$. The case $\lambda = 1$ corresponds to the (quadric) polarity with respect to a nonsingular quadric in $\mathbb{P}^n$ which we discussed in this section. The case $\lambda = -1$ corresponds to a *null-system* (or *null polarity* which we will discuss in Chapters 2 and 10. In terms of bilinear forms, a correlation is a quadric polarity (resp. null polarity) if it can be represented by a symmetric (skew-symmetric) bilinear form.

Theorem 1.1.4. *Any projective automorphism is equal to the product of two quadric polarities.*

Proof. Choose a basis in V to represent the automorphism by a Jordan matrix J. Let $J_k(\lambda)$ be its block of size k with λ at the diagonal. Let

$$B_k = \begin{pmatrix} 0 & 0\dots & 0 & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots & \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Then

$$C_k(\lambda) = B_k J_k(\lambda) = \begin{pmatrix} 0 & 0\dots & 0 & 0 & \lambda \\ 0 & 0 & \dots & \lambda & 1 \\ \dots & \dots & \dots & \dots & \\ 0 & \lambda & \dots & 0 & 0 \\ \lambda & 1 & \dots & 0 & 0 \end{pmatrix}.$$

Observe that the matrices B_k and $C_k(\lambda)$ are symmetric. Thus each Jordan block of J can be written as the product of symmetric matrices, hence J is the product of two symmetric matrices. It follows from the definition of composition in the group $\Sigma\mathrm{PGL}(V)$, that the product of matrices representing the bilinear forms associated to correlations is the matrix representing a projective transformation equal to the composition of the correlations. $\square$

1.1.3 The second polars

The $(d-2)$- polar of $X = V(f)$ is a quadric, called the *polar quadric* of X with respect to a. It is defined by the quadratic form

$$q = D_{a^{d-2}}(f) = (d-2)! \sum_{|\mathbf{i}|=d-2} (\mathbf{i}!)^{-1} a^{\mathbf{i}} \boldsymbol{\partial}^{\mathbf{i}}(f).$$

Using equation (1.8), we obtain

$$q = 2 \sum_{|\mathbf{i}|=2} (\mathbf{i}!)^{-1} \mathbf{t}^{\mathbf{i}} \boldsymbol{\partial}^{\mathbf{i}}(f)(a).$$

By (1.12), each $a \in X$ belongs to the polar quadric $P_{a^{d-2}}(X)$. Also, by Theorem 1.1.1,

$$\mathbb{T}_a(P_{a^{d-2}}(X)) = P_a(P_{a^{d-2}}(X)) = P_{a^{d-1}}(X) = \mathbb{T}_a(X). \tag{1.14}$$

This shows that the polar quadric is tangent to the hypersurface at the point a.

Let us see where $P_{a^2}(X)$ intersects X. By (1.12)

$$P_{a^2}(X) \cap X = \{b \in X : a \in P_{b^{d-2}}(X)\} \tag{1.15}$$

Consider the line $\ell = \overline{a,b}$ through two points a, b. Let $\varphi : \mathbb{P}^1 \to \mathbb{P}^n$ be its parametric equation. It follows from (1.7) and (1.8) that

$$i(X, \overline{a,b})_b \geq s+1 \Longleftrightarrow a \in P_{b^{d-k}}(X), \quad k \leq s. \tag{1.16}$$

For $s = 1$, by Theorem 1.1.1, this condition implies that b, and hence ℓ, belongs to the tangent plane $\mathbb{T}_a(X)$. For $s = 2$, this condition says that ℓ belongs to the second polar $P_{a^2}(X)$ if and only if $i(X, \overline{a,b})_b \geq 3$.

Assume that b is a singular point of X of multiplicity $s+1$. For a general point $a \in \mathbb{P}^n$, the line $\overline{a,b}$ intersects X with multiplicity $s+1$ at b. Hence (1.16) implies that $P_{b^{d-k}}(X) = \mathbb{P}^n$ for $k \leq s$, or, equivalently, b is a singular point of X of multiplicity $s+1$.

Definition 1.2. *A line is called a* flex tangent *to X at a point a if*

$$i(X, \ell)_a > 2.$$

Proposition 1.1.5. *Let ℓ be a line through a point a. Then ℓ is a flex tangent to X at a if and only if it is contained in the intersection of $\mathbb{T}_a(X)$ with the polar quadric $P_{a^{d-2}}(X)$.*

Note that the intersection of a quadric hypersurface $Q = V(q)$ with its tangent hyperplane H at a point $a \in Q$ is a cone in H over the quadric $\bar{Q}$ in the image $\bar{H}$ of H in $|E/Ka|$.

Corollary 1.1.6. *Assume $n \geq 3$. For each $a \in X$ there exists a flex tangent line. The union of the flex tangent lines containing the point a is the cone $\mathbb{T}_a(X) \cap P_{a^{d-2}}(X)$ in $\mathbb{T}_a(X)$.*

Example 1.1.3. Assume a is a singular point of X. By Theorem 1.1.1, this is equivalent to $P_{a^{d-1}}(X) = \mathbb{P}^n$. By (1.14), the polar quadric Q is also singular at a and thus it is a cone over its image under the projection from a. The union of flex tangents is equal to Q.

Example 1.1.4. Assume a is a nonsingular point of a surface $X \subset \mathbb{P}^3$. A hyperplane which is tangent to X at a cuts out in X a curve C with a singular point a. If a is an ordinary double point of C, there are two flex tangents corresponding to the two branches of C at a. The polar quadric Q is nonsingular at a. It is a cone over a quadric $\bar{Q}$ in $\mathbb{P}^1$. If $\bar{Q}$ consists of 2 points we have two flex tangents corresponding to the two branches of C at a. If $\bar{Q}$ consists of one point (corresponding to non-reduced hypersurface in $\mathbb{P}^1$), then we have one branch. The latter case happens only if Q is singular at some point $b \neq a$.

1.1.4 The Hessian hypersurface

Let $Q(a)$ be a polar quadric of $X = V(f)$ at some point $a \in \mathbb{P}^n$. The symmetric matrix defining the corresponding quadratic form is equal to the *Hessian matrix* of second partial derivatives of f

$$\mathrm{He}(f) = \Big(\frac{\partial^2 f}{\partial t_i \partial t_j}\Big)_{i,j=0,n}, \tag{1.17}$$

evaluated at the point a. The quadric $Q(a)$ is singular if and only if the determinant of the matrix is equal to zero (the singular points correspond to the null-space of the matrix). The hypersurface

$$\mathrm{He}(X) = V(\det \mathrm{He}(f)) \tag{1.18}$$

describes the set of points $a \in \mathbb{P}^n$ such that the polar quadric $P_{a^{d-2}}(X)$ is singular. It is called the *Hessian hypersurface* of X. Its degree is equal to $(d-2)(n+1)$ unless it coincides with $\mathbb{P}^n$.

Proposition 1.1.7. *The following is equivalent:*

(i) $\mathrm{He}(X) = \mathbb{P}^n$*;*

(ii) there exists a nonzero polynomial $g(z_0, \ldots, z_n)$ *such that*

$$g(\partial_0 f, \ldots, \partial_n f) \equiv 0.$$

Proof. This is a special case of a more general result about the *jacobian* of $n+1$ polynomial functions $f_0, \ldots, f_n$ defined by

$$J(f_0, \ldots, f_n) = \det\big((\frac{\partial f_i}{\partial t_j})\big).$$

Suppose $J(f_0, \ldots, f_n) \equiv 0$. Then the map $f : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ defined by the functions $f_0, \ldots, f_n$ is degenerate at each point (i.e. df_x is of rank $< n+1$ at each point x). Thus the closure of the image is a proper closed subset of $\mathbb{C}^{n+1}$. Hence there is an irreducible polynomial which vanishes identically on the image.

Conversely, assume that $g(f_0,\ldots,f_n)\equiv 0$ for some polynomial g which we may assume to be irreducible. Then

$$\frac{\partial g}{\partial t_i}=\sum_{j=0}^{n}\frac{\partial g}{\partial z_j}(f_0,\ldots,f_n)\frac{\partial f_j}{\partial t_i}=0,\ i=0,\ldots,n.$$

Since g is irreducible its set of zeros is nonsingular on a Zariski open set U. Thus the vector

$$\big(\frac{\partial g}{\partial z_0}(f_0(x),\ldots,f_n(x)),\ldots,\frac{\partial g}{\partial z_n}(f_0(x),\ldots,f_n(x)\big)$$

is a nontrivial solution of the system of linear equations with matrix $(\frac{\partial f_i}{\partial t_j}(x))$, where $x\in U$. Thus the determinant of this matrix must be equal to zero. This implies that $J(f_0,\ldots,f_n)=0$ on U hence it is identically zero. $\square$

Remark 1.1.1. It was claimed by O. Hesse that the vanishing of the Hessian implies that the partial derivatives are linearly dependent. Unfortunately, his attempted proof is wrong. The first counterexample was given by P. Gordan and M. Noether in [188]. Consider the polynomial

$$f=t_2t_0^2+t_3t_1^2+t_4t_0t_1=0.$$

Note that the partial derivatives

$$\frac{\partial f}{\partial t_2}=t_0^2,\quad \frac{\partial f}{\partial t_3}=t_1^2,\quad \frac{\partial f}{\partial t_4}=t_0t_1$$

are algebraically dependent. This implies that the Hessian is identically equal to zero. We have

$$\frac{\partial f}{\partial t_0}=2t_0t_2+t_4t_1,\quad \frac{\partial f}{\partial t_1}=2t_1t_3+t_4t_0.$$

Suppose that a linear combination of the partials is equal to zero. Then

$$c_0t_0^2+c_1t_1^2+c_2t_0t_1+c_3(2t_0t_2+t_4t_1)+c_4(2t_1t_3+t_4t_0)=0.$$

Collecting the terms in which t_2,t_3,t_4 enters we get

$$2c_3t_0=0,\quad 2c_4t_1=0,\quad c_3t_1+c_4t_0.$$

This gives $c_3=c_4=0$. Since the polynomials t_0^2,t_1^2,t_0t_1 are linearly independent we also get $c_0=c_1=c_2=0$.

The known cases when the assertion of Hesse is true are $d=2$ (any n) and $n\le 3$ (any d) (see [188], [272], [73]).

Recall that the set of singular quadrics in $\mathbb{P}^n$ is the *discriminant hypersurface* $\mathcal{D}_2(n)$ in $\mathbb{P}^{\frac{(n+1)(n+2)}{2}-1}$ defined by the equation

$$\det\begin{pmatrix} t_{00} & t_{01} & \dots & t_{0n} \\ t_{01} & t_{11} & \dots & t_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ t_{0n} & t_{1n} & \dots & t_{nn} \end{pmatrix} = 0. \tag{1.19}$$

By differentiating, we easily find that its singular points are defined by the determinants of $n \times n$ minors of the matrix. This shows that the singular locus of $\mathcal{D}_2(n)$ parametrizes quadrics defined by quadratic forms of rank $\leq n-1$ (or corank ≥ 2). Abusing the terminology we say that a quadric is of rank k if the corresponding quadratic form is of this rank. Note that

$$\dim \mathrm{Sing}(Q) = \text{corank } Q - 1.$$

Assume that $\mathrm{He}(f) \neq 0$. Consider the rational map $p : \mathbb{P}^n = |E| \to |S^2(E^\vee)| = \mathbb{P}^{\binom{n+2}{2}-1}$ defined by $a \mapsto P_{a^{d-2}}(X)$. Note that $P_{a^{d-2}}(f) = 0$ implies $P_{a^{d-1}}(f) = 0$ and hence $\sum_{i=0}^n b_i \partial_i f(a) = 0$ for all b. This shows that a is a singular point of X. Thus p is defined everywhere except maybe at singular points of X. So the map p is regular if X is nonsingular, and the preimage of the discriminant hypersurface is equal to the Hessian of X. The preimage of the singular locus $\mathrm{Sing}(\mathcal{D}_2(n))$ consists of the subset of points $a \in \mathrm{He}(f)$ such that $\dim \mathrm{Sing}(P_{a^{d-2}}(X)) > 0$. One expects that, in general case, this will be equal to the set of singular points of the Hessian hypersurface.

Here is another description of the Hessian hypersurface.

Proposition 1.1.8. *The Hessian hypersurface* $\mathrm{He}(X)$ *is the locus of singular points of the first polars of* X.

Proof. Let $a \in \mathrm{He}(X)$ and let $b \in \mathrm{Sing}(P_{a^{d-2}}(X))$. Then

$$D_b(D_{a^{d-2}}(f)) = D_{a^{d-2}}(D_b(f)) = 0.$$

Since $D_b(f)$ is of degree $d-1$, this means that $\mathbb{T}_a(P_b(X)) = \mathbb{P}^n$, i.e., a is a singular point of $P_b(X)$.

Conversely, if $a \in \mathrm{Sing}(P_b(X))$ for $b \in \mathbb{P}^n$, then $D_{a^{d-2}}(D_b(f)) = 0$, hence $D_b(D_{a^{d-2}}(f)) = 0$. This means that b is a singular point of the polar quadric with respect to a. Hence $a \in \mathrm{He}(X)$. $\square$

Let us find the affine equation of the Hessian hypersurface. Applying the Euler formula (9.9), we can write

$$t_0 f_{0i} = (d-1)\partial_i f - t_1 f_{1i} - \dots - t_n f_{ni},$$

$$t_0 \partial_0 f = df - t_1 \partial_1 f - \dots - t_n \partial_n f,$$

where f_{ij} denote the second partial derivative. Multiplying the first row of the Hessian determinant by t_0 and adding to it the linear combination of the remaining rows with the coefficients t_i, we get the following equality.

$$\det(\mathrm{He}(f)) = \frac{d-1}{t_0}\det\begin{pmatrix}\partial_0 f & \partial_1 f & \dots & \partial_n f\\ f_{10} & f_{11} & \dots & f_{1n}\\ \vdots & \vdots & \vdots & \\ f_{n0} & f_{n1} & \dots & f_{nn}\end{pmatrix}.$$

Repeating the same procedure but this time with the columns, we finally get

$$\det(\mathrm{He}(f)) = \frac{(d-1)^2}{t_0^2}\det\begin{pmatrix}\frac{d}{d-1}f & \partial_1 f & \dots & \partial_n f\\ \partial_1 f & f_{11} & \dots & f_{1n}\\ \vdots & \vdots & \vdots & \\ \partial_n f & f_{n1} & \dots & f_{nn}\end{pmatrix}. \tag{1.20}$$

Let $\phi(z_1,\dots,z_n)$ be the dehomogenization of f with respect to t_0, i.e.,

$$f(t_0,\dots,t_d) = t_0^d\phi(\frac{t_1}{t_0},\dots,\frac{t_n}{t_0}).$$

We have

$$\frac{\partial f}{\partial t_i} = t_0^{d-1}\phi_i(z_1,\dots,z_n),\ \frac{\partial^2\phi}{\partial t_i\partial t_j} = t_0^{d-2}\phi_{ij}(z_1,\dots,z_n),\quad i,j=1,\dots,n,$$

where

$$\phi_i = \frac{\partial f}{\partial z_i},\quad \phi_{ij} = \frac{\partial^2 f}{\partial z_i\partial z_j}.$$

Plugging these expressions in (1.20), we obtain, that up to a nonzero constant factor,

$$t_0^{-(n+1)(d-2)}\det(\mathrm{He}(\phi)) = \det\begin{pmatrix}\frac{d}{d-1}\phi(z) & \phi_1(z) & \dots & \phi_n(z)\\ \phi_1(z) & \phi_{11}(z) & \dots & \phi_{1n}(z)\\ \vdots & \vdots & \vdots & \\ \phi_n(z) & \phi_{n1}(z) & \dots & \phi_{nn}(z)\end{pmatrix}, \tag{1.21}$$

where $z=(z_1,\dots,z_n), z_i = t_i/t_0,\ i=1,\dots,n$.

Remark 1.1.2. If $f(x,y)$ is a real polynomial in three variables, the value of (1.21) at a point a of the hypersurface $V(f)$ multiplied by $\frac{-1}{f_1(a)^2+f_2(a)^2+f_3(a)^2}$ is equal to the *Gauss curvature* of $X(\mathbb{R})$ at the point a (see [164]).

1.1.5 Parabolic points

Let us see where $\mathrm{He}(X)$ intersects X. A glance at the expression (1.21) reveals the following fact.

Proposition 1.1.9. *Each singular point of X belongs to* $\mathrm{He}(X)$.

Let us see now when a nonsingular point $a \in X$ lies in its Hessian hypersurface $\mathrm{He}(X)$.

By Corollary 1.1.6, the flex tangent lines in $\mathbb{T}_a(X)$ sweep the intersection of $\mathbb{T}_a(X)$ with the polar quadric $P_{a^{d-2}}(X)$. If $a \in \mathrm{He}(X)$, then the polar quadric is singular at some point b.

If $n = 2$, a singular quadric is the union of two lines, so this means that one of the lines is a flex tangent line. A nonsingular point a of a plane curve X such that there exists a flex tangent at a is called an *inflection point* or a *flex* of X.

If $n > 2$, the flex tangents lines at a point $a \in X \cap \mathrm{He}(X)$ sweep a cone over a singular quadric in $\mathbb{P}^{n-2}$. Such a point is called a *parabolic point* of X. The closure of the set of parabolic points is the *parabolic hypersurface* in X.

Theorem 1.1.10. *Let X be a hypersurface of degree d in $\mathbb{P}^n$. If $n = 2$, then $\mathrm{He}(X) \cap X$ consists of singular and inflection points of X. In particular, each nonsingular curve of degree ≥ 3 has an inflection point, and the number of inflections points is less or equal than $3d(d-2)$ or infinite. If $n > 2$, then the set $X \cap \mathrm{He}(X)$ consists of singular points and parabolic points. The parabolic hypersurface in X is either the whole X or a subvariety of degree $(n+1)d(d-2)$ in $\mathbb{P}^n$.*

Example 1.1.5. Let X be a surface of degree d in $\mathbb{P}^3$. If a is a parabolic point of X, then $\mathbb{T}_a(X) \cap X$ is a singular curve whose singularity at a is unibranched. In fact, otherwise X has at least two distinct flex lines which cannot sweep a cone over a singular quadric in $\mathbb{P}^1$. The converse is also true. For example, a nonsingular quadric has no parabolic points, and all nonsingular points of a singular quadric are parabolic.

A generalization of a quadratic cone is a *developable surface*. It is a special kind of a *ruled surface* (see [164] and later Chapters) which are characterized by the condition that the tangent plane does not change along a ruling. The Hessian surface of a developable surface contains this surface. The residual surface of degree $2d - 8$ is called *Pro-Hessian surface*. A concrete example of a developable surface is the quartic surface

$$(x_0x_3 - x_1x_2)^2 - 4(x_1^2 - x_0x_2)(x_2^2 - x_1x_3) = 0.$$

It is the surface swept out by the tangent lines of a rational normal curve of degree 3. It is also the *determinantal surface* of a binary cubic, i.e. the surface parameterizing binary cubics $a_0x^3 + 4a_1x^2y + 6a_2xy^2 + a_3y^3$ which have a multiple root. The Pro-Hessian of any quartic developable surface is the surface itself [60].

Assume now that X is a curve. Let us see when it has infinitely many inflection points. Certainly, this happens when X contains a line component; each of its point is an inflection point. It must be also an irreducible component of $\mathrm{He}(X)$. The set of inflection points is a closed subset of X. So, if X has infinitely many inflection points, it must have an irreducible component consisting of inflection points. Each such component is contained in $\mathrm{He}(X)$. Conversely, each common irreducible component of X and $\mathrm{He}(X)$ consists of inflection points.

We will prove the converse in a little more general form taking care of not necessary reduced curves.

Proposition 1.1.11. *A polynomial $f(x_0, x_1, x_2)$ is a factor of its Hessian polynomial* $\mathrm{He}(f)$ *if and only if each factor of f entering with multiplicity 1 is a linear polynomial.*

Proof. Since each point on a non-reduced component of $X = V(f)$ is a singular point (i.e. all the first partials vanish), and each point on a line component is an inflection point, we see that the condition is sufficient for $X \subset \mathrm{He}(f)$. Suppose this happens and let R be a reduced irreducible component of the curve X which is contained in the Hessian. Take a nonsingular point of R and consider an affine equation of R with coordinates (x, y). We may assume that $\mathcal{O}_{R,x}$ is included in $\hat{\mathcal{O}}_{R,x} \cong K[[t]]$ such that $x = t, y = t^r\epsilon$, where $\epsilon(0) = 1$. Thus the equation of R looks like

$$f(x, y) = y - x^r + g(x, y), \tag{1.22}$$

where $g(x, y)$ does not contain terms $cy, c \in \mathbb{C}$. It is easy to see that $(0, 0)$ is an inflection point if and only if $r > 2$ with the flex tangent $y = 0$.

We use the affine equation of the Hessian (1.21), and obtain that the image of

$$h(x,y) = \det \begin{pmatrix} \frac{d}{d-1}f & f_1 & f_2 \\ f_1 & f_{11} & f_{12} \\ f_2 & f_{21} & f_{22} \end{pmatrix}$$

in $K[[t]]$ is equal to

$$\det \begin{pmatrix} 0 & -rt^{r-1} + g_1 & 1 + g_2 \\ -rt^{r-1} + g_1 & -r(r-1)t^{r-2} + g_{11} & g_{12} \\ 1 + g_2 & g_{12} & g_{22} \end{pmatrix}.$$

Since every monomial entering in g is divisible by y^2, xy or $x^i, i > r$, we see that g_y is divisible by t and g_x is divisible by t^{r-1}. Also g_{11} is divisible by t^{r-1}. This shows that

$$h(x,y) = \det \begin{pmatrix} 0 & at^{r-1} + \dots & 1 + \dots \\ at^{r-1} + \dots & -r(r-1)t^{r-2} + \dots & g_{12} \\ 1 + \dots & g_{12} & g_{22} \end{pmatrix},$$

where $\dots$ denotes terms of higher degree in t. We compute the determinant and see that it is equal to $r(r-1)t^{r-2} + \dots$. This means that its image in $K[[t]]$ is not equal to zero, unless the equation of the curve is equal to $y = 0$, i.e. the curve is a line. $\square$

In fact, we have proved more. We say that a nonsingular point of X is an inflection point of *order*. $r - 2$ and denote the order by $\mathrm{ordfl}_x X$ if one can choose an equation of the curve as in (1.22) with $r \geq 3$. It follows from the previous proof that $r - 2$ is equal to the multiplicity $i(X, \mathrm{He})_x$ of the intersection of the curve and its Hessian at the point x. It is clear that $\mathrm{ordfl}_x X = i(\ell, X)_x - 2$, where ℓ is the flex tangent line of X at x. We have

$$\sum_{x \in X} i(X, \mathrm{He})_x = \sum_{x \in X} \mathrm{ordfl}_x X = 3d(d-2). \tag{1.23}$$

1.1.6 The Steinerian hypersurface

Recall that Hessian hypersurface of a hypersurface $X = V(f)$ is the locus of points a such that the polar quadric $P_{a^{d-2}}(X)$ is singular. The *Steinerian hypersurface* $\mathrm{St}(X)$ of X is the locus of singular points of the polar quadrics. Thus

$$\mathrm{St}(X) = \bigcup_{a\in \mathrm{He}(X)} \mathrm{Sing}(P_{a^{d-2}}(X)). \tag{1.24}$$

The proof of Theorem 1.1.8 shows that it can be equivalently defined as

$$\mathrm{St}(X) = \{a \in \mathbb{P}^n : P_a(X) \text{ is singular}\}. \tag{1.25}$$

We also have

$$\mathrm{He}(X) = \bigcup_{a\in \mathrm{St}(X)} \mathrm{Sing}(P_a(X)). \tag{1.26}$$

A point $b = [b_0, \ldots, b_n] \in \mathrm{St}(X)$ satisfies the equation

$$\mathrm{He}(f)(a) \cdot \begin{pmatrix} b_0 \\ \vdots \\ b_n \end{pmatrix} = 0, \tag{1.27}$$

where $a \in \mathrm{He}(X)$. This equation defines a subvariety $\mathrm{HS}(X)$ of $\mathbb{P}^n \times \mathbb{P}^n$ given by $n+1$ equations of bidegree $(d-2, 1)$. When the Steinerian map is defined, it is just its graph. The projection to the second factor is a closed subscheme of $\mathbb{P}^n$ with support at $\mathrm{St}(X)$. This gives a scheme-theoretical definition of the Steinerian hypersurface which we will accept from now on. It also makes clear why $\mathrm{St}(X)$ is a hypersurface, not obvious from the definition. The expected dimension of the image of the second projection is $n-1$.

The following argument confirms our expectation. It is known that the locus of singular hypersurfaces of degree d in $|V|$ is a hypersurface

$$\mathcal{D}_n(d) \subset |S^d E^\vee|$$

of degree $(n+1)(d-1)^n$ defined by the *discriminant* of a general degree d homogeneous polynomial in $n+1$ variables (the *discriminant hypersurface*). Let L be the projective subspace of $|S^{d-1}E^\vee|$ which consists of first polars of X. Assume that no polar $P_a(X)$ is equal to $\mathbb{P}^n$. Then

$$\mathrm{St}(X) \cong L \cap \mathcal{D}_n(d-1).$$

So, unless L is contained in $\mathcal{D}_n(d-1)$ we get a hypersurface. Moreover we obtain

$$\deg(\mathrm{St}(X)) = (n+1)(d-2)^n. \tag{1.28}$$

Assume that the quadric $P_{a^{d-2}}(X)$ is of corank 1 (i.e. the matrix $\mathrm{He}(f)(a)$ is of rank n). Then it has a unique singular point $b = [b_0, \ldots, b_n]$, whose coordinates can be chosen to be any column or a row of the adjugate matrix $\mathrm{adj}(\mathrm{He}(f))$ evaluated at the point a. Thus $\mathrm{St}(X)$ is the image of the Hessian hypersurface under the rational map

$$\mathrm{st} : \mathrm{He}(X) - \to \mathrm{St}(X), \quad a \mapsto \mathrm{Sing}(P_{a^{d-2}}(X)),$$

given by polynomials of degree $n(d-2)$. We call it the *Steinerian map*. Of course, it is not defined when all polar quadrics are of corank > 1. Also, if the first polar hypersurface $P_a(X)$ has an isolated singular point for a general point a, we get a rational map

$$\mathrm{st}^{-1} : \mathrm{St}(X) - \to \mathrm{He}(X), \quad a \mapsto \mathrm{Sing}(P_a(X)).$$

These maps are obvioulsy inverse to each other. It is a difficult question to determine the sets of indeterminacy points for both maps.

Proposition 1.1.12. *The Steinerian hypersurface coincides with the whole $\mathbb{P}^n$ if and only if X has a point of multiplicity ≥ 3.*

Proof. The first polars of X form a linear system of hypersurfaces of degree $d-1$. By Bertini's Theorem, a singular point of a general member of the linear system is one of the base points. Thus $\mathrm{St}(X) = \mathbb{P}^n$ implies that X has a singular point. Without loss of generality, we may assume that the points is $[1, 0, \ldots, 0]$. Write the equation of X in the form

$$f = t_0^k g_{d-k}(t_1, \ldots, t_n) + t_0^{k-1} g_{d+1-k}(t_1, \ldots, t_n) + \cdots + g_d(t_1, \ldots, t_n) = 0,$$

where the subscript indicates the degree of the polynomial. Then the first polar $P_a(X)$ has the equation

$$a_0 \sum_{i=0}^{k} k t_0^{k-1-i} g_{d-k+i} + \sum_{s=1}^{n} a_s \sum_{i=0}^{k} t_0^{k-i} \frac{\partial g_{d-k+i}}{\partial t_s} = 0.$$

The largest power of t_0 in this expression is at most k. The degree of the equation is $d-1$. Thus a is a singular point of $P_a(X)$ if and only if $k \leq d-3$, or, equivalently, when a is at least triple point of X. $\square$

Assume that $a = [v]$ be point on a hypersurface $X = V(f)$ of degree $d > 1$. Applying Euler's formula to the partial derivatives of f we find

$$(d-1)\frac{\partial f}{\partial t_i} = \sum_{j=0}^{n} t_j \frac{\partial^2 f}{\partial t_i t_j}, \ i = 0, \ldots, n.$$

This implies

$$(d-1)\nabla(f)(v) = \mathrm{He}(f)(v) \cdot v, \tag{1.29}$$

where $\nabla(f)(v)$ denotes the gradient vector of f at v (note that we do not put the transpose over v since, without ambiguity, v must be considered as a column vector). Assume a is a singular point of X. Then $\nabla(f)(v) = 0$ and, using (1.27), we infer that $a \in \mathrm{He}(X)$ and $a \in \mathrm{St}(X)$. This gives

Proposition 1.1.13. *The intersection* $\mathrm{He}(X) \cap \mathrm{St}(X)$ *contains the singular locus of* X.

One can assign one more variety to a hypersurface $X = V(f)$. This is the *Cayleyan variety*. It is defined as the image $\mathrm{Cay}(X)$ of the rational map

$$\mathrm{HS}(X) - \to G_1(\mathbb{P}^n), \quad (a,b) \mapsto \overline{a,b},$$

where $G_1(\mathbb{P}^n)$ denotes the Grassmannian of lines in $\mathbb{P}^n$. The map is not defined at the intersection of the diagonal with $\mathrm{HS}(X)$. We know that $\mathrm{HS}(a,a) = 0$ means that $P_{a^{d-1}}(X) = 0$, and the latter means that a is a singular point of X. Thus the map is a regular map for a nonsingular hypersurface X.

Note that in the case $n = 2$, the Cayleyan variety is a plane curve in the dual plane, the *Cayleyan curve* of X.

Proposition 1.1.14. *Let* X *be a hypersurface of degree* $d \geq 3$ *with no singular points of multiplicity* ≥ 3. *Then*

$$\deg \mathrm{Cay}(X) = \begin{cases} \binom{n+1}{2}(d-2)(1+(d-2)^{n-1}) & \text{if } d > 3, \\ \frac{1}{2}\binom{n+1}{2}(d-2)(1+(d-2)^{n-1}) & \text{if } d = 3, \end{cases}$$

where the degree is considered with respect to the Plücker embedding of the Grassmannian $G_1(\mathbb{P}^n)$.

Proof. By Proposition 1.1.12, $\mathrm{St}(X) \neq \mathbb{P}^2$, hence $\mathrm{HS}(X)$ is a complete intersection of 3 hypersurfaces in $\mathbb{P}^n \times \mathbb{P}^n$ of bidegree $(d-2,1)$. It is known that the set of lines intersecting a codimension 2 linear subspace L is a hyperplane section of the Grassmannian $G_1(\mathbb{P}^n)$ in its Plücker embedding. Write $\mathbb{P}^n = |V|$ and $L = |W|$. Let $\omega = w_1 \wedge \ldots \wedge w_{n-1}$ for some basis $(w_1, \ldots, w_{n-1})$ of W. The locus of pairs of points $a = \mathbb{C}v_1, b = \mathbb{C}v_2$ lying on a line intersecting L is given by the equation $v_1 \wedge v_2 \wedge \omega = 0$. This is a hypersurface L of bidegree $(1,1)$ in $\mathbb{P}^n \times \mathbb{P}^n$. Let h_1, h_2 be the natural generators of $H^*(\mathbb{P}^n \times \mathbb{P}^n, \mathbb{Z})$. We have

$$\#\mathrm{HS}(X) \cap L = ((d-2)h_1 + h_2)^{n+1}(h_1+h_2) = \binom{n+1}{2}(d-2)^n + \binom{n+1}{2}(d-2)$$

$$= \binom{n+1}{2}(d-2)((d-2)^{n-1}+1).$$

If $d = 3$, we will see later that $\mathrm{He}(X) = \mathrm{St}(X)$ and the Steinerian map is an involution. Thus to get the degree we have to divide the above number by 2. $\square$

Remark 1.1.3. From the point of view of the classical invariant theory, the homogeneous forms defining the Hessian and Steinerian hypersurfaces of $V(f)$ are examples of covariants of f. The form defining the Cayleyan of a plane curve is an example of a contravariant.

1.2 The dual hypersurface

1.2.1 The polar map

The linear space of first polars $P_a(X)$ defines a linear subsystem of the complete linear system $\left|\mathcal{O}_{\mathbb{P}^n}(d-1)\right|$ of hypersurfaces of degree $d-1$ in $\mathbb{P}^n$. Its dimension is equal to n if the first partial derivatives of f are linearly independent. By Proposition 1.1.2 this happens if and only if X is not a cone. We assume that this is the case. Let us identify the linear system of first polars with $|E| = \mathbb{P}^n$ by assigning to each $a \in \mathbb{P}^n$ the polar hypersurface $P_a(X)$. Let $p_X : \mathbb{P}^n - \to \check{\mathbb{P}}^n$ be the rational map defined by the linear system of polars. It is called the *polar map*. In coordinates, the polar map is given by

$$[t_0, \ldots, t_n] \mapsto \big[\frac{\partial f}{\partial t_0}, \ldots, \frac{\partial f}{\partial t_n}\big].$$

Recall that a hyperplane $H_a = V(\sum a_i \xi_i)$ in the dual projective space $\check{\mathbb{P}}^n$ is the point $a = [a_0, \ldots, a_n] \in \mathbb{P}^n$. The preimage of the hyperplane H_a under p_X is the polar $P_a(f) = V(\sum a_i \frac{\partial f}{\partial t_i})$.

If X is nonsingular, the polar map is a regular map given by polynomials of degree $d-1$.

One can view the polar map as the rational map that sends a point x to the polar hyperplane $P_{x^{d-1}}(X) = H$. A point in the preimage of a hyperplane H is called a *pole* of H with respect to X.

Proposition 1.2.1. *Assume X is nonsingular. The ramification divisor* $\mathrm{Ram}(p_X)$ *of the polar map is equal to* $\mathrm{He}(X)$.

Proof. Note for any finite map $\phi : X \to Y$ of nonsingular varieties, the ramification divisor $\mathrm{Ram}(\phi)$ is defined locally by the determinant of the linear map of locally free sheaves $\phi^*(\Omega^1_Y) \to \Omega^1_X$. The image of $\mathrm{Ram}(\phi)$ in Y is called the *branch divisor*. Both of the divisors may be nonreduced. We have the *Hurwitz formula*

$$K_X = \phi^*(K_Y) + \mathrm{Ram}(\phi). \tag{1.30}$$

The map ϕ is étale outside $\mathrm{Ram}(\phi)$, i.e., for any point $x \in X$ the homomorphism of local ring $\mathcal{O}_{Y,\phi(x)} \to \mathcal{O}_{X,x}$ defines an isomorphism of their formal completions. In particular, the preimage $\phi^{-1}(Z)$ of a nonsingular subvariety $Z \subset Y$ is nonsingular outside the support of $\mathrm{Ram}(\phi)$. Applying this to the polar map we see that the singular points of $P_a(X) = p_X^{-1}(H_a)$ are contained in the ramification locus $\mathrm{Ram}(p_X)$ of the polar map. On the other hand, we know that the set of singular points of first polars is the Hessian $\mathrm{He}(X)$. This shows that $\mathrm{He}(X) \subset \mathrm{Ram}(p_X)$. Applying the Hurwitz formula, we have $K_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(-n-1)$, $K_{\check{\mathbb{P}}^n} = \mathcal{O}_{\check{\mathbb{P}}^n}(-n-1)$, $p_X^{-1}(K_{\check{\mathbb{P}}^n}) = \mathcal{O}_{\mathbb{P}^n}((-n-1)(d-1))$. This gives $\deg(\mathrm{Ram}(p_X)) = (n+1)(d-2) = \deg(\mathrm{He}(X))$. This shows that $\mathrm{He}(X) = \mathrm{Ram}(p_X)$. □

What is the branch divisor? One can show that the preimage of a hyperplane H_a is singular if and only if it is tangent to the branch locus of the map. The preimage of H_a is the polar hypersurface $P_a(X)$. Thus the set of hyperplanes tangent to the branch divisor is equal to the Steinerian $\mathrm{St}(X)$. This shows that the branch locus equals the dual variety of $\mathrm{St}(X)$. Another implication of this is the following.

Corollary 1.2.2. *Assume X is nonsingular. For any point $a \in \mathrm{He}(X)$ the polar hyperplane of X with the pole at a is tangent to the Steinerian* $\mathrm{St}(X)$ *at a.*

1.2.2 Dual varieties

Recall that the *dual variety* $X^\vee$ of a subvariety X in $\mathbb{P}^n = |E|$ is the closure in the dual projective space $\check{\mathbb{P}}^n = |E^\vee|$ of the locus of hyperplanes in $\mathbb{P}^n$ which are tangent to X at some nonsingular point of X.

When $X = V(f)$ is a hypersurface, we see that the dual variety is the image of X under the rational map given by the first polars. In fact, $(\partial_0 f(x), \ldots, \partial_n f(x))$ in $\check{\mathbb{P}}^n$ is the hyperplane $V(\sum_{i=0}^n \partial_i f(x) t_i)$ in $\mathbb{P}^n$ which is tangent to X at the point x.

The following result is called the *projective duality*. Many modern text-books contain a proof (see [183], [203], [429]).

Theorem 1.2.3.

$$(X^\vee)^\vee = X.$$

It follows from any proof in loc. cit. that, for any nonsingular point $y \in X^\vee$ and any nonsingular point $x \in X$,

$$\mathbb{T}_x(X) \subset H_y \Leftrightarrow \mathbb{T}_y(X^\vee) \subset H_x.$$

The set of all hyperplanes in $\check{\mathbb{P}}^n$ containing the linear subspace $\mathbb{T}_y(X^\vee)$ is the dual linear space of $\mathbb{T}_y(X^\vee)$ in $\mathbb{P}^n$. Thus the fibre of the *duality map* (or *Gauss map*)

$$\gamma : X^{\mathrm{ns}} \to X^\vee, \quad x \mapsto \mathbb{T}_x(X), \tag{1.31}$$

over a nonsingular point $y \in X^\vee$ is an open subset of the projective subspace in $\mathbb{P}^n$ equal to the dual of the tangent space $\mathbb{T}_y(X^\vee)$. Here and later X^{ns} denotes the set of nonsingular points of a variety X. In particular, if $X^\vee$ is a hypersurface, the dual space of $\mathbb{T}_y(X^\vee)$ must be a point, and hence the map γ is birational.

Let us apply this to our case when X is a nonsingular hypersurface. Then the map given by first polars is a regular map $\mathbb{P}^n \to \check{\mathbb{P}}^n$ defined by homogeneous polynomials of degree $d-1$. It is a finite map (after applying the Veronese map it becomes a linear projection map). Therefore, its fibres are finite sets. This shows that the dual of a nonsingular hypersurface is a hypersurface. Thus, the duality map, equal to the restriction of the polar map, is a birational isomorphism

$$d : X \underset{\mathrm{bir}}{\cong} X^\vee.$$

The degree of the dual hypersurface $X^\vee$ (if it is a hypersurface) is called the *class* of X. For example, the class of any plane curve of degree > 1 is well-defined.

Example 1.2.1. Let $\mathcal{D}_d(n)$ be the discriminant hypersurface in $|S^d E^\vee|$. We would like to describe explicitly the tangent hyperplane of $\mathcal{D}_d(n)$ at its nonsingular point. Let

$$\tilde{\mathcal{D}}_d(n) = \{(X, x) \in |\mathcal{O}_{\mathbb{P}^n}(d)| \times \mathbb{P}^n : x \in \mathrm{Sing}(X)\}.$$

Let us see that $\tilde{\mathcal{D}}_d(n)$ is nonsingular and the projection to the first factor

$$\pi : \tilde{\mathcal{D}}_d(n) \to \mathcal{D}_d(n) \tag{1.32}$$

is a resolution of singularities. In particular, π is an isomorphism over the open set $\mathcal{D}_d(n)^{\mathrm{ns}}$ of nonsingular points of $\mathcal{D}_d(n)$.

The fact that $\tilde{\mathcal{D}}_d(n)$ is nonsingular follows easily from considering the projection to $\mathbb{P}^n$. For any point $x \in \mathbb{P}^n$ the fibre of the projection is the projective space of hypersurfaces which have a singular point at x (this amounts to $n+1$ linear conditions on the coefficients). Thus $\tilde{\mathcal{D}}_d(n)$ is a projective bundle over $\mathbb{P}^n$ and hence is nonsingular.

Let us see where π is an isomorphism. Let $A_{\mathbf{i}}, |\mathbf{i}| = d$, be the projective coordinates in $\left|\mathcal{O}_{\mathbb{P}^n}(d)\right| = |S^d E^\vee|$ corresponding to the coefficients of a hypersurface of degree d and let $t_0, \ldots, t_n$ be projective coordinates in $\mathbb{P}^n$. Then $\tilde{\mathcal{D}}_d(n)$ is given by $n+1$ bihomogeneous equations of bidegree $(1, d-1)$:

$$\sum_{|\mathbf{i}|=d} i_s A_{\mathbf{i}} t^{\mathbf{i}-e_s} = 0, \quad s = 0, \ldots, n, \tag{1.33}$$

Here e_s is the s-th unit vector in $\mathbb{Z}^{n+1}$.

A point $(X, x) = (V(f), [v_0]) \in |\mathcal{O}_{\mathbb{P}^n}(1)| \times \mathbb{P}^n$ belongs to $\tilde{\mathcal{D}}_d(n)$ if and only if, replacing $A_{\mathbf{i}}$ with the coefficient of f at $\mathbf{t}^i$ and t_i with the i-th coefficient of v_0, we get the identities.

We identify the tangent space of $|S^d E^\vee| \times |E|$ at a point (X, x) with the space $S^d E^\vee/\mathbb{C}f \oplus E/\mathbb{C}v_0$. In coordinates, a vector in the tangent space is a pair $(g, [v])$, where $g = \sum_{|\mathbf{i}|=d} a_{\mathbf{i}} \mathbf{t}^{\mathbf{i}}, v = (x_0, \ldots, x_n)$ considered modulo pairs $(\lambda f, \mu v_0)$. Differentiating equations (1.33), we see that the tangent space is defined by the $(n+1) \times \binom{n+d}{d}$-matrix

$$M = \begin{pmatrix} \ldots & i_0 x^{\mathbf{i}-e_0} & \ldots & \sum_{|\mathbf{i}|=d} i_0 i_0 A_{\mathbf{i}} x^{\mathbf{i}-e_0-e_0} & \ldots & \sum_{|\mathbf{i}|=d} i_0 i_n A_{\mathbf{i}} x^{\mathbf{i}-e_0-e_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ \ldots & i_n x^{\mathbf{i}-e_n} & \ldots & \sum_{|\mathbf{i}|=d} i_n i_0 A_{\mathbf{i}} x^{\mathbf{i}-e_n-e_0} & \ldots & \sum_{|\mathbf{i}|=d} i_n i_n A_{\mathbf{i}} x^{\mathbf{i}-e_n-e_n}, \end{pmatrix}$$

where $x^{\mathbf{i}-e_s} = 0$ if $\mathbf{i} - e_s$ is not a non-negative vector. It is easy to interpret solutions of these equations as pairs (g, v) from above such that

$$\nabla(g)(v_0) + \mathrm{He}(f)(v_0) \cdot v = 0. \tag{1.34}$$

Note that $\nabla(f)(b) = 0$ since $[v_0]$ is a singular point of $V(f)$ and $\mathrm{He}(f)(v_0) \cdot v_0 = 0$ as follows from (1.29). This confirms that pairs $(\lambda f, \mu v_0)$ are always solutions. The tangent map $d\pi$ at the point $(V(f), [v_0])$ is given by the projection $(g, v) \mapsto g$, where (g, v) is a solution of (1.34). Its kernel consist of pairs $(\lambda f, v)$ modulo pairs $(\lambda f, \mu v_0)$. For such pairs the equations (1.34) give

$$\mathrm{He}(f)(v_0) \cdot v = 0. \tag{1.35}$$

We may assume that $v_0 = (1, 0, \ldots, 0)$. Since $[v_0]$ is a singular point of $V(f)$ we can write $f = t_0^{d-2} f_2(t_1, \ldots, t_n) + \ldots$. Computing the Hessian matrix at the point v_0 we see that it is equal to

$$\begin{pmatrix} 0 & \dots & \dots & 0 \\ 0 & a_{11} & \dots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & a_{n1} & \dots & a_{nn} \end{pmatrix}, \tag{1.36}$$

where $f_2(t_1, \dots, t_n) = \sum_{0 \le i,j \le n} a_{ij} t_i t_j$. Thus a solution of (1.35), not proportional to v_0 exists if and only if $\det \mathrm{He}(f_2) = 0$. By definition, this means that the singular point of X at x is not ordinary double point. Thus we obtain that the projection map (1.32) is an isomorphism over the open subset of $\mathcal{D}_d(n)$ representing hypersurfaces with an isolated ordinary singularity.

We can also find the description of the tangent space of $\mathcal{D}_d(n)$ at its point $X = V(f)$ representing a hypersurface with a unique ordinary singular point x. It follows from calculation of the hessian matrix in (1.36), that its corank at the ordinary singular point is equal to 1. Since the matrix is symmetric, the dot-product of a vector in its nullspace is orthogonal to the column of the matrix. Since we know that $\mathrm{He}(f)(v_0) \cdot v_0 = 0$, this implies that the dot-product $\nabla(g)(v_0) \cdot v_0$ is equal to zero. By Euler's formula this gives $g(v_0) = 0$. The converse is also true. This proves that

$$T(\mathcal{D}_d(n))_X = \{g \in S^d(E^\vee)/\mathbb{C}f : g(x) = 0\}. \tag{1.37}$$

Now we are ready to compute the dual variety of $\mathcal{D}_d(n)$. The condition $g(b) = 0$, where $\mathrm{Sing}(X) = \{b\}$ is equivalent to $D_{b^d}(f) = 0$. Thus the tangent hyperplane, considered, as a point in the dual space $|S^d E| = |(S^d E^\vee)^\vee|$ corresponds to the envelope $b^d = (\sum_{s=0}^n b_s \partial_i)^d$. The set of such envelopes is the Veronese variety $\nu_d(|E|)$. Thus

$$\mathcal{D}_d(n)^\vee \cong \nu_d(\mathbb{P}^n), \tag{1.38}$$

Of course, it is predictable. Recall that the Veronese variety is embedded naturally in $|\mathcal{O}_{\mathbb{P}^n}(d)|^\vee$. Its hyperplane section can be naturally identified with a hypersurface of degree d in $\mathbb{P}^n$. A tangent hyperplane is a hypersurface with a singular point, i.e. a point in $\mathcal{D}_d(n)$. Thus the dual of $\nu_d(\mathbb{P}^n)$ is $\mathcal{D}_d(n)$, and hence, by duality, the dual of $\mathcal{D}_d(n)$ is $\nu_d(\mathbb{P}^n)$.

Example 1.2.2. Let $Q = V(q)$ be a nonsingular quadric in $\mathbb{P}^n$. Let $A = (a_{ij})$ be a symmetric matrix defining Q, i.e. $q(\mathbf{t}) = \mathbf{t} \cdot A \cdot \mathbf{t}$. The tangent hyperplane of Q at a point $x = [x_0, \dots, x_n] \in \mathbb{P}^n$ is the hyperplane

$$t_0 \sum_{j=0}^n a_{0j} x_j + \dots + t_n \sum_{j=0}^n a_{nj} x_j = 0.$$

Thus the vector of coordinates $\mathbf{y} = (y_0, \dots, y_n)$ of the tangent hyperplane is equal to the vector $A \cdot \mathbf{x}$. Since A is invertible, we can write $\mathbf{x} = A^{-1} \cdot \mathbf{y}$. We have

$$0 = \mathbf{x} \cdot A \cdot \mathbf{x} = (\mathbf{y} \cdot A^{-1}) A (A^{-1} \cdot \mathbf{y}) = \mathbf{y} \cdot A^{-1} \cdot \mathbf{y} = 0.$$

Here we treat $\mathbf{x}$ or $\mathbf{y}$ as a row-matrix or as a column-matrix in order the matrix multiplication makes sense. Since $A^{-1} = \det(A)^{-1}\mathrm{adj}(A)$, where $\mathrm{adj}(A)$ is the adjugate matrix, we obtain that the dual variety of Q is also a quadric given by the adjugate matrix of the matrix defining Q.

The description of the tangent space of the discriminant hypersurface from Example 1.2.1 has the following nice application.

Proposition 1.2.4. *Let X be a hypersurface of degree d in $\mathbb{P}^n$. Suppose x is a nonsingular point of the Steinerian hypersurface $\mathrm{St}(X)$. Then $\mathrm{Sing}(P_x(X))$ consists of an ordinary singular point y and*

$$\mathbb{T}_x(\mathrm{St}(X)) = P_{y^{d-1}}(X) = \{a \in \mathbb{P}^n : y \in P_a(X)\}.$$

Proof. The linear system L of the first polars of X intersects the discriminant hypersurface $\mathcal{D}_{d-1}(n)$ at the point $P_x(X)$. Since $\mathrm{St}(X) = p_X^{-1}(L \cap \mathcal{D}_{d-1}(n))$ is nonsingular at x, the hypersurface $P_x(X)$ is a nonsingular point of $\mathcal{D}_{d-1}(n)$, and hence its singular set consists of an ordinary double point y. This follows from the computations from Example 1.2.1. Corollary 1.2.2 and the description of the tangent space of $\mathcal{D}_{d-1}(n)$ at its nonsingular point proves the assertion. □

1.2.3 The Plücker formulas

Let $C = V(f)$ be an irreducible plane curve of degree d. If C is nonsingular, its first polar $P_a(C)$ with respect to a general point in $\mathbb{P}^2$ intersects C at $d(d-1)$ points b such that $a \in \mathbb{T}_b(C)$. This shows that the pencil of lines through a contains $d(d-1)$ tangent lines to C. A pencil of lines in $\mathbb{P}^2$ is the same as a line in the dual plane. Thus we see that the dual curve $C^\vee$ has $d(d-1)$ intersection points with a general line. In other words

$$\deg(C^\vee) = d(d-1). \tag{1.39}$$

If C is singular, the degree of $C^\vee$ must be smaller. In fact, all polars $P_a(C)$ pass through singular points of C and hence the number of nonsingular points b such that $a \in \mathbb{T}_b(C)$ is smaller than $d(d-1)$. The difference is equal to the sum of intersection numbers of a general polar and the curve at singular points

$$d(d-1) - \deg(C^\vee) = \sum_{x \in \mathrm{Sing}(C)} i(C, P_a(C))_x. \tag{1.40}$$

Let us compute the intersection numbers assuming that C has only ordinary nodes and cusps. Assume x is an ordinary node. Choose a coordinate system such that $x = [1, 0, 0]$ and write the equation in the form $f = t_0^{d-2} f_2(t_1, t_2) + \ldots$. We may assume that $f_2(t_1, t_2) = t_1 t_2$. Computing the partials and dehomogenizing the equations, we find that $P_a(f) = a_1 \phi_x + a_2 \phi_y$, where $\phi = xy + \ldots$ is the affine equation of the curve, and ϕ_x, ϕ_y its partials in x and y. Thus, we need to compute the dimension of the vector space

$$\mathbb{C}[x, y]/(\phi, a_1\phi_x + a_2\phi_y) = \mathbb{C}[x, y]/(xy + \ldots, a_1 x + a_2 y + \ldots),$$

where ... denotes the terms of higher degree. It is easy to see that this number is equal to the intersection number at a node with a general line through the node. The number is equal to 2.

If x is an ordinary cusp, the affine equation of C is $y^2 + x^3 + \dots$ and we have to compute the dimension of the vector space

$$\mathbb{C}[x,y]/(f, a_1 f_x + a_2 f_y) = \mathbb{C}[x,y]/(y^2 + x^3 + \dots, a_1 x^2 + a_2 y + \dots).$$

It is easy to see that this number is equal to the intersection number at a cusp with a parabola whose tangent is equal to the line $y = 0$. The number is equal to 3.

Thus we obtain

Theorem 1.2.5. *Let C be an irreducible plane curve of degree d. Assume that C has only ordinary double points and ordinary cusps as singularities. Then*

$$\deg(C^\vee) = d(d-1) - 2\delta - 3\kappa,$$

where δ is the number of nodes and κ is the number of cusps.

Note that the dual curve $C^\vee$ of a nonsingular curve of degree $d > 2$ is always singular. This follows from the formula for the genus of a nonsingular plane curve and the fact that C and $C^\vee$ are birationally isomorphic. The polar map $C \to C^\vee$ is equal to the normalization map. A singular point of $C^\vee$ corresponds to a line which is either tangent to C at several points, or is a flex tangent. We skip a local computation which shows that a line which is a flex tangent at one point with $\mathrm{ordfl} = 1$ (an *honest flex tangent*) gives an ordinary cusp of $C^\vee$ and a line which is tangent at two points which are not inflection points (*honest bitangent*) gives a node. Thus we obtain that the number $\check{\delta}$ of nodes of $C^\vee$ is equal to the number of honest bitangents of C and the number $\check{\kappa}$ of ordinary cusps of $C^\vee$ is equal to the number of honest flex tangents to $C^\vee$.

Assume that C is nonsingular and $C^\vee$ has no other singular points except ordinary nodes and cusps. We know that the number of inflection points is equal to $3d(d-2)$. Applying Theorem 1.2.5 to $C^\vee$, we get that

$$\check{\delta} = \frac{1}{2}\big(d(d-1)(d(d-1)-1) - d - 9d(d-2)\big) = \frac{1}{2}d(d-2)(d^2-9). \tag{1.41}$$

This is the (expected) number of bitangents of a nonsingular plane curve. For example, we expect that a nonsingular plane quartic has 28 bitangents.

We refer for discussions of Plücker formulas to many modern text-books (e.g. [163], [173], [197], [183]).

1.3 Polar polyhedra

1.3.1 Apolar schemes

Let E be a complex vector space of dimension $n+1$. Recall from section 1.1 that we have a natural pairing

$$S^k E \times S^d E^\vee \to S^{d-k} E^\vee, \quad (\psi, f) \mapsto D_\psi(f), \quad d \geq k,$$

which extends the canonical pairing $E \times E^\vee \to \mathbb{C}$. By choosing a basis in E and the dual basis in $E^\vee$, we view the ring $\mathrm{Sym}^\bullet E^\vee$ as the polynomial algebra $\mathbb{C}[t_0, \dots, t_n]$ and $\mathrm{Sym}^\bullet E$ as the ring of differential operators $\mathbb{C}[\partial_0, \dots, \partial_n]$. The polarity pairing is induced by the natural action of operators on polynomials.

Definition 1.3. *A homogeneous form $\psi \in S^k E$ is called* apolar to a homogeneous form $f \in S^d E^\vee$ if $D_\psi(f) = 0$. We extend this definition to hypersurfaces in the obvious way.

Lemma 1.3.1. *For any $\psi \in S^k E, \psi' \in S^m E$ and $f \in S^d E^\vee$,*

$$D_{\psi'}(D_\psi(f)) = D_{\psi\psi'}(f).$$

Proof. By linearity and induction on the degree, it suffices to verify the assertions in the case when $\psi = \partial_i$ and $\psi' = \partial_j$. In this case they are obvious. □

Corollary 1.3.2. *Let $f \in S^d E^\vee$. Let $\mathrm{AP}_k(f)$ be the subspace in $S^k E$ spanned by apolar forms of degree k to f. Then*

$$\mathrm{AP}(f) = \bigoplus_{k=0}^{\infty} AP_k(f)$$

is a homogeneous ideal in the symmetric algebra $\mathrm{Sym}^\bullet E$.

Definition 1.4. *The quotient ring*

$$A_f = \mathrm{Sym}^\bullet E/\mathrm{AP}(f)$$

is called the apolar ring of f.

The ring A_f inherits the grading of $\mathrm{Sym}^\bullet E$. Since any polynomial $\psi \in S^r E$ with $r > d$ is apolar to f, we see that A_f is killed by the ideal $\mathfrak{m}_+^{d+1} = (\partial_0, \dots, \partial_n)^{d+1}$. Thus A_f is an Artinian graded local algebra over $\mathbb{C}$. Since the pairing between $S^d E$ and $S^d E^\vee$ has values in $S^0 E^\vee = \mathbb{C}$, we see that $\mathrm{AP}_d(f)$ is of codimension 1 in $S^d E$. Thus $(A_f)_d$ is a vector space of dimension 1 over $\mathbb{C}$ and coincides with the *socle* of A_f, i.e. the ideal of elements of A_f annulated by its maximal ideal.

Note that the latter property characterizes Gorenstein graded local Artinian rings, see [156], [231].

Proposition 1.3.3. *(F. S. Macaulay). The correspondence $f \mapsto A_f$ is a bijection between $|S^d E^\vee|$ and graded Artinian quotient algebras* $\mathrm{Sym}^\bullet E/I$ *with one-dimensional socle.*

Proof. Let us show how to reconstruct $\mathbb{C}f$ from $\mathrm{Sym}^\bullet E/I$. Since $(\mathrm{Sym}^\bullet E/I)_d$ is one-dimensional, the multiplication of d vectors in E composed with the projection to $S^d E/I_d$ defines a linear map $S^d E \to S^d E/I_d$. Choosing a basis $(\mathrm{Sym}^\bullet E/I)_d$, we obtain a linear function f on $S^d E$. It corresponds to an element of $S^d E^\vee$.

□

Recall that any closed subscheme $Z \subset \mathbb{P}^n$ is defined by a unique saturated homogeneous ideal I_Z in $\mathbb{C}[t_0, \ldots, t_n]$. Its locus of zeros in the affine space $\mathbb{A}^{n+1}$ is the affine cone C_Z over Z isomorphic to Spec$(\mathbb{C}[t_0, \ldots, t_n]/I_Z)$.

Definition 1.5. *Let $f \in S^dE^\vee$. A subscheme $Z \subset |E^\vee| = \mathbb{P}(E)$ is called* apolar to f if its homogeneous ideal I_Z is contained in AP(f), or, equivalently, Spec(A_f) is a closed subscheme of the affine cone C_Z of Z.

This definition agrees with the definition of an apolar homogeneous form ψ. A homogeneous form $\psi \in S^kE$ is apolar to f if and only if the hypersurface $V(\psi)$ is apolar to $V(f)$.

Consider the natural pairing

$$(A_f)_k \times (A_f)_{d-k} \to (A_f)_d \cong \mathbb{C} \tag{1.42}$$

defined by multiplication of polynomials. It is well defined because of Lemma 1.3.1. The left kernel of this pairing consists of $\psi \in S^kE \mod \text{AP}(f) \cap S^kE$ such that $D_{\psi\psi'}(f) = 0$ for all $\psi' \in S^{d-k}E$. By Lemma 1.3.1, $D_{\psi\psi'}(f) = D_{\psi'}(D_\psi(f)) = 0$ for all $\psi' \in S^{d-k}E$. This implies $D_\psi(f) = 0$. Thus $\psi \in$ AP(f) and hence is zero in A_f. This shows that the pairing (6.13) is a perfect pairing. This is one of the nice features of a Gorenstein artinian algebra (see [156], 21.2).

It follows that the Hilbert polynomial

$$H_{A_f}(t) = \sum_{i=0}^{d} \dim(A_f)_i t^i = a_d t^d + \cdots + a_0$$

is a reciprocal monic polynomial, i.e. $a_i = a_{d-i}, a_d = 1$. It is an important invariant of a homogeneous form f.

Example 1.3.1. Let $f = l^d$ be the d-th power of a linear form $l \in E^\vee$. For any $\psi \in S^kE = (S^kE^\vee)^*$ we have

$$D_\psi(l^d) = d(d-1)\ldots(d-k+1)l^{d-k}\psi(l) = d!l^{[d-k]}\psi(l),$$

where we set $l^{[i]} = \frac{1}{i!}l^i$. Here we view $\psi \in S^dE$ as a homogeneous function on $E^\vee$. In coordinates, $l = \sum_{i=0}^n a_i t_i, \psi = \psi(\partial_0, \ldots, \partial_n)$ and $\psi(l) = \psi(a_0, \ldots, a_n)$. Thus we see that $AP_k(f), k \le d$, consists of polynomials of degree k vanishing at l. Assume for simplicity that $l = t_0$. The ideal $AP(f)$ is generated by $\partial_1, \ldots, \partial_n, \partial_0^{d+1}$. The Hilbert polynomial is equal to $1 + t + \cdots + t^d$.

1.3.2 Sums of powers

For any point $a \in |E^\vee|$ we continue to denote by H_a the corresponding hyperplane in $|E|$.

Suppose $f \in S^dE^\vee$ is equal to a sum of powers of nonzero linear forms

$$f = l_1^d + \cdots + l_s^d. \tag{1.43}$$

This implies that for any $\psi \in S^k E$,

$$D_\psi(f) = D_\psi(\sum_{i=1}^{s} l_i^d) = \sum_{i=1}^{s} \psi(l_i) d(d-1)\cdots(d-k+1) l_i^{d-k}. \tag{1.44}$$

In particular, taking $d = k$, we obtain that

$$\langle l_1^d, \ldots, l_s^d \rangle_{S^d E}^{\perp} = \{\psi \in S^d E : \psi(l_i) = 0, i = 1, \ldots, s\} = (I_Z)_d,$$

where Z is the closed subscheme of points $\{[l_1], \ldots, [l_s]\} \subset |E^\vee|$ corresponding to the linear forms l_i.

This implies that the codimension of the linear span $\langle l_1^d, \ldots, l_s^d \rangle$ in $S^d E^\vee$ is equal to the dimension of $(I_Z)_d$, hence the forms $l_1^d, \ldots, l_s^d$ are linearly independent if and only if the points $[l_1], \ldots, [l_s]$ impose independent conditions on hypersurfaces of degree d in $\mathbb{P}(E) = |E^\vee|$.

Suppose $f \in \langle l_1^d, \ldots, l_s^d \rangle$, then $(I_Z)_d \subset \mathrm{AP}_d(f)$. Conversely, if this is true, we have

$$f \in \mathrm{AP}_d(f)^\perp \subset (I_Z)_d^\perp = \langle l_1^d, \ldots, l_s^d \rangle.$$

If we additionally assume that $(I_{Z'})_d \not\subset \mathrm{AP}_d(f)$ for any proper subset Z' of Z, we obtain, after replacing the forms $l_i's$ by proportional ones, that

$$f = l_1^d + \cdots + l_s^d.$$

Definition 1.6. *A* polar s-polyhedron of f is a set of hyperplanes $H_i = V(l_i), i = 1, \ldots, s$, in $|E|$ such that

$$f = l_1^d + \cdots + l_s^d,$$

and, considered as points $[l_i]$ in $\mathbb{P}(E)$, the hyperplanes H_i impose independent conditions in the linear system $|\mathcal{O}_{\mathbb{P}(E)}(d)|$.

Note that this definition does not depend on the choice of linear forms defining the hyperplanes. Nor does it depend on the choice of the equation defining the hypersurface $V(f)$.

The following propositions follow from the above discussion.

Proposition 1.3.4. *Let $f \in S^d E^\vee$. Then $Z = \{[l_1], \ldots, [l_s]\}$ is a polar s-polyhedron of f if and only if the following properties are satisfied*

(i) $I_Z(d) \subset \mathrm{AP}_d(f)$*;*

(ii) $I_{Z'}(d) \not\subset \mathrm{AP}_d(f)$ *for any proper subset Z' of Z.*

Proposition 1.3.5. *A set $Z = \{[l_1], \ldots, [l_s]\}$ is a polar s-polyhedron of $f \in S^d E^\vee$ if and only if Z, considered as a closed subscheme of $|E^\vee|$, is apolar to f but no proper subscheme of Z is apolar to f.*

1.3.3 Generalized polar polyhedra

Proposition 1.3.5 allows one to generalize the definition of a polar polyhedron. A polar polyhedron can be viewed as a reduced closed subscheme Z of $\mathbb{P}(E) = |E|^\vee$ consisting of s points. Obviously, $h^0(\mathcal{O}_Z) = \dim H^0(|E|^\vee, \mathcal{O}_Z) = s$. More generally, we may consider non-reduced closed subschemes Z of $|E|^\vee$ of dimension 0 satisfying $h^0(\mathcal{O}_Z) = s$. The set of such subschemes is parameterized by a projective algebraic variety $\mathrm{Hilb}^s(|E|^\vee)$ called the *punctual Hilbert scheme* of $|E|^\vee$ of 0-cycles of length s.

Any $Z \in \mathrm{Hilb}^s(\mathbb{P}(E))$ defines the subspace

$$I_Z(d) = \mathbb{P}(H^0(\mathbb{P}(E), \mathcal{I}_Z(d)) \subset H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(d)) = S^dE.$$

The exact sequence

$$0 \to H^0(\mathbb{P}(E), \mathcal{I}_Z(d)) \to H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(d)) \to H^0(\mathbb{P}(E), \mathcal{O}_Z) \qquad (1.45)$$
$$\to H^1(\mathbb{P}(E), \mathcal{I}_Z(d)) \to 0$$

shows that the dimension of the subspace

$$\langle Z\rangle_d = \mathbb{P}(H^0(\mathbb{P}(E), \mathcal{I}_Z(d))^\perp) \subset \mathbb{P}(S^dE^\vee) \qquad (1.46)$$

is equal to $h^0(\mathcal{O}_Z) - h^1(\mathcal{I}_Z(d)) - 1 = s - 1 - h^1(\mathcal{I}_Z(d))$. If $Z = Z_{red} = \{p_1, \ldots, p_s\}$, then $\langle Z\rangle_d = \langle v_d(p_1), \ldots, v_d(p_s)\rangle$, where $v_d : \mathbb{P}(E) \to \mathbb{P}(S^dE)$ is the Veronese map. Hence $\dim\langle Z\rangle = s - 1$ if the points $v_d(p_1), \ldots, v_d(p_s)$ are linearly independent. We say that Z is *linearly d-independent* if $\dim\langle Z\rangle_d = s - 1$.

Definition 1.7. *A* generalized s-polyhedron *of f is a linearly d-independent subscheme $Z \in \mathrm{Hilb}^s(\mathbb{P}(E))$ which is apolar to f.*

Recall that Z is apolar to f if, for each $k \geq 0$,

$$I_Z(k) = H^0(\mathbb{P}(E), \mathcal{I}_Z(k)) \subset \mathrm{AP}_k(f). \qquad (1.47)$$

In view of this definition a polar polyhedron is a reduced generalized polyhedron. The following is a generalization of Proposition 1.3.4.

Proposition 1.3.6. *A linearly independent subscheme $Z \in \mathrm{Hilb}^s(\mathbb{P}(E))$ is a generalized polar s-polyhedron of $f \in S^dE^\vee$ if and only if*

$$I_Z(d) \subset \mathrm{AP}_d(f).$$

Proof. We have to show that the inclusion in the assertion implies $I_Z(d) \subset \mathrm{AP}_k(f)$ for any $k \leq d$. For any $\psi' \in S^{d-k}E$ and any $\psi \in I_Z(k)$, the product $\psi\psi'$ belongs to $I_Z(k)$. Thus $D_{\psi\psi'}(f) = 0$. By the duality, $D_\psi(f) = 0$, i.e. $\psi \in \mathrm{AP}_k(f)$. $\square$

Example 1.3.2. Let $Z = m_1p_1 + \cdots + m_kp_k \in \mathrm{Hilb}^s(\mathbb{P}(E))$ be the union of *fat points* p_k, i.e. at each $p_i \in Z$ the ideal $\mathcal{I}_{Z,p_i}$ is equal to the m_i-th power of the maximal ideal. Obviously,

$$s = \sum_{i=1}^{k} \binom{n+m_i-1}{m_i-1}.$$

Then the linear system $|\mathcal{O}_{\mathbb{P}(E)}(d) - Z)|$ consists of hypersurfaces of degree d which have singularity at p_i of multiplicity $\geq m_i$ for each $i = 1, \ldots, k$. One can show (see [231], Theorem 5.3) that Z is apolar to f if and only if

$$f = l_1^{d-m_i+1} g_1 + \ldots + l_k^{d-m_k+1} g_k,$$

where $p_i = V(l_i)$ and g_i is a homogeneous polynomial of degree $m_i - 1$ or the zero polynomial.

Remark 1.3.1. It is not known whether the set of generalized s-polyhedra of f is a closed subset of $\mathrm{Hilb}^s(\mathbb{P}(E))$. It is known to be true for $s \leq d+1$ since in this case $\dim I_Z(d) = t := \dim S^d E - s$ for all $Z \in \mathrm{Hilb}^s(\mathbb{P}(E))$ (see [231], p.48). This defines a regular map of $\mathrm{Hilb}^s(\mathbb{P}(E))$ to the Grassmannian $G(t, S^d E)$ and the set of generalized s-polyhedra is equal to the preimage of a closed subset consisting of subspaces contained in $\mathrm{AP}_d(f)$. Also we see that $h^1(\mathcal{I}_Z(d)) = 0$, hence Z is always linearly d-independent.

1.3.4 Secant varieties

The notion of a polar polyhedron has a simple geometric interpretation. Let

$$v_d : |E^\vee| \to |S^d E^\vee|, \ l \mapsto l^d,$$

be the Veronese map of the dual projective space. Denote by Ver_d^n its image. Then $f \in S^d E^\vee \setminus \{0\}$ represents a point $[f]$ in $\mathbb{P}(S^d E^\vee)$. A set of hyperplanes $H_i = V(l_i), i = 1, \ldots, s$, represents a set of points $[l_i^d]$ in the Veronese variety Ver_d^n. It is a polar s-polyhedron of f if and only if $[f]$ belongs to the linear span $\langle [l_1^d], \ldots, [l_s^d] \rangle$ of dimension $s-1$, a $(s-1)$-secant of the Veronese variety, and does not belong to its proper subspace.

Recall that for any irreducible nondegenerate projective variety $X \subset \mathbb{P}^r$ of dimension n its t-*secant variety* $\mathrm{Sec}_t(X)$ is defined to be the Zariski closure of the set of points in $\mathbb{P}^r$ which lie in the linear span of dimension t of some set of $t+1$ linearly independent points in X.

Counting constants easily gives

$$\dim \mathrm{Sec}_t(X) \leq \min((n+1)(t+1) - 1, r).$$

The subvariety $X \subset \mathbb{P}^r$ is called t-*defective* if the inequality is strict. An example of a 1-defective variety is a Veronese surface in $\mathbb{P}^5$.

A fundamental result about secant varieties is the following Lemma whose modern proof can be found, for example in [429], Proposition 1.10.

Lemma 1.3.7. *(A. Terracini). Let $p_1, \ldots, p_{t+1}$ be general $t+1$ points in X and p be a general point in their span. Then*

$$\mathbb{T}_p(\mathrm{Sec}_t(X)) = \overline{\mathbb{T}_{p_1}(X), \ldots, \mathbb{T}_{p_{t+1}}(X)}.$$

The inclusion part

$$\overline{\mathbb{T}_{p_1}(X),\ldots,\mathbb{T}_{p_{t+1}}(X)} \subset \mathbb{T}_p(\mathrm{Sec}_t(X))$$

is easy to prove. We assume for simplicity that $t = 1$. Then $\mathrm{Sec}_1(X)$ contains the cone $C(p_1, X)$ which is swept out by the lines $\overline{p_1, q}, q \in X$. Therefore, $\mathbb{T}_p(C(p_1, X)) \subset \mathbb{T}_p(\mathrm{Sec}_1(X))$. However, it is easy to see that $\mathbb{T}_p(C(p_1, X))$ contains $\mathbb{T}_{p_1}(X)$.

Corollary 1.3.8. $\mathrm{Sec}_t(X) \neq \mathbb{P}^r$ *if and only if for any* $t + 1$ *general points of* X *there exists a hyperplane section of* X *singular at these points. In particular, if* $r \leq (n+1)(t+1)-1$, *the variety* X *is* t*-defective if and only if for any* $t+1$ *general points of* X *there exists a hyperplane section of* X *singular at these points.*

Example 1.3.3. Let $X = \mathrm{Ver}_d^n \subset \mathbb{P}^{\binom{d+n}{n}-1}$ be the image of $\mathbb{P}^n$ under a Veronese map defined by homogeneous polynomials of degree d. Assume $(n+1)(t+1) \geq \binom{d+n}{n}-1$. A hyperplane section of X is isomorphic to a hypersurface of degree d in $\mathbb{P}^n$. Thus $\mathrm{Sec}_t(\mathrm{Ver}_d^n) \neq |S^d E^\vee|$ if and only if for any $t + 1$ general points in $\mathbb{P}^n$ there exists a hypersurface of degree d singular at these points.

Take $n = 1$. Then $r = d$ and $r \leq (n + 1)(t + 1) - 1 = 2t + 1$ for $t \geq (d - 1)/2$. Since $t + 1 > d/2$ there are no homogeneous forms of degree d which have $t + 1$ multiple roots. Thus the Veronese curve $R_d = v_d(\mathbb{P}^1) \subset \mathbb{P}^d$ is not t-degenerate for $t \geq (d - 1)/2$.

Take $n = 2$ and $d = 2$. For any two points in $\mathbb{P}^2$ there exists a conic singular at these points, namely the double line through the points. This explains why a Veronese surface V_2^2 is 1-defective.

Another example is $\mathrm{Ver}_4^2 \subset \mathbb{P}^{14}$ and $t = 4$. The expected dimension of $\mathrm{Sec}_4(X)$ is equal to 14. For any 5 points in $\mathbb{P}^2$ there exists a conic passing through these points. Taking it with multiplicity 2 we obtain a quartic which is singular at these points. This shows that Ver_4^2 is 4-defective.

The following Corollary of Terracini's Lemma is called the *First Main Theorem on apolarity* in [155]. The authors gave an algebraic proof of this Theorem without using (or probably without knowing) Terracini's Lemma.

Corollary 1.3.9. *A general homogeneous form* $f \in S^d E^\vee$ *admits a polar* s*-polyhedron if and only if there exist linear forms* $l_1, \ldots, l_s \in E^\vee$ *such that for any nonzero* $\psi \in S^d E$ *the ideal* $AP(\psi) \subset \mathrm{Sym}^\bullet E^\vee$ *does not contain* $\{l_1^{d-1}, \ldots, l_s^{d-1}\}$.

Proof. A general form $f \in S^d E^\vee$ admits a polar s-polyhedron if and only if the secant variety $\mathrm{Sec}_{s-1}(\mathrm{Ver}_d^n)$ is equal to the whole space. This means that the span of the tangent spaces at some points $q_i = V(l_i^d), i = 1, \ldots, s$, is equal to the whole space. By Terracini's Lemma, this is equivalent to that the tangent spaces of the Veronese variety at the points q_i are not contained in a hyperplane defined by some $\psi \in S^d E = (S^d E^\vee)^*$. It remains to use that the tangent space of the Veronese variety at q_i is equal to the projective space of all homogeneous forms of the form $l_i^{d-1} l, l \in E^\vee \setminus \{0\}$ (see Exercises). Thus, for any nonzero $\psi \in S^d E$, it is impossible that $P_{l_i^{d-1} l}(\psi) = 0$ for all l and for all i. But $P_{l_i^{d-1} l}(\psi) = 0$ for all l if and only if $P_{l_i^{d-1}}(\psi) = 0$. This proves the assertion. $\square$

The following fundamental result is due to J. Alexander and A. Hirschowitz [4].

Theorem 1.3.10. Ver_d^n *is* t*-defective if and only if*

$$(n,d,t) = (2,2,1),(2,4,4),(3,4,8),(4,3,6),(4,4,13).$$

In all these cases the secant variety $\mathrm{Sec}_t(\mathrm{Ver}_d^n)$ *is a hypersurface.*

For the sufficiency of the condition, only the case $(4,3,6)$ is not trivial. It asserts that for 7 general points in $\mathbb{P}^3$ there exists a cubic hypersurface which is singular at these points. Other cases are easy. We have seen already the first two cases. The third case follows from the existence of a quadric through 9 general points in $\mathbb{P}^3$. The square of its equation defines a quartic with 9 points. The last case is similar. For any 14 general points there exists a quadric in $\mathbb{P}^4$ containing these points.

Corollary 1.3.11. *Assume* $s(n+1) \geq \binom{d+n}{n}$*. Then a general homogeneous polynomial* $f \in \mathbb{C}[t_0,\ldots,t_n]_d$ *can be written as a sum of* d*-th powers of* s *linear forms unless* $(n,d,s) = (2,2,2),(2,4,5),(3,4,9),(4,3,7),(4,4,14)$.

1.3.5 The Waring problems

The well-known Waring problem in number theory asks about the smallest number $s(d)$ such that each natural number can be written as a sum of $s(d)$ d-th powers of natural numbers. It also asks in how many ways it can be done. Its polynomial analog asks about the smallest number $s(d,n)$ such that a general homogeneous polynomial of degree d in $n+1$ variables can be written as a sum of s d-th powers of linear forms.

The Alexander-Hirschowitz Theorem completely solves this problem. We have $s(d,n)$ is equal to the smallest natural number s_0 such that $s_0(n+1) \geq \binom{n+d}{n}$ unless $(n,d) = (2,2),(2,4),(3,4),(4,3),(4,4)$, where $s(d,n) = s_0+1$.

Other versions of the Waring problem ask the following questions:

- (W1) Given a homogeneous form $f \in S^dE^\vee$, study the subvariety $\mathrm{VSP}(f;s)^o$ of $\mathbb{P}(E)^{(s)}$ (the *variety of power sums*) which consists of polar s-polyhedra of f or more general the subvariety $\mathrm{VSP}(f;s)$ of $\mathrm{Hilb}^s(\mathbb{P}(E))$ parameterizing generalized s-polyhedra.

- (W2) For given s find the equations of the closure $\mathrm{PS}(s,d;n)$ in $S^dE^\vee$ of the locus of homogeneous forms of degree d which can be written as a sum of s powers of linear forms.

Note that $\mathrm{PS}(s,d;n)$ is the affine cone over the secant variety $\mathrm{Sec}_{s-1}(\mathrm{Ver}_d^n)$.

In the language of secant varieties, the variety $\mathrm{VSP}(f;s)^o$ is the set of linearly independent sets of s points $p_1,\ldots,p_s$ in Ver_d^n such that $[f] \in \langle p_1,\ldots,p_s\rangle$ and does not belong to the span of the proper subset of the set of these points. The variety $\mathrm{VSP}(f;s)$ is the set of linearly independent $Z \in \mathrm{Hilb}^s(\mathbb{P}(E))$ such that $[f] \in \langle Z\rangle$. Note that we have a natural map

$$\mathrm{VSP}(f;s) \to G(s,S^dE), \quad Z \mapsto \langle Z\rangle_d,$$

where $G(s, S^dE)$ is the Grassmannian of s-dimensional subspaces of S^dE. This map is not injective in general.

Also note that for a general form f the variety $\mathrm{VSP}(f;s)$ is equal to the closure of $\mathrm{VSP}(f;s)^o$ in the Hilbert scheme $\mathrm{Hilb}^s(\mathbb{P}(E))$ (see [231], 7.2). It is not true for an arbitrary form f. One can also embed $\mathrm{VSP}(f;s)^o$ in $\mathbb{P}(S^dE)$ by assigning to $\{l_1, \ldots, l_s\}$ the product $l_1 \cdots l_s$. Thus we can compactify $\mathrm{VSP}(f;s)^o$ by taking its closure in $\mathbb{P}(S^dE)$. In general, this closure is not isomorphic to $\mathrm{VSP}(f;s)$.

Proposition 1.3.12. *Assume $n = 2$. For general $f \in S^dE^\vee$ the variety $\mathrm{VSP}(f;s)$ is either empty or a smooth irreducible variety of dimension $3s - \binom{2+d}{d}$.*

Proof. We consider $\mathrm{VSP}(f;s)$ as the closure of $\mathrm{VSP}(f;s)^o$ in the Hilbert scheme $\mathrm{Hilb}^s(\mathbb{P}(E))$. Recall that $Z \in \mathrm{Hilb}^s(\mathbb{P}(E))$ is a generalized polar polyhedron of f if and only if $f \in I_Z(d)^\perp$ but this is not true for any proper closed subscheme Z' of Z. Consider the incidence variety

$$X = \{(Z, f) \in \mathrm{Hilb}^s(\mathbb{P}(E)) \times S^dE^\vee : Z \in \mathrm{VSP}(f;s)\}.$$

It is known that for any nonsingular surface the punctual Hilbert scheme is nonsingular (see [166]). Let U be the open subset of the first factor such that for any point $Z \in U$, $\dim I_Z(d) = \dim S^dE - s$. The fibre of the first projection over $Z \in U$ is an open Zariski subset of the linear space $I_Z(d)^\perp$. This shows that X is irreducible and nonsingular. The fibres of the second projection are the varieties $\mathrm{VSP}(f;s)$. Thus for an open Zariski subset of $S^dE^\vee$ the varieties $\mathrm{VSP}(f;s)$ are empty or irreducible and nonsingular.

$\square$

1.4 Dual homogeneous forms

1.4.1 Catalecticant matrices

Let $f \in S^dE^\vee$. Consider the linear map (the *apolarity map*)

$$\mathrm{ap}_f^k : S^kE \to S^{d-k}E^\vee, \quad \psi \mapsto D_\psi(f). \tag{1.48}$$

Its kernel is the space $\mathrm{AP}_k(f)$ of forms of degree k which are apolar to f.

By the polarity duality, the dual space of $S^{d-k}E^\vee$ can be identified with $S^{d-k}E$. Applying Lemma 1.3.1, we obtain

$${}^t(\mathrm{ap}_f^k) = \mathrm{ap}_f^{d-k}. \tag{1.49}$$

Assume that $f = \sum_{i=1}^s l_i^d$ for some $l_i \in E^\vee$. It follows from (1.44) that

$$\mathrm{ap}_f^k(S^kE) \subset \langle l_1^{d-k}, \ldots, l_s^{d-k} \rangle,$$

and hence

$$\mathrm{rank}(\mathrm{ap}_f^k) \le s. \tag{1.50}$$

If we choose a basis in E and a basis in $E^\vee$, then ap_f^k is given by a matrix of size $\binom{k+n}{k} \times \binom{n+d-k}{d-k}$ whose entries are linear forms in coefficients of f.

Choose a basis $\xi_0, \ldots, \xi_n$ in E and the dual basis $t_0, \ldots, t_n$ in $E^\vee$. Consider a monomial basis in $S^k E$ (resp. in $S^{d-k}E^\vee$) which is lexicographically ordered. The matrix of ap_f^k with respect to these bases is called the k-th *catalecticant matrix* of f and is denoted by $\mathrm{Cat}_k(f)$. Its entries $c_{\mathbf{uv}}$ are parameterized by pairs $(\mathbf{u}, \mathbf{v}) \in \mathbb{N}^{n+1} \times \mathbb{N}^{n+1}$ with $|\mathbf{u}| = d - k$ and $|\mathbf{v}| = k$. If we write

$$f = d! \sum_{|\mathbf{i}|=d} \frac{1}{\mathbf{i}!} a_{\mathbf{i}} \mathbf{t}^{\mathbf{i}},$$

then

$$c_{\mathbf{uv}} = a_{\mathbf{u}+\mathbf{v}}.$$

This follows easily from the formula

$$\partial_0^{i_0} \cdots \partial_n^{i_n}(t_0^{j_0} \cdots t_n^{j_n}) = \begin{cases} \frac{\mathbf{j}!}{(\mathbf{j}-\mathbf{i})!} \mathbf{t}^{\mathbf{j}-\mathbf{i}} & \text{if } \mathbf{j} - \mathbf{i} \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Considering $a_{\mathbf{i}}$ as independent variables $t_{\mathbf{i}}$, we obtain the definition of a general catalecticant matrix $\mathrm{Cat}_k(d, n)$.

Example 1.4.1. Let $n = 1$. Write $f = \sum_{i=0}^d \binom{d}{i} a_i t_0^{d-i} t_1^i$. Then

$$\mathrm{Cat}_k(f) = \begin{pmatrix} a_0 & a_1 & \ldots & a_k \\ a_1 & a_2 & \ldots & a_{k+1} \\ \vdots & \vdots & \ldots & \vdots \\ a_{d-k} & a_{d-k+1} & \ldots & a_d \end{pmatrix}.$$

A matrix of this type is called a *Hankel matrix* or *persymmetric matrix*. It follows from (1.50) that $f \in \mathrm{PS}(s, d; 1)$ implies that all $(s+1) \times (s+1)$ minors of $\mathrm{Cat}_k(f)$ are equal to zero. Thus we obtain that $\mathrm{Sec}_{s-1}(\mathrm{Ver}_d^1)$ is contained in the subvariety of $\mathbb{P}^d$ defined by $(s+1) \times (s+1)$-minors of the matrices

$$\mathrm{Cat}_k(d, 1) = \begin{pmatrix} t_0 & t_1 & \ldots & t_k \\ t_1 & t_2 & \ldots & t_{k+1} \\ \vdots & \vdots & \ldots & \vdots \\ t_{d-k} & t_{d-k+1} & \ldots & t_d \end{pmatrix}, \quad k = 1, \ldots, \min\{d-s, s\}.$$

For example, if $s = 1$, we obtain that the Veronese curve $\mathrm{Ver}_d^1 \subset \mathbb{P}^d$ satisfies the equations $t_i t_j - t_k t_l = 0$, where $i + j = k + l$. It is well known that these equations generate the homogeneous ideal of the Veronese curve.

Assume $d = 2k$. Then the Hankel matrix is a square matrix of size $k + 1$. Its determinant vanishes if and only if f admits a nonzero apolar form of degree k. The set of such f's is a hypersurface in $\mathbb{C}[t_0, t_1]_{2k}$. It contains the Zariski open subset of forms which can be written as a sum of k powers of linear forms (see section 1.5.1).

For example, take $k = 2$. Then the equation

$$\det \begin{pmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{pmatrix} = 0 \tag{1.51}$$

describes binary quartics

$$f = a_0 t_0^4 + 4a_1 t_0^3 t_1 + 6a_2 t_0^2 t_1^2 + 4a_3 t_0 t_1^3 + a_4 t_1^4$$

which lie in the Zariski closure of the locus of quartics represented in the form $(\alpha_0 t_0 + \beta_0 t_1)^4 + (\alpha_1 t_0 + \beta_1 t_1)^4$. Note that a quartic of this form has simple roots unless it has a root of multiplicity 4. Thus any binary quartic with simple roots satisfying equation (1.51) can be represented as a sum of two powers of linear forms.

The cubic hypersurface in $\mathbb{P}^4$ defined by equation (1.51) is equal to the 1-secant variety of a Veronese curve in $\mathbb{P}^4$.

Note that

$$\dim \mathrm{AP}_i(f) = \dim \mathrm{Ker}(\mathrm{ap}_f^i) = \binom{n+k}{i} - \text{rank Cat}_i(f).$$

Therefore,

$$\dim(A_f)_i = \text{rank Cat}_i(f),$$

and

$$H_{A_f}(t) = \sum_{i=0}^{d} \text{rank Cat}_i(f) t^i. \tag{1.52}$$

It follows from (1.49) that

$$\text{rank Cat}_i(f) = \text{rank Cat}_{d-i}(f)$$

confirming that $H_{A_f}(t)$ is a reciprocal monic polynomial.

Suppose $d = 2k$ is even. Then the coefficient at t^k in $H_{A_f}(t)$ is equal to the rank of $\text{Cat}_k(f)$. The matrix $\text{Cat}_k(f)$ is a square matrix of size $\binom{n+k}{k}$. One can show that for a general f, this matrix is invertible. A polynomial f is called *degenerate* if $\det(\text{Cat}_k(f)) = 0$. Thus, the set of degenerate polynomials is a hypersurface (*catalecticant hypersurface*) given by the equation

$$\det(\text{Cat}_k(2k, n)) = 0. \tag{1.53}$$

The polynomial $\det(\text{Cat}_k(2k, n))$ in variables $t_{\mathbf{i}}, |\mathbf{i}| = d$, is called the *catalecticant determinant*.

Example 1.4.2. Let $d = 2$. It is easy to see that the catalecticant polynomial is the discriminant polynomial. Thus a quadratic form is degenerate if and only if it is degenerate in the usual sense. The Hilbert polynomial of a quadratic form f is

$$H_{A_f}(t) = 1 + rt + t^2,$$

where r is the rank of the quadratic form.

Example 1.4.3. Suppose $f = t_0^d + \cdots + t_s^d, s \le n$. Then $t_0^i, \ldots, t_s^i$ are linearly independent for any i, and hence rank $\mathrm{Cat}_i(f) = s$ for $0 < i < d$. This shows that

$$H_{A_f}(t) = 1 + s(t + \cdots + t^{d-1}) + t^d.$$

Let $\mathcal{P}$ be the set of reciprocal monic polynomials of degree d. One can stratify the space $S^d E^\vee$ by setting, for any $p \in \mathcal{P}$,

$$S^d E_p^\vee = \{f \in S^d E^\vee : H_{A_f} = p\}.$$

If $f \in \mathrm{PS}(s, d; n)$ we know that

$$\text{rank } \mathrm{Cat}_k(f) \le h(s, d, n)_k = \min(s, \tbinom{n+k}{n}, \tbinom{n+d-k}{n}).$$

One can show that for a general enough f, we have the equality (see [231], Lemma 1.7). Thus there is a Zariski open subset of $\mathrm{PS}(s, d; n)$ which belongs to the strata $S^d E_p^\vee$, where $p = \sum_{i=0}^d h(s, d, n)_i t^i$.

1.4.2 Dual homogeneous forms

In Chapter 1 we introduced the notion of a dual quadric. If $Q = V(q)$, where q is a nondegenerate quadratic form, then the dual variety $\check{Q}$ is a quadric defined by the quadratic form $\check{q}$ whose matrix is the adjugate matrix of q. For any homogeneous form of even degree $f \in S^{2k} E^\vee$ one can define the dual homogeneous form $\check{f} \in S^{2k} E$ in a similar fashion using the notion of the catalecticant matrix.

Let

$$\mathrm{ap}_f^k : S^k E \to S^k E^\vee \tag{1.54}$$

be the apolarity map (1.48). We can view this map as a symmetric bilinear form

$$\Omega_f : S^k E \times S^k E \to \mathbb{C}, \quad \Omega_f(\psi_1, \psi_2) = \mathrm{ap}_f^k(\psi_1)(\psi_2) = \langle \psi_2, \mathrm{ap}_f^k(\psi_1) \rangle. \tag{1.55}$$

Its matrix with respect to a monomial basis in $S^k E$ and its dual monomial basis in $S^k E^\vee$ is the catalecticant matrix $\mathrm{Cat}_k(f)$.

Let us identify Ω_f with the associated quadratic form on $S^k E$ (the restriction of Ω_f to the diagonal). This defines a linear map

$$\Omega : S^{2k} E^\vee \to S^2 S^k E^\vee, \quad f \mapsto \Omega_f.$$

There is also a natural left inverse map of Ω

$$P : S^2 S^k E^\vee \to S^{2k} E^\vee$$

defined by multiplication $S^k E^\vee \times S^k E^\vee \to S^{2k} E^\vee$. All these maps are $\mathrm{GL}(E)$-equivariant and realize the linear representation $S^{2k} E^\vee$ as a direct summand in the representation $S^2 S^k E^\vee$.

Theorem 1.4.1. *Assume that $f \in S^{2k} E^\vee$ is nondegenerate. There exists a unique homogeneous form $\check{f} \in S^{2k} E$ (the* dual homogeneous form*) such that*

$$\Omega_{\check{f}} = \check{\Omega}_f.$$

Proof. We know that $\check{\Omega}_f$ is defined by the adjugate matrix $\mathrm{adj}(\mathrm{Cat}_k(f)) = (c^*_{\mathbf{uv}})$ so that

$$\check{\Omega}_f = \sum c^*_{\mathbf{uv}} \xi^{\mathbf{u}} \xi^{\mathbf{v}}.$$

Let

$$\check{f} = \sum_{|\mathbf{u}+\mathbf{v}|=2k} \tfrac{d!}{(\mathbf{u}+\mathbf{v})!} c^*_{\mathbf{uv}} \xi^{\mathbf{u}+\mathbf{v}}.$$

Recall that the entries $c_{\mathbf{uv}}$ of the catalecticant matrix depend only on the sum of the indices. Thus the entries of the adjugate matrix $\mathrm{adj}(\mathrm{Cat}_k(f)) = (c^*_{\mathbf{uv}})$ depend only on the sum of the indices. For any $t^{\mathbf{i}} \in S^k E^\vee$, we have

$$P_{t^{\mathbf{i}}}(\check{f}) = \sum_{\mathbf{u},\mathbf{v},\mathbf{u}+\mathbf{v} \geq \mathbf{i}} \tfrac{d!}{(\mathbf{u}+\mathbf{v})!} c^*_{\mathbf{uv}} \tfrac{(\mathbf{u}+\mathbf{v})!}{(\mathbf{u}+\mathbf{v}-\mathbf{i})!} \xi^{\mathbf{u}+\mathbf{v}-\mathbf{i}} = \sum_{|\mathbf{j}|=k} \tfrac{d!}{\mathbf{j}!} c^*_{\mathbf{ij}} \xi^{\mathbf{j}}$$

This checks that the matrix of the linear map $S^k E^\vee \to S^k E$ defined by $\Omega_{\check{f}}$ is equal to the matrix $\mathrm{adj}(\mathrm{Cat}_k(f))$. Thus the quadratic form $\Omega_{\check{f}}$ is equal to the dual of the quadratic form Ω_f. ☐

Recall that the locus of zeros of a quadratic from $q \in S^2 E^\vee$ consists of vectors $v \in E$ such that the value of the polarized bilinear form $b_q : E \to E^\vee$ at v vanishes at v. Dually, the set of zeros of $\check{q} \in S^2 E$ consists of linear function $l \in E^\vee$ such that the value of $b_{\check{q}} : E^\vee \to E$ at l is equal to zero. The same is true for the dual form $\check{f}$. Its locus of zeros consists of linear forms l such that $\Omega_f^{-1}(l^k) \in S^k E$ vanishes on l. The degree k homogeneous form $\Omega_f^{-1}(l^k)$ is classically known as the *anti-polar* of l (with respect to f).

Definition 1.8. *Two linear forms $l, m \in E^\vee$ are called* conjugate *with respect to a nondegenerate form $f \in S^{2k} E^\vee$ if*

$$\Omega_{\check{f}}(l^k, m^k) = \check{f}(l^k m^k) = 0.$$

Proposition 1.4.2. *Suppose f is given by (1.43), where the powers l_i^k are linearly independent in $S^k E^\vee$. Then each pair l_i, l_j is conjugate with respect to f.*

Proof. It follows from computation of Ω_f in the proof of Proposition 1.4.3 that it suffices to check the assertion for quadratic forms. Choose a coordinate system such that $l_i = t_0, l_j = t_1$ and $f = t_0^2 + t_2^2 + \cdots + t_s^2$. Then $\check{f} = \xi_0^2 + \cdots + \xi_s^2$, where $\xi_0, \ldots, \xi_s$ are dual coordinates. Now the assertion is easily checked. ☐

1.4.3 The Waring rank of a homogeneous form

Since any quadratic form q can be reduced to a sum of squares, one can characterize its rank as the smallest number r such that

$$q = l_1^2 + \cdots + l_r^2$$

for some linear forms $l_1, \ldots, l_r$.

Definition 1.9. *Let $f \in S^d E^\vee$. Its* Waring rank $\mathrm{wrk}(f)$ *is the smallest number r such that*

$$f = l_1^d + \cdots + l_r^d \tag{1.56}$$

for some linear forms $l_1, \ldots, l_r$.

Proposition 1.4.3. *Let Ω_f be the quadratic form on $S^k E$ associated to $f \in S^{2k}E^\vee$. Then the Waring rank of f is greater or equal than the rank of Ω_f.*

Proof. Suppose (1.43) holds with $d = 2k$. Since Ω_f is linear with respect to f, we have $\Omega_f = \sum \Omega_{l_i^{2k}}$. If we choose coordinates such that l_i is a coordinate function t_0, we easily compute the catalecticant matrix of l_i^{2k}. It is equal to the matrix with 1 at the upper left corner and zero elsewhere. The corresponding quadratic form is equal to $(t_0^k)^2$. Thus $\Omega_{l_i^{2k}} = (l_i^k)^2$ and we obtain

$$\Omega_f = \sum_{i=1}^r \Omega_{l_i^{2k}} = \sum_{i=1}^r (l_i^k)^2.$$

Thus the rank of f is greater or equal than the rank of Ω_f. □

Corollary 1.4.4. *Suppose f is a nondegenerate form of even degree $2k$, then*

$$\mathrm{wrk}(f) \geq \binom{k+n}{n}.$$

A naive way to compute the Waring rank is by counting constants. Consider the map

$$s : (E^\vee)^r \to \mathbb{C}^{\binom{d+n}{n}}, \quad (l_1, \ldots, l_r) \mapsto \sum l_i^d. \tag{1.57}$$

If $r(n+1) \geq \binom{d+n}{n}$ one expects that this map is surjective and hence $\mathrm{wrk}(f) \leq r$ for general f. Here "general" means that the coefficients of f belong to an open Zariski subset of the affine space $\mathbb{C}^{\binom{d+n}{n}}$. It follows from Theorem 1.3.10 that the only exceptional cases when it is false and the map s fails to be surjective are the following cases:

- $n = 2, d = 2, r = 2, \mathrm{wrk}(f) = 3;$
- $n = 2, d = 4, r = 5, \mathrm{wrk}(f) = 6;$
- $n = 3, d = 4, r = 9, \mathrm{wrk}(f) = 10;$
- $n = 4, d = 3, r = 7, \mathrm{wrk}(f) = 8;$
- $n = 4, d = 4, r = 14, \mathrm{wrk}(f) = 15;$

Proposition 1.4.5. *Let f be a general homogeneous form of even degree $2k$. Then*

$$\mathrm{wrk}(f) > \mathrm{rank}\, \Omega_f,$$

except in the following cases, where the equality takes place,:

- $k = 1$;
- $n = 1$;
- $n = 2, k \leq 4$;
- $n = 3, k = 2$.

Proof. The first case is obvious. It follows from considering the map (1.57) that $\mathrm{wrk}(f) \geq \binom{n+2k}{n}/(n+1)$. On the other hand the rank of Ω_f for general f is equal to $\dim S^k E = \binom{n+k}{n}$.

We know that the case $n = 1$ is not exceptional so that we can compute the Waring rank of f by counting constants and get $\mathrm{wrk}(f) = k + 1 = \mathrm{rank}\ \Omega_f$.

If $n = 2$, we get $\mathrm{wrk}(f) \geq (2k+2)(2k+1)/6 = (k+1)(2k+1)/3$ and $\mathrm{rank}\ \Omega_f = \binom{k+2}{2} = (k+2)(k+1)/2$. We have $(k+1)(2k+1)/3 > (k+2)(k+1)/2$ if $k > 4$. By Theorem 1.3.10,

$$\mathrm{wrk}(f) = \begin{cases} 6 & \text{if k = 2,} \\ 10 & \text{if k = 3,} \\ 15 & \text{if k = 4.} \end{cases}$$

This shows that $\mathrm{wrk}(f) = \mathrm{rank}\ \Omega_f$ in all these cases.

If $n = 3$, we get

$$\mathrm{wrk}(f) \geq (2k+3)(2k+2)(2k+1)/24 > \binom{k+3}{3} = (k+3)(k+2)(k+1)/6$$

unless $k = 2$.

Finally, it is easy to see that for $n > 3$

$$\mathrm{wrk}(f) \geq \tfrac{1}{n+1}\binom{2k+n}{n} > \binom{k+n}{n}$$

for $k > 1$. □

1.4.4 Mukai's skew-symmetric form

Let $\omega \in \bigwedge^2 E$ be a skew-symmetric bilinear form on $E^\vee$. It admits a unique extension to a Poisson bracket $\{,\}_\omega$ on $\mathrm{Sym}^\bullet E^\vee$ which restricts to a skew-symmetric bilinear form

$$\{,\}_\omega : S^{k+1}E^\vee \times S^{k+1}E^\vee \to S^{2k}E^\vee. \tag{1.58}$$

Recall that a *Poisson bracket* on a commutative algebra A is a skew-symmetric bilinear map $A \times A \to A, (f,g) \mapsto \{f,g\}$ such that its left and right partial maps $A \to A$ are derivations.

Let $f \in S^{2k}E^\vee$ be a nondegenerate form and $\check{f} \in S^{2k}E = (S^{2k}E^\vee)^\vee$ be its dual form. For each ω as above define $\sigma_{\omega,f} \in \bigwedge^2(S^{k+1}E)$ by

$$\sigma_{\omega,f}(f,g) = \check{f}(\{f,g\}_\omega).$$

Theorem 1.4.6. *Let f be a nondegenerate form in $S^{2k}E^\vee$ of Waring rank N. Assume that $N = \text{rank}\,\Omega_f = \binom{n+k}{n}$. For any $Z = \{\ell_1, \ldots, \ell_N\} \in \text{VSP}(f;N)^o$ let $\langle Z\rangle_{k+1}$ be the linear span of the powers l_i^{k+1} in $S^{k+1}E^\vee$. Then*

(i) $\langle Z\rangle_{k+1}$ *is isotropic with respect to each form* $\sigma_{\omega,f}$*;*

(ii) $\text{ap}_f^{k-1}(S^{k-1}E) \subset \langle Z\rangle_{k+1}$*;*

(iii) $\text{ap}_f^{k-1}(S^{k-1}E)$ *is contained in the radical of each* $\sigma_{\omega,f}$*.*

Proof. To prove the first assertion it is enough to check that $\sigma_{\omega,f}(l_i^{k+1}, l_j^{k+1}) = 0$ for all i,j. We have

$$\sigma_{\omega,f}(l_i^{k+1}, l_j^{k+1}) = \check{f}(\{l_i^k, l_j^k\}_\omega) = \check{f}(l_i^k l_j^k)\omega(l_i, l_j).$$

Since ℓ_i^k are linearly independent, by Proposition 1.4.2, $\check{f}(l_i^k l_j^k) = \Omega_{\check{f}}(l_i^k, l_j^k) = 0$. This checks the first assertion.

For any $\psi \in S^{k-1}E$,

$$D_\psi(f) = D_\psi(\sum_{i=1}^N l_i^{2k}) = \sum_{i=1}^N D_\psi(l_i^{2k}) = \tfrac{(2k)!}{(k+1)!}\sum_{i=1}^N D_\psi(l_i^{k-1})l_i^{k+1}.$$

This shows that $\text{ap}_f^{k-1}(S^{k-1}E)$ is contained in $\langle Z\rangle_{k+1}$. It remains to check that for any $\psi \in S^{k-1}E, g \in S^{k+1}E^\vee$ and any $\omega \in \bigwedge^2 E$, one has $\sigma_{\omega,f}(D_\psi(f), g) = 0$. Choose coordinates $t_0, \ldots, t_n$ in $E^\vee$ and the dual coordinates $\xi_0, \ldots, \xi_n$ in E. The space $\bigwedge^2 E$ is spanned by the forms $\omega_{ij} = \xi_i \wedge \xi_j$. We have

$$\{D_\psi(f), g\}_{\omega_{ij}} = D_{\xi_i}(D_\psi(f))D_{\xi_j}(g) - D_{\xi_j}(D_\psi(f))D_{\xi_i}(g)$$
$$= D_{\xi_i\psi}(f)D_{\xi_j}(g) - D_{\xi_j\psi}(f)D_{\xi_i}(g) = D_{\psi\xi_i}(f)D_{\xi_j}(g) - D_{\psi\xi_j}(f)D_{\xi_i}(g).$$

For any $a, b \in S^kE^\vee$,

$$\check{f}(ab) = \Omega_{\check{f}}(a,b) = \langle \Omega_f^{-1}(a), b\rangle.$$

Thus

$$\sigma_{\omega_{ij},f}(D_\psi(f), g) = \check{f}(D_{\psi\xi_i}(f)D_{\xi_j}(g) - D_{\psi\xi_j}(f)D_{\xi_i}(g))$$
$$= \langle \psi\xi_i, D_{\xi_j}(g)\rangle - \langle \psi\xi_j, D_{\xi_i}(g)\rangle = D_\psi(D_{\xi_i\xi_j}(g) - D_{\xi_j\xi_i}(g)) = D_\psi(0) = 0.$$

□

Let $Z = \{[l_1], \ldots, [l_s]\} \in \text{VSP}(f;s)^o$ be a polar s-polyhedron of a nondegenerate form $f \in S^{2k}E^\vee$ and, as before, let $\langle Z\rangle_{k+1}$ be the linear span of $(k+1)$-th powers of the linear forms l_i. Let

$$L(Z) = \langle Z\rangle_{k+1}/\text{ap}_f^{k-1}(S^{k-1}E). \tag{1.59}$$

It is a subspace of $W = S^{k+1}E^\vee/\text{ap}_f^{k-1}(S^{k-1}E)$. By (1.49),

$$W^\vee = \text{ap}_f^{k-1}(S^{k-1}E)^\perp = \text{AP}_{k+1}(f),$$

where we identify the dual space of $S^{k+1}E^\vee$ with $S^{k+1}E$. Now observe that $\langle Z\rangle_{k+1}^\perp$ is equal to $I_{\mathcal{P}}(k+1)$, where we identify Z with the reduced closed subscheme of the dual projective space $\mathbb{P}(E)$. This allows one to extend the definition of $L(Z)$ to any generalized polar s-polyhedron $Z \in \mathrm{VSP}(f;s)$:

$$L(Z) = I_Z(k+1)^\perp/\mathrm{ap}_f^{k-1}(S^{k-1}E) \subset S^{k+1}E^\vee/\mathrm{ap}_f^{k-1}(S^{k-1}E).$$

Proposition 1.4.7. *Let f be a nondegenerate homogeneous form of degree $2k$ of Waring rank equal to $N_k = \binom{n+k}{k}$. Let $Z, Z' \in \mathrm{VSP}(f;s)$. Then*

$$L(Z) = L(Z') \iff Z = Z'.$$

Proof. It is enough to show that

$$I_Z(k+1) = I_{Z'}(k+1) \Longrightarrow Z = Z'.$$

Suppose $Z \neq Z'$. Choose a subscheme Z_0 of Z of length $N_k - 1$ which is not a subscheme of Z'. Since $\dim I_{Z_0}(k) \geq \dim S^kE^\vee - h^0(\mathcal{O}_Z) = \binom{n+k}{k} - N_k + 1 = 1$, we can find a nonzero $\psi \in I_{Z_0}(k)$. The sheaf $\mathcal{I}_Z/\mathcal{I}_{Z_0}$ is concentrated at one point x and is annihilated by the maximal ideal $\mathfrak{m}_x$. Thus $\mathfrak{m}_x\mathcal{I}_{Z_0} \subset \mathcal{I}_Z$. Let $\xi \in E$ be a linear form on $E^\vee$ vanishing at x but not vanishing at any subscheme of Z'. This implies that $\xi\psi \in I_Z(k+1) = I_{Z'}(k+1)$ and hence $\psi \in I_{Z'}(k) \subset \mathrm{AP}_k(f)$ contradicting the nondegeneracy of f. □

It follows from Theorem 1.4.6 that each $\omega \in \bigwedge^2 E$ defines a skew-symmetric 2-form $\sigma_{\omega,f}$ on $S^{k+1}E$ which factors through a skew-symmetric 2-form $\bar{\sigma}_{\omega,f}$ on $W = S^{k+1}E/\mathrm{ap}_f^{k-1}(S^{k-1}E)$. We call it the *Mukai 2-form*. For each $\mathcal{P} \in \mathrm{VSP}(f;N_k)^o$ the subspace $L(Z) \subset W$ is isotropic with respect to $\bar{\sigma}_{\omega,f}$.

1.5 First examples

1.5.1 Binary forms

This is the case $n = 1$. Let U be a 2-dimensional linear space and $f \in S^dU^\vee \setminus \{0\}$. The hypersurface $V(f)$ can be identified with a positive divisor $D = \sum m_ix_i$ of degree d on $|U| \cong \mathbb{P}^1$. Thus we can identify the space $|S^dU^\vee|$ with the symmetric power $|U|^{(d)} := |U|^d/\mathfrak{S}_d$ and with the Hilbert scheme $\mathrm{Hilb}^d(|U|)$. A generalized s-polyhedron of f is a positive divisor $Z = \sum_{i=1}^k m_i[l_i]$ of degree s in $\mathbb{P}(U) = |U|^\vee$ such that $[f] \in \langle Z\rangle = \mathbb{P}(H^0(\mathbb{P}(E), \mathcal{I}_Z(d))^\perp)$. Note that in our case Z is automatically linearly independent (because $H^1(\mathcal{I}_Z(d)) = 0$). Obviously, $H^0(\mathbb{P}(U), \mathcal{I}_Z(d))$ consists of polynomials of degree d which are divisible by $\psi = \xi_1^{m_1}\cdots\xi_k^{m_k}$, where $\xi_i \in \mathrm{AP}_1(l_i)$. In coordinates, if $l_i = a_it_0 + b_it_1$, then $\xi_i = b_i\partial_0 - a_i\partial_1$. Thus f is orthogonal to this space if and only if $P_{\psi\psi'}(f) = 0$ for all $\psi' \in S^{d-s}(U)$. By the apolarity duality we obtain that $D_\psi(f) = 0$, hence $\psi \in \mathrm{AP}_s(f)$. This gives the following.

Theorem 1.5.1. *A positive divisor $Z = V(l_1^{m_1}\cdots l_k^{m_k})$ of degree s is a generalized s-polyhedron of f if and only if $\xi_1^{m_1}\cdots\xi_k^{m_k} \in \mathrm{AP}_s(f)$.*

Corollary 1.5.2. *Assume* $n = 1$. *Then*

$$\mathrm{VSP}(f; s) = |\mathrm{AP}_s(f)|.$$

Note that the kernel of the map

$$S^k U \to S^{d-k} U^\vee, \quad \psi \mapsto D_\psi(f)$$

is of dimension $\geq \dim S^k U - \dim S^{d-k} U^\vee = k + 1 - (d - k + 1) = 2k - d$. Thus $D_\psi(f) = 0$ for some nonzero $\psi \in S^k U$, whenever $2k > d$. This shows that f has always generalized polar k-polyhedron for $k > d/2$. If d is even, a binary form has an apolar $d/2$-form if and only if $\det \mathrm{Cat}_{d/2}(f) = 0$. This is a divisor in the space of all binary d-forms.

Example 1.5.1. Take $d = 3$. Assume that f admits a polar 2-polyhedron. Then

$$f = (a_1 t_0 + b_1 t_1)^3 + (a_2 t_0 + b_2 t_1)^3.$$

It is clear that f has 3 distinct roots. Thus, if $f = (a_1 t_0 + b_1 t_1)^2 (a_2 t_0 + b_2 t_1)^2$ has a double root, it does not admit a polar 2-polyhedron. However, it admits a generalized 2-polyhedron defined by the divisor $2p$, where $p = (b_1, -a_1)$. In the secant variety interpretation, we know that any point in $|S^3 E^\vee|$ either lies on a unique secant or on a unique tangent line of the Veronese cubic curve. The space $\mathrm{AP}_2(f)$ is always one-dimensional. It is generated either by a binary quadric $(-b_1\xi_0 + a_1\xi_1)(-b_2\xi_0 + a_2\xi_1)$ or by $(-b_1\xi_0 + a_1\xi_1)^2$.

Thus $\mathrm{VSP}(f; 2)^o$ consists of one point or empty but $\mathrm{VSP}(f; 2)$ always consists of one point. This example shows that $\mathrm{VSP}(f; 2) \neq \overline{\mathrm{VSP}(f; 2)}^o$ in general.

1.5.2 Quadrics

It follows from Example 1.3.3 that $\mathrm{Sec}_t(\mathrm{Ver}_2^n) \neq |S^2 E^\vee|$ if only if there exists a quadric with $t + 1$ singular points in general position. Since the singular locus of a quadric $V(q)$ is a linear subspace of dimension equal to $\mathrm{corank}(q) - 1$, we obtain that $\mathrm{Sec}_n(\mathrm{Ver}_2^n) = |S^2 E^\vee|$, hence any general quadratic form can be written as a sum of $n + 1$ squares of linear forms $l_0, \ldots, l_n$. Of course, linear algebra gives more. Any quadratic form of rank $n + 1$ can be reduced to sum of squares of the coordinate functions. Assume that $q = t_0^2 + \cdots + t_n^2$. Suppose we also have $q = l_0^2 + \cdots + l_n^2$. Then the linear transformation $t_i \mapsto l_i$ preserves q and hence is an orthogonal transformation. Since polar polyhedra of q and λq are the same, we see that the projective orthogonal group $\mathrm{PO}(n+1)$ acts transitively on the set $\mathrm{VSP}(f; n+1)^o$ of polar $(n+1)$-polyhedra of q. The stabilizer group G of the coordinate polar polyhedron is generated by permutations of coordinates and diagonal orthogonal matrices. It is isomorphic to the semi-direct product $2^n \rtimes \mathfrak{S}_{n+1}$ (the Weyl group of root systems of types B_n, D_n), where we use the notation 2^k for the 2-elementary abelian group $(\mathbb{Z}/2\mathbb{Z})^k$. Thus we obtain

Theorem 1.5.3. *Let* q *be a quadratic form in* $n + 1$ *variables of rank* $n + 1$. *Then*

$$\mathrm{VSP}(q; n + 1)^o \cong \mathrm{PO}(n + 1)/2^n \rtimes \mathfrak{S}_{n+1}.$$

The dimension of $\mathrm{VSP}(q; n + 1)^o$ *is equal to* $\frac{1}{2}n(n + 1)$.

Example 1.5.2. Take $n = 1$. Using the Veronese map $\nu_2 : \mathbb{P}^1 \to \mathbb{P}^2$, we consider a nonsingular quadric $Q = V(q)$ as a point p in $\mathbb{P}^2$ not lying on the conic $C = V(t_0t_2 - t_1^2)$. A polar 2-gon of q is a pair of distinct points p_1, p_2 on C such that $p \in \langle p_1, p_2\rangle$. The set of polar 2-gons can be identified with the pencil of lines through p with the two tangent lines to C deleted. Thus $W(q, 2)^o = \mathbb{P}^1 \setminus \{0, \infty\} = \mathbb{C}^*$. There are two generalized 2-gons $2p_0$ and $2p_\infty$ defined by the tangent lines. Each of them gives the representation of q as l_1l_2, where $V(l_i)$ are the tangents. We have $\mathrm{VSP}(f; 2) = \overline{\mathrm{VSP}(f; 2)}^o \cong \mathbb{P}^1$.

In the next chapter we will discuss a good compactification of this space in the case $n = 2$.

Let $q \in S^2E^\vee$ be a nondegenerate quadratic form. For each $Z \in \mathrm{VSP}(q; n + 1)$ the linear space $L(Z) = \langle Z\rangle_2/\mathbb{C}q \subset S^2E^\vee/\mathbb{C}q$ is of dimension n. It is an isotropic subspace of $W = S^2E^\vee/\mathbb{C}q$ with respect to any Mukai's 2-form $\bar{\sigma}_{\omega,q}$. This defines a map

$$\mu : \mathrm{VSP}(q; n + 1) \to G(n, W), \quad Z \mapsto L(Z). \tag{1.60}$$

By Proposition 1.4.7, the map is injective. The image of $\mathrm{VSP}(q, n + 1)^o$ is contained in the locus $G(n, W)^\mu$ of subspaces which are isotropic with respect to any Mukai's 2-form $\bar{\sigma}_{q,\omega}$. Since for general f the variety $\mathrm{VSP}(q, n+1)$ is the closure of $\mathrm{VSP}(q, n+1)^o$ in the Hilbert scheme, the image of $\mathrm{VSP}(q; n + 1)$ is contained $G(n, W)^\mu$. Since all nonsingular quadrics are isomorphic, the assertion is true for any nondegenerate quadratic form f.

Recall that the Grassmann variety $G(n, W)$ carries the natural rank n vector bundle $\mathcal{S}$, the *tautological bundle*. Its fibre over a point $L \in G(n, W)$ is equal to L. It is a subbundle of the trivial bundle $W_{G(n,W)}$ associated to the vector space W. We have a natural exact sequence

$$0 \to \mathcal{S} \to W_{G(n,V)} \to \mathcal{Q} \to 0,$$

where $\mathcal{Q}$ is the *universal quotient bundle*, whose fibre over L is equal to W/L. We can consider each element σ of $\bigwedge^2 W^\vee$ as a section of the trivial bundle $\bigwedge^2 W^\vee_{G(n,W)}$. Restricting σ to the subbundle $\mathcal{S}$, we get a section of the vector bundle $\bigwedge^2 \mathcal{S}^*$. Thus we can view a Mukai's 2-form $\bar{\sigma}_{q,\omega}$ as section $s_{q,\omega}$ of $\bigwedge^2 \mathcal{S}^\vee$.

It follows from above that the image of the map (1.60) is contained in the set of common zeros of the sections $s_{q,\omega}$ of $\bigwedge^2 \mathcal{S}^\vee$.

Corollary 1.5.4. *Let q be a nondegenerate quadratic form on a three-dimensional vector space E. Then the image of $\mathrm{VSP}(q; 3)$ in $G(2, W)$, embedded in the Plücker space $|\bigwedge^2 W|$, is a smooth irreducible 3-fold equal to the intersection X of $G(2, W)$ with a linear space of codimension 3.*

Proof. We have $\dim W = 5$, so $G(2, W) \cong G(2, 5)$ is of dimension 6. Hyperplanes in the Plücker space are elements of the space $|\bigwedge^2 W^\vee|$. Note that the functions $s_{q,\omega}$ are linearly independent. In fact, a basis ξ_0, ξ_1, ξ_2 in E gives a basis $\omega_{01} = \xi_0 \wedge \xi_1, \omega_{02} = \xi_0 \wedge \xi_2, \omega_{12} = \xi_1 \wedge \xi_2$ in $\bigwedge^2 E$. Thus the space of sections $s_{q,\omega}$ is spanned by 3 sections s_{01}, s_{02}, s_{12} corresponding to the forms ω_{ij}. Without loss of generality, we may assume that $q = t_0^2 + t_1^2 + t_2^2$. If we take $a = t_0t_1 + t_2^2, b = -t_0^2 + t_1^2 + t_2^2$, we see that $s_{01}(a,b) \neq 0, s_{12}(a,b) = 0, s_{02}(a,b) = 0$. Thus a linear dependence between the functions s_{ij} implies the linear dependence between two of the functions. It is easy to see that no two functions are proportional. So our 3 functions $s_{ij}, 0 \leq i < j \leq 2$ span a 3-dimensional subspace of $\bigwedge^2 W^\vee$ and hence define a codimension 3 projective subspace L in the Plücker space $|\bigwedge^2 W|$. The image of $\mathrm{VSP}(q;3)$ under the map (1.60) is contained in the intersection $G(2,E) \cap L$. This is a 3-dimensional subvariety of $G(2,W)$, and hence contains $\mu(\mathrm{VSP}(q;3))$ as an irreducible component. We skip an argument, based on counting constants, which proves that the subspace L belongs to an open Zariski subspace of codimension 3 subspaces of $\bigwedge^2 W$ for which the intersection $L \cap G(2,W)$ is smooth and irreducible (see [137]). $\square$

If $n > 2$, the vector bundle $\bigwedge^2 \mathcal{S}^\vee$ is of rank $r = \binom{n}{2} > 1$. The zero locus of its nonzero section is of expected codimension equal to r. We have $\binom{n+1}{2}$ sections s_{ij} of $\bigwedge^2 \mathcal{S}$ and $\dim G(n,E) = n(\binom{n+2}{2} - n - 1)$. For example, when $n = 3$, we have 6 sections s_{ij} each vanishing on a codimension 3 subvariety of 18-dimensional Grassmannian $G(3,9)$. So there must be some dependence between the functions s_{ij}.

Remark 1.5.1. One can also consider the varieties $\mathrm{VSP}(q;s)$ for $s > n+1$. For example, we have

$$\begin{aligned} t_0^2 - t_2^2 &= \tfrac{1}{2}(t_0+t_1)^2 + \tfrac{1}{2}(t_0-t_1)^2 - \tfrac{1}{2}(t_1+t_2)^2 - \tfrac{1}{2}(t_1^2-t_2)^2 \\ t_0^2 + t_1^2 + t_2^2 &= (t_0+t_2)^2 + (t_0+t_1)^2 + (t_1+t_2)^2 - (t_0+t_1+t_2)^2. \end{aligned}$$

This shows that $\mathrm{VSP}(q;n+2), \mathrm{VSP}(q;n+3)$ are not empty for any nondegenerate quadric Q in $\mathbb{P}^n, n \geq 2$.

Exercises

1.1 Show that the first polar $P_a(X)$ contains singular points of X. Suppose X is a plane curve and $x \in X$ is its ordinary double point. Show that the pair consisting of the tangent line of $P_a(X)$ at x and the line $\overline{a,x}$ is harmonically conjugate (see section 2.1.2) to the pair of tangents to the branches of X at x in the pencil of lines through x. If x is an ordinary cusp, then tangent line of $P_a(X)$ at x is equal to the cuspidal tangent of X at x.

1.2 Show that a line contained in a hypersurface X belongs to all polars of X with respect to any point on this line.

1.3 Find the multiplicity of the intersection of a plane curve C with its Hessian at an ordinary double point and at an ordinary cusp of C. Show that the Hessian has a triple point at the cusp.

1.4 Suppose a hypersurface X in $\mathbb{P}^n$ has a singular point x of multiplicity $m > 1$. Prove that $\mathrm{He}(X)$ has this point as a point of multiplicity $\geq (n+1)m - 2n$.

1.5 Suppose a hyperplane is tangent to a hypersurface X along a closed subvariety Y of codimension 1. Show that Y is contained in $\mathrm{He}(X)$.

1.6 Suppose f is the product of d distinct linear forms $l_i(t_0,\ldots,t_n)$. Let A be the matrix of size $(n+1) \times d$ whose i-th column is formed by the coefficients of l_i (defined, of course up to proportionality). Let Δ_I be the maximal minor of A corresponding to a subset I of $[1,\ldots,d]$ and f_I be the product of linear forms $l_i, i \notin I$. Show that

$$\mathrm{He}(f) = (-1)^n(d-1)f^{n-1}\sum_I \Delta_I^2 f_I^2.$$

([288], p. 660).

1.7 Let $n=2$. Assume $\mathrm{He}(V(f)) = \mathbb{P}^2$. Show that f is the union of concurrent lines.

1.8 Show that the locus of the points on the plane where the first polars of a plane curve X are tangent to each other is the Hessian of X and the set of common tangents is the Cayleyan curve .

1.9 Show that each flex tangent of a plane curve X, considered as a point in the dual plane, lies on the Cayleayan of X.

1.10 Show that the class of the Steinerian $\mathrm{St}(X)$ of a plane curve X of degree d is equal to $3(d-1)(d-2)$ but its dual is not equal to $\mathrm{Cay}(X)$.

1.11 Let $\mathcal{D}_{m,n} \subset \mathbb{P}^{mn-1}$ be the image in the projective space of the variety of $m \times n$ matrices of rank $\le \min\{m,n\}-1$. Show that the variety

$$\tilde{\mathcal{D}}_{m,n} = \{(A,x) \in \mathbb{P}^{mn-1} \times \mathbb{P}^n : A \cdot x = 0\}$$

is a resolution of singularities of $\mathcal{D}_{m,n}$. Find the dual variety of $\mathcal{D}_{m,n}$.

1.12 Find the dual variety of the Segre variety $\mathbb{P}^n \times \mathbb{P}^n \hookrightarrow \mathbb{P}^{n^2+2n}$.

1.13 Prove that the degree of the dual variety of a nonsingular hypersurface of degree d in $\mathbb{P}^n$ is equal to $d(d-1)^{n-1}$.

1.14 Let X be the union of k nonsingular conics in general position. Show that $X^\vee$ is also the union of k nonsingular conics in general position. Check the Plücker formulas in this case.

1.15 Let X has only δ ordinary nodes and κ ordinary cusps as singularities. Assume that the dual curve $X^\vee$ has also only $\check{\delta}$ ordinary nodes and $\check{\kappa}$ ordinary cusps as singularities. Find $\check{\delta}$ and $\check{\kappa}$ in terms of d, δ, κ.

1.16 Give an example of a self-dual (i.e. $X^\vee \cong X$) plane curve of degree > 2.

1.17 Let $f \in S^2E^\vee$. Show that the map $E \to E^\vee$ defined by $\psi \mapsto D_\psi(f)$ corresponds to the symmetric bilinear form $E \times E \to \mathbb{C}$ associated to Q.

1.18 Show that the embedded tangent space of the Veronese variety Ver_d^n at a point represented by the form l^d is equal to the projectivization of the linear space of homogeneous polynomials of degree d of the form $l^{d-1}m$.

1.19 Show using the following steps that Ver_3^4 is 6-defective by proving that for 7 general points p_i in $\mathbb{P}^4$ there is a cubic hypersurface with singular points at the p_i's.

(i) Show that there exists a Veronese curve R_4 of degree 4 through the seven points.

(ii) Show that the secant variety of R_4 is a cubic hypersurface which is singular along R_4.

1.20 Let q be a nondgenerate quadratic form in $n+1$ variables. Show that $\mathrm{VSP}(q;n+1)^o$ embedded in $G(n,E)$ is contained in the linear subspace of codimension n.

1.21 Compute the catalecticant matrix $\mathrm{Cat}_2(f)$, where f is a homogeneous form of degree 4 in 3 variables.

1.22 Let $f \in S^{2k}E^\vee$ and Ω_f be the corresponding quadratic form on S^kE. Show that the quadric $V(\Omega_f)$ in $|S^kE|$ is characterisezed by the following two properties:

- Its preimage under the Veronese map $\nu_k : |E| \to |S^kE|$ is equal to $V(f)$;

- Ω_f is apolar to any quadric in $|S^kE^\vee|$ which contains the image of the Veronese map $|E^\vee| = \mathbb{P}(E) \to |S^kE^\vee| = |S^kE|$.

1.23 Let C_k be the locus in $|S^{2k}E^\vee|$ of hypersurfaces $V(f)$ such that $\det \mathrm{Cat}_k(f) = 0$. Show that C_k is a rational variety. [Hint: Consider the rational map $C_k - \to |E|)$ which assigns to $V(f)$ the point defined by the subspace $\mathrm{AP}_k(f)$ and study its fibres].

1.24 Give an example of a polar 4-gon of the cubic $t_0t_1t_2 = 0$.

1.25 Find all binary forms of degree d for which $\mathrm{VSP}(f;2)^o = \emptyset$.

1.26 Let f be a form of degree d in $n+1$ variables. Show that the variety $\mathrm{VSP}(f;\binom{n+d}{d})^o$ is an irreducible variety of dimension $n\binom{n+d}{d}$.

1.27 Describe the variety $\mathrm{VSP}(f;4)$, where f is a nondegenerate quadratic form in 3 variables.

1.28 Show that a smooth y point of a hypersurface X belongs to the intersection of the polar hypersurfaces $P_x(X)$ and $P_{x^2}(X)$ if and only if the line connecting x and y intersects X at the point y with multiplicity ≥ 3.

Historical Notes

Although the polar lines of conics were known to mathematicians of Ancient Greece, the first systematic study of polars of curves of higher degree started in the works of E. Bobilier [32] and J. Plücker [318]. However, some of theory were known before to G. Monge and J. Poncelet. According to the historical account in [160], vol. II, (see also [97], p. 60) the name "polaire" was introduced by J. Gergonne. As was customary for him, J. Steiner stated many properties of polar curves without proofs [394]. Other historical information can be found in [28] and [308], p.279. The Hessian curve was first introduced by J. Steiner [394] who called it the *Kerncurve* (later the term *Kernfläche* was used for Hessian surfaces). The current name was coined by J. Sylvester in honor of O. Hesse whose paper [209] provided many fundamental properties of the curve. The Steinerian curve originates in the works of J. Steiner in more general setting of nets of plane curves (not necessary the net of polars). The name was given by G. Salmon and L. Cremona. The Cayleyan curve was introduced by A. Cayley in [48] who called it the *pippiana*. The name was proposed by L. Cremona.

There are many beautiful results in the hessians in the classical literature, many of them can be found in standard text-books of that time (e.g. [82], [160], [356]). Excellent surveys of these results can be found in [28] and [308].

The theory of dual varieties, generalization of Plücker formulae to arbitrary dimension is still a popular subject of modern algebraic geometry. It is well-documented in modern literature and for this reason this topic is barely touched here.

The theory of apolarity is one of the forgotten topics of classical algebraic geometry. It originates from the works of Rosanes [341] and Reye [331]. According to G. Salmon ([354], p. 346) the term "apolar" is due to Reye. We refer for survey of classical results to [308] and to a modern exposition of some of these results to [137] which we followed in these notes.

Chapter 2

Conics

2.1 Self-polar triangles

2.1.1 The Veronese quartic surface

Recall that the *Veronese variety* is defined to be the image of the map

$$|E| \to |S^d E|, \quad [v] \mapsto [v^d].$$

If we identify $|S^d E|$ with the dual space of $S^d E^\vee = H^0(|E|, \mathcal{O}_{|E|}(d))$, then the map is given by the complete linear system $|\mathcal{O}_{|E|}(d)|$. The Veronese variety is of dimension n and degree d^n. More generally, one defines a Veronese variety as the image of $\mathbb{P}^n$ in $\mathbb{P}^{N(d,n)-1}$ under the map given by the complete linear system $|\mathcal{O}_{\mathbb{P}^n}(d)|$ and a choice of a basis in $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$.

Let $d = n = 2$, this is the case of the *Veronese quartic surface* Ver_2^2. The preimage of a hyperplane $H \in |S^2 E^\vee|$ in $|S^2 E| \cong \mathbb{P}^5$ is a conic $C \in |S^2 E^\vee|$. There are three sorts of hyperplanes corresponding to the cases: C is nonsingular, C is a line-pair, C is a double line. In the first case H intersects the Veronese surface $\mathrm{Ver}_2^2 = v_2(\mathbb{P}^2)$ transversally, in the second case H is tangent to Ver_2^2 at a single point, and in the third case H is tangent to Ver_2^2 along a conic.

Choosing a basis in E we can identify the space $S^2 E^\vee$ with the space of symmetric 3×3 matrices. The Veronese surface is identified with matrices of rank 1. Its equations are given by 2×2 minors. The variety of matrices of rank ≤ 2 is the cubic hypersurface $\mathcal{D}_2(2)$ given by the determinant. It singular along the Veronese surface.

Since any nonzero matrix of rank ≤ 2 can be written as a sum of matrices of rank 1, we see that $\mathcal{D}_2(2)$ is equal to the first secant variety of Ver_2^2.

A linear projection of Ver_2^2 from a point not lying in $\mathcal{D}_2(2)$ is an isomorphism onto a quartic surface V_4 in $\mathbb{P}^4$, called the *projected Veronese surface*.

The image of Ver_2^2 under a linear projection from a point Q lying in $\mathcal{D}_2(2)$ but not lying on the surface is a non-normal quartic surface V_4' in $\mathbb{P}^4$. To see this we may assume that $Q = V(t_0^2 + t_1^2)$. The plane of conics $V(at_0^2 + bt_0t_1 + ct_1^2)$ contains Q and intersects Ver_2^2 along the conic of double lines $V((\alpha t_0 + \beta t_1)^2)$. The projection maps this conic two-to-one to a double line of the image of Ver_2^2.

The image of Ver_2^2 under a linear projection from its point is a cubic scroll in $\mathbb{P}^3$, the image of $\mathbb{P}^2$ under a map given by the linear system of conics with one base point.

2.1.2 Polar lines

Let C be a nonsingular conic. For any point $a \in \mathbb{P}^2$ the first polar $P_a(C)$ is a line, the *polar line* of a. For any line ℓ there exists a unique point a such that $P_a(C) = l$. The point a is called the *pole* of ℓ. The point a considered as a line in the dual plane is the polar line of the point ℓ with respect to the dual conic $\check{C}$.

A set of three non-colinear lines ℓ_1, ℓ_2, ℓ_3 is called a *self-polar triangle* with respect to C if each ℓ_i is the polar line of C with respect to the point of intersection of the other two lines.

Recall that two unordered pairs $\{a, b\}, \{c, d\}$ of points in $\mathbb{P}^1$ are called *harmonically conjugate* if

$$-2\beta\beta' + \alpha\gamma' + \alpha'\gamma = 0, \tag{2.1}$$

where $V(\alpha t_0^2 + 2\beta t_0 t_1 + \gamma t_1^2) = \{a, b\}$ and $V(\alpha' t_0^2 + 2\beta' t_0 t_1 + \gamma' t_1^2) = \{c, d\}$. It follows that this definition does not depend on the order of points in each pair.

It is easy to check that (2.1) is equivalent to the polarity condition

$$D_{cd}(q) = D_{ab}(q') = 0, \tag{2.2}$$

where $V(q) = \{a, b\}, V(q') = \{c, d\}$.

Proposition 2.1.1. *Let ℓ_1, ℓ_2, ℓ_3 be a self-polar triangle of C and $a = \ell_1 \cap \ell_2$. Assume $a \notin C$. Then the pairs of points $\ell_3 \cap C$ and $(b, c) = (\ell_1 \cap \ell_3, \ell_2 \cap \ell_3)$ on the line ℓ_3 are harmonically conjugate. Conversely, if $\{c, d\}$ is a pair of points on ℓ_3 which is harmonically conjugate to the pair $C \cap \ell_3$, then the lines $\langle a, b\rangle, \langle a, c\rangle, \ell_3$ form a self-polar triangle of C.*

Proof. Consider the pair $C \cap \ell_3$ as a quadric q in ℓ_3. We have $c \in P_b(C)$, thus $D_{bc}(q) = 0$. Restricting to ℓ_3 and using (2.2), we see that b, c form a harmonic pair with respect to q. Conversely, if $D_{bc}(q) = 0$, the polar line $P_b(C)$ contains a and intersects ℓ_3 at c, hence coincides with $\overline{a, c}$. Similarly, $P_c(C) = \overline{a, b}$. □

The polar line $\ell = P_a(C)$ intersects the conic C at two points x, y such that $a \in \mathbb{T}_x(C) \cap \mathbb{T}_y(C)$.

Borrowing terminology from the Euclidean geometry, we call three non-collinear lines in $\mathbb{P}^2$ a *triangle*. The lines themselves will be called the *sides* of the triangle. The three intersection points of pairs of sides are called the *vertices* of the triangle.

Let

$$f = a_{00}t_0^2 + a_{11}t_1^2 + a_{22}t_2^2 + 2a_{01}t_0t_1 + 2a_{02}t_0t_2 + 2a_{12}t_1t_2 = 0$$

be the equation of a nonsingular conic C. Choose projective coordinates in $\mathbb{P}^2$ such that $\ell_i = V(t_i)$. Then

$$P_{[1,0,0]}(X) = \ell_1 = V(\frac{\partial f}{\partial t_0}) = V(a_{00}t_0 + a_{01}t_1 + a_{02}t_2), \tag{2.3}$$
$$P_{[0,1,0]}(X) = \ell_2 = V(\frac{\partial f}{\partial t_1}) = V(a_{11}t_1 + a_{01}t_0 + a_{12}t_2),$$
$$P_{[0,0,1]}(X) = \ell_2 = V(\frac{\partial f}{\partial t_2}) = V(a_{22}t_2 + a_{02}t_0 + a_{12}t_1)$$

implies that

$$f = \frac{1}{2}(t_0^2 + t_1^2 + t_2^2). \tag{2.4}$$

Conversely, any conic $V(l_1^2 + l_2^2 + l_3^2)$ where l_i are three linearly independent linear forms, defines a self-polar triangle with the sides $V(l_i)$.

Any triangle in $\mathbb{P}^2$ defines the dual triangle in the dual plane $\check{\mathbb{P}}^2$. Its sides are the pencils of lines with the base point of one of the vertices.

Proposition 2.1.2. *The dual of a self-polar triangle of a conic C is a self-polar triangle of the dual conic $\check{C}$.*

Proof. Choose the coordinate system such that the self-polar triangle is the coordinate triangle. Then $C = V(t_0^2 + t_1^2 + t_2^2)$ and the assertion is easily verified. $\square$

All of this is immediately generalized to nonsingular quadrics Q in $\mathbb{P}^n$ for arbitrary n. We leave the generalization to the reader. For example, the problem of reducing a quadratic form to the sum of squares (or to *principal axes*) is nothing more as the problem of finding a self-conjugate $n+1$-polyhedron of Q.

2.1.3 The variety of self-polar triangles

Let C be a nonsingular conic. The group of projective transformations of $\mathbb{P}^2$ leaving C invariant is isomorphic to the projective complex orthogonal group

$$\mathrm{PO}_3 = \mathrm{O}_3/(\pm I_3) \cong \mathrm{SO}_3.$$

It is also isomorphic to the group PSL_2 via the Veronese map

$$\nu_2 : \mathbb{P}^1 \to \mathbb{P}^2, \quad [t_0, t_1] \mapsto [t_0^2, t_0t_1, t_1^2].$$

Obviously, PO_3 acts transitively on the set of self-polar triangles of C. We may assume that C is given by (2.4). The stabilizer subgroup of the self-polar triangle defined by the coordinate lines is equal to the subgroup generated by permutation matrices and orthogonal diagonal matrices. It is easy to see that it is isomorphic to the semi-direct product $(\mathbb{Z}/2\mathbb{Z})^2 \rtimes \mathfrak{S}_3$. An easy exercise in group theory gives that this group is isomorphic to the permutation group $\mathfrak{S}_4$. Thus we obtain the following.

Theorem 2.1.3. *The set of self-polar triangles of a nonsingular conic has a structure of a homogeneous space SO_3/Γ, where Γ is a finite subgroup isomorphic to $\mathfrak{S}_4$.*

Let us describe a natural compactification of the homogeneous space SO_3/Γ. Let V be a Veronese surface in $\mathbb{P}^5$. We view $\mathbb{P}^5$ as the projective space of conics in $\mathbb{P}^2$ and V as its subvariety of double lines. A trisecant plane of V is spanned by three linearly independent double lines. A conic $C \in \mathbb{P}^5$ belongs to this trisecant if and only if the corresponding three lines form a self-polar triangle of C. Thus the set of self-polar triangles of C can be identified with the set of trisecant planes of the Veronese surface which contain C. The latter will also include *degenerate self-polar triangles* corresponding to the case when the trisecant plane is tangent to the Veronese surface at some of its points of intersections. Projecting from C to $\mathbb{P}^4$ we will identify the set of self-polar triangles (maybe degenerate) with the set of trisecant lines of the projected Veronese surface $\bar{X}$. This is a closed subvariety of the Grassmann variety $G_1(\mathbb{P}^4)$ of lines in $\mathbb{P}^4$.

Let E be a linear space of odd dimension $2k+1$ and let $G(2,E) := G_1(|E|)$ be the Grassmannian of lines in $|E|$. Consider its Plücker embedding $\bigwedge^2 : G(2,E) \hookrightarrow G_1(\bigwedge^2 E) = |\bigwedge^2 E|$. Any nonzero $\omega \in (\bigwedge^2 E)^\vee = \bigwedge^2 E^\vee$ defines a hyperplane H_ω in $|\bigwedge^2 E|$. Consider ω as a linear map $\alpha_\omega : E \to E^\vee$ defined by $\alpha_\omega(v)(w) = \omega(v,w)$. The map α_ω is skew-symmetric in the sense that its transpose map coincides with $-\alpha_\omega$. Thus its determinant is equal to zero, and $\mathrm{Ker}(\alpha_\omega) \neq \{0\}$. Let v_0 be a nonzero element of the kernel. Then for any $v \in E$ we have $\omega(v_0, v) = \alpha_\omega(v)(v_0) = 0$. This shows that ω vanishes on all decomposable 2-vectors $v_0 \wedge v$. This implies that the intersection of the hyperplane H_ω with $G(2,E)$ contains all lines which intersect the linear subspace $C_\omega = |\mathrm{Ker}(\alpha_\omega)| \subset |E|$ which we call the *pole* of the hyperplane H_ω.

Now recall the following result from linear algebra (see Exercise 2.1). Let A be a skew-symmetric matrix of odd size $2k+1$. Its principal submatrices A_i of size $2k$ (obtained by deleting the i-th row and the i-th column) are skew-symmetric matrices of even size. Let Pf_i be the pfaffians of A_i (i.e. $\det(A_i) = \mathrm{Pf}_i^2$). Assume that $\mathrm{rank}(A) = 2k$, or, equivalently, not all Pf_i vanish. Then the system of linear equations $A \cdot x = 0$ has one-dimensional null-space generated by the vector $(a_1, \ldots, a_{2k+1})$, where $a_i = (-1)^{i+1}\mathrm{Pf}_i$.

Let us go back to Grassmannians. Suppose we have an $s+1$-dimensional subspace W in $\bigwedge^2 E^\vee$ spanned by $\omega_0, \ldots, \omega_s$. Suppose that for any $\omega \in W$ we have rank $\alpha_\omega = 2k$, or equivalently, the pole C_ω of H_ω is a point. It follows from the theory of determinant varieties that the subvariety

$$\{\mathbb{C}\omega \in |\bigwedge^2 E^\vee| : \mathrm{corank}\ \alpha_\omega \geq i\}$$

is of codimension $\binom{i}{2}$ in $|\bigwedge^2 E^\vee|$ [204], [254]. Thus, if $s < 4$, a general W will satisfy the assumption. Consider a regular map $\Phi : |W| \to |E|$ defined by $\omega \mapsto C_\omega$. If we take $\omega = t_0\omega_0 + \cdots + t_s\omega_s$ so that $t = (t_0, \ldots, t_s)$ are projective coordinate functions in $|W|$, we obtain that Φ is given by $2k+1$ principal pfaffians of the matrix A_t defining ω.

We shall apply the preceeding to the case $\dim E = 5$. Take a general 3-dimensional subspace W of $\bigwedge^2 E^\vee$. The map $\Phi : |W| \to |E \cong \mathbb{P}^4$ is defined by homogeneous polynomials of degree 2. Its image is a projected Veronese surface S. Any trisecant line of S passes through 3 points on S which are the poles of elements w_1, w_2, w_3

from W. These elements are linearly independent since otherwise their poles lie on the conic image of a line under Φ. But no trisecant line can be contained in a conic plane section of S. We consider $\omega \in W$ as a hyperplane in the Plücker space $|\bigwedge^2 E|$. Thus any trisecant line is contained in all hyperplanes defined by W. Now we are ready to prove the following.

Theorem 2.1.4. *Let $\bar{X}$ be the closure in $G_1(\mathbb{P}^4)$ of the locus of trisecant lines of a projected Veronese surface. Then $\bar{X}$ is equal to the intersection of $G_1(\mathbb{P}^4)$ with three linearly independent hyperplanes. In particular, $\bar{X}$ is a Fano 3-fold of degree 5 with canonical sheaf $\omega_{\bar{X}} \cong \mathcal{O}_{\bar{X}}(-2)$.*

Proof. We have already shown that the locus of poles of a general 3-dimensional linear space W of hyperplanes in the Plücker space is a projected Veronese surface S and its trisecant variety is contained in $Y = \cap_{w \in W} H_w \cap G_1(\mathbb{P}^4)$. So, its closure $\bar{X}$ is also contained in Y. On the other hand, we know that $\bar{X}$ is irreducible and 3-dimensional (it contains an open subset isomorphic to a homogeneous space $X = \mathrm{SO}(3)/\mathfrak{S}_4$). By Bertini's Theorem the intersection of $G_1(\mathbb{P}^4)$ with a general linear space of codimension 3 is an irreducible 3-dimensional variety. This proves that $Y = \bar{X}$. By another Bertini's theorem, Y is smooth. The rest is the standard computation of the canonical class of the Grassmann variety and the adjunction formula. It is known that the canonical class of the Grassmannian $G = G_m(\mathbb{P}^n)$ of m-dimensional subspaces of $\mathbb{P}^n$ is equal to

$$K_G = \mathcal{O}_G(-n-1) \tag{2.5}$$

(see Exercise 3.2). By the adjunction formula, the canonical class of $\bar{X} = G_1(\mathbb{P}^4) \cap H_1 \cap H_2 \cap H_3$ is equal to $\mathcal{O}_{\bar{X}}(-2)$. □

Corollary 2.1.5. *The homogeneous space $X = \mathrm{SO}(3)/\mathfrak{S}_4$ admits a smooth compactification $\bar{X}$ isomorphic to the intersection of $G_1(\mathbb{P}^4)$, embedded via Plücker in $\mathbb{P}^9$, with a linear subspace of codimension 3. The boundary $\bar{X} \setminus X$ is an anticanonical divisor cut out by a hypersurface of degree 2.*

Proof. The only unproven assertion is one about the boundary. We use that the 3-dimensional group $G = \mathrm{SL}(2)$ acts transitively on a 3-dimensional variety X minus the boundary. For any point $x \in X$ consider the map $\mu_x : G \to X, g \mapsto g \cdot x$. Its fibre over the point x is the isotropy subgroup G_x of x. The differential of this map defines a linear map $\mathfrak{g} = T_e(G) \to T_x(X)$. When we let x vary in X, we get a map of vector bundles

$$\phi : \mathfrak{g}_X = \mathfrak{g} \times X \to T(X).$$

Now take the determinant of this map

$$\bigwedge^3 \phi = \bigwedge^3 \mathfrak{g} \times X \to \bigwedge^3 T(X) = K_X^\vee,$$

where K_X is the canonical line bundle of X. The left-hand side is the trivial line bundle over X. The map $\bigwedge^3(\phi)$ defines a section of the anticanonical line bundle. The zeros of this section are the points where the differential of the map μ_x is not injective, i.e., where $\dim G_x > 0$. But this is exactly the boundary of X. In fact, the boundary consists of orbits of dimension smaller than 3, hence the isotropy of each such orbit is of positive dimension. This shows that the boundary is contained in our anticanonical divisor. Obviously, the latter is contained in the boundary. Thus we see that the boundary is equal to the intersection of $G_1(\mathbb{P}^4)$ with a quadric hypersurface.

□

Remark 2.1.1. There is another construction of the variety $\bar{X}$ of self-polar triangles due to S. Mukai and H. Umemura [289]. Let V_6 be the space of homogeneous binary forms $f(t_0, t_1)$ of degree 6. The group $\mathrm{SL}(2)$ has a natural linear representation in V_6 via linear change of variables. Let $f = t_0t_1(t_0^4 - t_1^4)$. The zeros of this polynomials are the vertices of a regular octahedron inscribed in $S^2 = \mathbb{P}^1$. The stabilizer subgroup of f in $\mathrm{SL}(2)$ is isomorphic to the binary octahedron group $\Gamma \cong \mathfrak{S}_4$. Consider the projective linear representation of $\mathrm{SL}(2)$ in $|V_6| \cong \mathbb{P}^5$. In the loc. cit. it is proven that the closure $\bar{X}$ of this orbit in $|V_6|$ is smooth and $B = \bar{X} \setminus X$ is the union of two orbits $Kt_0^5t_1$ and Kt_0^6. The first orbit is of dimension 2. The isotropy subgroup of the first orbit is isomorphic to the multiplicative group $\mathbb{C}^*$. The second orbit is one-dimensional and is contained in the closure of the first one. The isotropy subgroup is isomorphic to the subgroup of upper triangular matrices. They also show that B is equal to the image of $\mathbb{P}^1 \times \mathbb{P}^1$ under a $\mathrm{SL}(2)$-equivariant map given by a linear system of curves of bidegree $(5, 1)$. Thus B is of degree 10, hence is cut out by a quadric. The image of the second orbit is a smooth rational rational curve in B and is equal to the singular locus of B. The fact that the two varieties are isomorphic follows from the theory of Fano 3-folds. It can be shown that there is a unique Fano threefold V with $\mathrm{Pic}(V) = \mathbb{Z}\frac{1}{2}K_V$ and $K_V^3 = 40$. We will discuss this variety in a later chapter.

2.1.4 Conjugate triangles

Let $C = V(f)$ be a nonsingular conic, ℓ be a line in $\mathbb{P}^2$, and p be its pole with respect to C. From the point view of linear algebra, the one-dimensional subspace defining p is orthogonal to the two-dimensional subspace defining ℓ with respect to the symmetric bilinear form defined by f.

Given a triangle with sides ℓ_1, ℓ_2, ℓ_3, the poles of the sides are the vertices of the triangle which is called the *conjugate triangle*. Its sides are the polar lines of the vertices of the original triangle. It is clear that this defines a duality in the set of triangles. Clearly, a triangle is *self-conjugate* if and only if it is a self-polar triangle.

Let ℓ_1, ℓ_2, ℓ_3 be three tangents to C at the points p_1, p_2, p_3, respectively. They form a triangle which can be viewed as a *circumscribed triangle*. It follows from Theorem 1.1.1 that the conjugate triangle has vertices p_1, p_2, p_3. It can be viewed as an *inscribed triangle*. The lines $\ell_1' = \langle p_2, p_3 \rangle, \ell_1' = \langle p_2, p_3 \rangle, \ell_1' = \langle p_2, p_3 \rangle$ are polar lines with respect to the points q_1, q_2, q_3, respectively.

Two lines in $\mathbb{P}^2$ are called *conjugate* with respect to C if the pole of one of the lines belongs to the other line. It is a reflexive relation on the set of lines. Obviously, two triangles are conjugate if and only if each of the sides of the first triangle is conjugate to a side of the second triangle.

Now let us consider the following problem. Given two triangles without common sides, find a conic C such that the triangles are conjugate to each other with respect to the conic C. Using equations (2.3) it is easy to get a necessary and sufficient condition for this to be true. Let v_1, v_2, v_3 be the coordinate vectors of the vertices p_1, p_2, p_3 of the first triangle and w_1, w_2, w_3 the coordinate vectors of the sides of the second triangle. Let A be the symmetric matrix of the quadratic form defining the conic. We have

$$A \cdot v_i = \lambda_i w_i, \ i = 1, 2, 3 \tag{2.6}$$

for some nonzero constants $\lambda_1, \lambda_2, \lambda_3$. Taking the dot-products, we get

$$v_j \cdot (A \cdot v_i) = \lambda_i v_j \cdot w_i, \quad 1 \leq i < j \leq 3.$$

Since the matrix A is symmetric, we get 3 linear equations

$$\begin{aligned} \lambda_1 v_2 \cdot w_1 - \lambda_2 v_1 \cdot w_2 &= 0 \\ \lambda_1 v_3 \cdot w_1 - \lambda_3 v_1 \cdot w_3 &= 0 \\ \lambda_2 v_3 \cdot w_2 - \lambda_3 v_2 \cdot w_3 &= 0. \end{aligned}$$

Computing the determinant of the matrix of the coefficients of the system of linear equations, we obtain a necessary condition for the existence of the symmetric matrix A:

$$(v_2 \cdot w_1)(v_1 \cdot w_3)(v_3 \cdot w_2) - (v_3 \cdot w_1)(v_1 \cdot w_2)(v_2 \cdot w_3) = 0. \tag{2.7}$$

Conversely, taking a nonzero solution $(\lambda_1, \lambda_2, \lambda_3)$ of these equations, we obtain that the matrix

$${}^t[v_1 v_2 v_3] \cdot [w_1 w_2 w_3] \cdot D = {}^tV \cdot W \cdot D,$$

where D is the diagonal matrix corresponding to the solution, is a symmetric nonsingular matrix. Equation (2.6) can be rewritten in equivalent form

$${}^tV \cdot A \cdot V = {}^tV \cdot W \cdot D,$$

so that we can take

$$A = W \cdot D \cdot V^{-1}$$

which is obviously symmetric and nonsingular. Note equation (2.6) implies that each triangle is self-conjugate with respect to some conic. Each such conic can be diagonalized in the basis defined by the three vertices of the triangle.

Note that, by choosing projective coordinates such that $V = I_3$, the condition (2.6) is equivalent to that the matrix $W \cdot D$ is symmetric for some invertible diagonal matrix D.

Note that, without much change, we can extend this argument to the projective space $\mathbb{P}^3$ to find a necessary and sufficient condition in order that 3 points in general linear position are conjugate to 3 hyperplanes in general linear position with respect to a nonsingular quadric. As above, we assume that the coordinate vectors of the three point is equal to $V = [e_1 e_2 e_3]$, where e_1, e_2, e_3 are the first unit vectors in $\mathbb{C}^4$. Then we find a diagonal matrix D such that $W \cdot D$ with the last row deleted is symmetric. Then we extend $W \cdot D = (a_{ij})$ to a square symmetric matrix A by adding one more column $(a_{14}, a_{24}, a_{34}, a_{44})$, where a_{44} is chosen such that A is invertible. The matrix A does the job (see [406]).

Remark 2.1.2. Consider a triangle (with an order on the set of sides, or vertices) as a point in $(\mathbb{P}^2)^3$ defined by its vertices. Then its sides are points in the dual plane which are expressed as polynomials of degree 2 in coordinates of the vertices. Thus equation (2.6) implies that the closure of the locus of the ordered pairs of conjugate triangles with respect to some conic is a hypersurface in $(\mathbb{P}^2)^3 \times (\mathbb{P}^2)^3 = (\mathbb{P}^2)^6$ given by a equation of multi-degree $(1,1,1,2,2,2)$ which is anti-symmetric in the first and the last three variables.

2.2 Poncelet relation

2.2.1 Darboux's theorem

Let C be a conic, and let $T = \{\ell_1, \ell_2, \ell_3\}$ be a circumscribed triangle. A conic C' which has T as an inscribed triangle is called the *Poncelet related conic*. Since passing through a point impose one condition, we have ∞^2 Poncelet related conics corresponding to a fixed triangle T. Varying T we expect to get ∞^5 conics, so that any conic is Poncelet related to C with respect to some triangle. But surprisingly this is wrong! A theorem of Darboux asserts that there is a pencil of divisors $p_1 + p_2 + p_3$ such that the triangles T with sides tangent to C at the points p_1, p_2, p_3 define the same Poncelet related conic.

We shall prove it here. In fact, for the future use we shall prove a more general result.

Instead of circumscribed triangles we shall consider circumscribed n-polygons. An n-*polygon* P in $\mathbb{P}^2$ is an ordered set of $n \geq 3$ points $(p_1, \ldots, p_n)$ in $\mathbb{P}^2$ such that no three points p_i, p_{i+1}, p_{i+2} are colinear. The points p_i are the *vertices* of P, the lines $\overline{p_i, p_{i+1}}$ are called the *sides* of P (here $p_{n+1} = p_1$). We say that two polygons are equal if the sets of their sides are equal. The number of n-polygons with the same set of vertices is equal to $n!/2n = (n-1)!/2$.

We say that P circumscribes a nonsingular conic C if each side is tangent to C. Given any ordered set $(q_1, \ldots, q_n)$ of n points on C, let ℓ_i be the tangent lines to C at the points q_i. Then they are the sides of the n-polygon P with vertices $p_i = \ell_i \cap \ell_{i+1}, i = 1, \ldots, n$ ($\ell_{n+1} = \ell_1$). This polygon circumscribes C. This gives a one-to-one correspondence between n-polygons circumscribing C and ordered sets of n points on C.

Let $P = (p_1, \ldots, p_n)$ be an n-polygon that circumscribes a nonsingular conic C. A conic S is called *Poncelet n-related* to C with respect to P if all points p_i lie on C.

Let us start with any two conics C and S. We choose a point p_1 on S and a tangent ℓ_1 to C passing through p_1. It intersects S at another point p_2. We repeat this construction. If the process stops after n steps (i.e. we are not getting new points p_i), we get an inscribed n-polygon in S which circumscribes C. In this case S is Poncelet related to C. The *Darboux Theorem* which will prove later says that if the process stops, then we can construct infinitely many n-polygons with this property starting from an arbitrary point on S.

Consider the following correspondence on $C \times S$:

$$R = \{(x, y) \in C \times S : \overline{x, y} \text{ is tangent to } C \text{ at } x\}.$$

Since, for any $x \in C$ the tangent to C at x intersects S at two points, and, for any $y \in S$ there are two tangents to C passing through y, we get that E is of bidegree $(2, 2)$. This means if we identify C, S with $\mathbb{P}^1$, then R is a curve of bidegree $(2, 2)$. As is well-known R is a curve of arithmetic genus 1.

Lemma 2.2.1. *The curve R is nonsingular if and only if the conics C and S intersect at four distinct points. In this case, R is isomorphic to the double cover of C (or S) ramified over the four intersection points.*

Proof. Consider the projection map $\pi_S : R \to S$. This is a map of degree 2. A branch point $y \in S$ is a point such that there only one tangent to C passing through y. Obviously, this is possible only if $y \in C$. It is easy to see that R is nonsingular if and only if the double cover $\pi_S : R \to S \cong \mathbb{P}^1$ has four branch points. This proves the assertion. $\square$

Note that the second projection map $\pi_C : R \to C$ must also have 4 branch points, if R is nonsingular. A point $x \in C$ is a branch point if and only if the tangent of C at x is tangent to S. So we obtain that two conics intersect transversally if and only if there are four different common tangents.

Take a point $(x[0], y[0]) \in R$ and let $(x[1], y[1]) \in R$ be defined as follows: $y[1]$ is the second point on S on the tangent to $x[0]$, $x[1] \neq x[0]$ is the point where the tangent of C at $[x[1]$ contains $y[1]$. This defines a self-map $\tau_{C,S} : R \to R$. This map has no fixed points on R and hence, if we fix a group law on R, is a translation map t_a with respect to a point a. Obviously, we get an n-polygon if and only if t_a is of order n, i.e. the order of a in the group law is n. As soon as this happens we can use the automorphism for constructing n-polygons starting from an arbitrary point $(x[0], y[0])$. This is the Darboux Theorem which we have mentioned in above.

Theorem 2.2.2. *(G. Darboux) Let C and S be two nondegenerate conics intersecting transversally. Then C and S are Poncelet n-related if and only if the automorphism $\tau_{C,S}$ of the associated elliptic curve R is of order n. If C and S are Poncelet n related, then starting from any point $x \in C$ and any point $y \in S$ there exists an n-polygon with a vertex at y and one side tangent to C at y which circumscribes C and inscribed in S.*

In order to give a more explicit answer when two conics are Poncelte related one needs to recognize when the automorphism $\tau_{C,S}$ is of finite order. Let us choose projective coordinates such that C is the Veronese conic $t_0t_2 - t_1^2 = 0$, the image of $\mathbb{P}^1$ under the map $[t_0, t_1] \mapsto [t_0^2, t_0t_1, t_2^2]$. By using a projective transformation leaving C invariant we may assume that the four intersection points $p_1, \ldots, p_4$ of C and S correspond to the points $[1, 0], [1, 1], [0, 1], [1, a] \in \mathbb{P}^1$, where $a \neq 0, 1$. Then R is isomorphic the elliptic curve given by the affine equation

$$y^2 = x(x - 1)(x - a).$$

The conic S belongs to the pencil of conics with base points $p_1, \ldots, p_4$:

$$(t_0t_2 - t_1^2) + \lambda t_1(at_0 - (1 + a)t_1 + t_2) = 0.$$

We choose the zero point in the group law on R to be the point $(x[0], y[0]) = (p_4, p_4) \in C \times S$. Then the automorphism $\tau_{C,S}$ sends this point to $(x[1], y[1])$, where

$$y[1] = (\lambda a, \lambda(1+a)+1, 0), \quad x[1] = ((a+1)^2\lambda^2, 2a(1+a)\lambda, 4a^2).$$

Thus $x[1]$ is the image of the point $(1, \frac{2a}{(a+1)\lambda}) \in \mathbb{P}^1$ under the Veronese map. The point $y[1]$ corresponds to one of the two roots of the equation

$$y^2 = \frac{2a}{(a+1)\lambda}(\frac{2a}{(a+1)\lambda} - 1)(\frac{2a}{(a+1)\lambda} - a).$$

So we need a criterion characterizing points $(x, \pm\sqrt{x(x-1)(x-a)}$ of finite order. Note that different choice of the sign corresponds to the inversion involution on the elliptic curve. So, the order of the points corresponding to two different choices of the sign are the same. We have the following:

Theorem 2.2.3. *(A. Cayley). Let R be an elliptic curve with affine equation*

$$y^2 = g(x),$$

where $g(x)$ is a cubic polynomial with three distinct nonzero roots. Write

$$y = \sum_{k=0}^{\infty} a_k x^k.$$

Then a point $(0, \sqrt{g(0)})$ is of order n if and only if

$$\begin{vmatrix} a_2 & a_3 & \dots & a_{m+1} \\ a_3 & a_4 & \dots & a_{m+2} \\ \vdots & \vdots & \vdots & \\ a_{m+1} & a_{m+2} & \dots & a_{2m} \end{vmatrix} = 0, \quad n = 2m+1,$$

$$\begin{vmatrix} a_3 & a_4 & \dots & a_{m+1} \\ a_4 & a_5 & \dots & a_{m+2} \\ \vdots & \vdots & \vdots & \\ a_{m+1} & a_{m+2} & \dots & a_{2m} \end{vmatrix} = 0, \quad n = 2m.$$

Proof. We fix a square root c_0 of $g(0)$ and consider the point $p = (0, c_0)$. A necessary and sufficient condition for p to be a n-torsion point is that there exists a rational function f on R with a zero of order n at p and a pole of order n at the infinity point $(\infty, 0)$. We shall assume that $n = 2k-1$ is odd. The other case is considered similarly. Since f is regular on the affine part, it must be a restriction of a polynomial $f(x, y)$ of some degree d. Since the infinity is an inflection point, the degree of f must be equal to $k-1$ and $f(x, y)$ must have a zero of order $2k-1$ at $(0, c_0)$ and a pole of order $k-2$ at infinity. Now we expand $y = \sum_{k=0}^{\infty} a_k x^k$ and put

$$y_m = a_0 + a_1 x + \dots + a_{k-1} x^{k-1}.$$

We have

$$\begin{aligned} y - y_k &= a_k x^k + \cdots + a_{2k-2}x^{2k-2} + \ldots \\ x(y - y_{k-1}) &= a_{k-1}x^k + \cdots + a_{2k-3}x^{2k-2} + \ldots \\ \ldots &= \ldots \\ x^{k-2}(y - y_2) &= a_2 x^k + \cdots + a_k x^{2k-2} + \ldots . \end{aligned}$$

We can find $n-1$ coefficients $c_0, c_1, \ldots, c_{k-2}$ such that the polynomial

$$f(x,y) = c_0(y - y_k) + c_1 x(y - y_{k-1}) + \cdots + c_{k-2}x^{k-2}(y - y_2)$$

vanishes at $x = 0$ of order $2k - 1$ if and only if

$$\begin{vmatrix} a_k & a_{k-1} & \ldots & a_2 \\ a_{k+1} & a_k & \ldots & a_3 \\ \ldots & \ldots & \ldots & \ldots \\ a_{2k-2} & a_{2k-3} & \ldots & a_k \end{vmatrix} = 0.$$

It is easy to see that this determinant is equal to one of the determinants from the assertion of the theorem. $\square$

To apply the proposition we have to take

$$\alpha = \frac{2a}{(a+1)\lambda}, \quad \beta = 1 + \frac{2a}{(a+1)\lambda}, \quad \gamma = a + \frac{2a}{(a+1)\lambda}.$$

Let us consider the variety $\mathcal{P}_n$ of pairs of conics (C, S) such that S is Poncelet n-related to C. We assume that C and S intersect transversally. We already know that $\mathcal{P}_n$ is a hypersurface in $\mathbb{P}^5 \times \mathbb{P}^5$. Obviously, $\mathcal{P}_n$ is invariant with respect to the diagonal action of the group $\mathrm{SL}(3)$ (acting on the space of conics). Thus the equation of $\mathcal{P}_n$ is an invariant of a pair of conics. This invariant was computed by F. Gerbradi [184]. It is of bidegree $(\frac{1}{4}T(n), \frac{1}{2}T(n))$, where $T(n)$ is equal to the number of elements of order n in the abelian group $(\mathbb{Z}/n\mathbb{Z})^2$.

Let us look at the quotient of $\mathcal{P}_n$ by $\mathrm{PSL}(3)$. Consider the rational map $\beta : \mathbb{P}^5 \times \mathbb{P}^5 \to (\mathbb{P}^2)^{(4)}$ which assigns to (C, S) the point set $C \cap S$. The fibre of β over a subset B of 4 points in general linear position is isomorphic to an open subset of $\mathbb{P}^1 \times \mathbb{P}^1$, where $\mathbb{P}^1$ is the pencil of conics with base point B. Since we can always transform such B to the set of points $\{[1,0,0],[0,1,0],[0,0,1],[1,1,1]\}$, the group $\mathrm{PSL}(3)$ acts transitively on the open subset of such 4-point sets. Its stabilizer is isomorphic to the permutation group $\mathfrak{S}_4$ generated by the following matrices:

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix}.$$

The orbit space $\mathcal{P}_n/\mathrm{PSL}(3)$ is isomorphic to a curve in an open subset of $\mathbb{P}^1 \times \mathbb{P}^1/\mathfrak{S}_4$, where $\mathfrak{S}_4$ acts diagonally. By considering one of the projection maps, we obtain that

$\mathcal{P}_n/\mathrm{PSL}(3)$ is an open subset of a cover of $\mathbb{P}^1$ of degree N equal to the number of Poncelet n-related conics in a given pencil of conics with 4 distinct base points with respect to a fixed conic from the pencil. This number was computed by F. Gerbardi [184] and is equal to $\frac{1}{2}T(n)$. A modern account of Gerbardi's result is given in [17]. A smooth compactification of $\mathcal{P}_n/\mathrm{PSL}(3)$ is the modular curve $X^0(n)$ which parametrizes the isomorphism classes of the pairs (R, e), where R is an elliptic curve and e is a point of order n in R.

Proposition 2.2.4. *Let C and S be two nonsingular conics. Consider each n-polygon inscribed in C as a subset of its vertices, and also as a positive divisor of degree n on C. The closure of the set of n-polygons inscribed in C and circumscribing S is either empty, or a g_n^1, i.e. a linear system of divisors of degree n.*

Proof. First observe that two polygons inscribed in C and circumscribing S which share a common vertex must coincide. In fact, the two sides passing through the vertex in each polygon must be the two tangents of S passing through the vertex. They intersect C at another two common vertices. Continuing in this way we see that the two polygons have the same set of vertices. Now consider the Veronese embedding v_n of $C \cong \mathbb{P}^1$ in $\mathbb{P}^n$. An effective divisor of degree n is a plane section of the Veronese curve $\mathrm{Ver}_n^1 = v_n(\mathbb{P}^1)$. Thus the set of effective divisors of degree n on C can be identified with the dual projective space $\check{\mathbb{P}}^n$. A hyperplane in $\check{\mathbb{P}}^n$ is the set of hyperplanes in $\mathbb{P}^n$ which pass through a fixed point in $\mathbb{P}^n$. The degree of an irreducible curve $X \subset \check{\mathbb{P}}^n$ of divisors is equal to the cardinality of the set of divisors containing a fixed general point of Ver_n^1. In our case it is equal to 1. $\square$

2.2.2 Poncelet curves

Let C and S be two Poncelet n-related conics in the plane $\mathbb{P}^2 = |E|$. Recall that this means that there exist n points $p_1, \ldots, p_n$ on C such that the tangent lines $\ell_i = \mathbb{T}_{p_i}(C)$ meet on S. One can drop the condition that S is a conic. We say that a plane curve S of degree $n-1$ is *Poncelet related* to the conic C if there exist n points as above such that the tangents to C at these points meet on S.

Before we prove an analog of Darboux's Theorem for Poncelet related curves of higher degree we have to relate these curves to curves of jumping lines of some special rank 2 vector bundles on the projective plane, so called the *Schwarzenberger bundles*.

Let us write $\mathbb{P}^1 = |U|$ for some vector space of dimension 2 and $\mathbb{P}^2 = |V|$ for some vector space of dimension 3. A closed embedding $\nu : \mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ has the image isomorphic to a nonsingular conic, a Veronese curve. This defines an isomorphism

$$V^\vee = H^0(|V|, \mathcal{O}_{|V|}(1)) \cong H^0(|U|, \mathcal{O}_{|U|}(2)) = S^2U^\vee,$$

whose transpose defines an isomorphism $V \cong S^2U$. This gives a bijective correspondence between nonsingular conics and linear isomorphisms $V \to S^2U$. Also, since $\dim \bigwedge^2 U = 1$, a choice of a basis in $\bigwedge^2 U$ defines a linear isomorphism $U \cong U^\vee$. This gives an isomorphism of projective spaces $|U| \cong |U|^\vee$ which does not depend on a choice of a basis in $\bigwedge^2 U$. Thus a choice of a nonsingular conic in $|V|$ defines also an isomorphism $|V^\vee| \to |S^2U|$ which must be given by a nonsingular conic in $|V^\vee|$. This is of course the dual conic.

Fix an isomorphism $\mathbb{P}^2 \cong |S^2U|$ defined by a choice of a conic C in $\mathbb{P}^2$. Consider the multiplication map $S^2U \otimes S^{n-2}(U) \to S^nU$. It defines a rank 2 vector bundle $\mathcal{S}_{n,C}$ on $\mathbb{P}^2$ whose fibre at the point $x = [q] \in |S^2U|$ is equal to the quotient space $S^n(U)/qS^{n-2}(U)$. One easily see that it admits a resolution of the form

$$0 \to S^{n-2}(U)(-1) \to S^nU \to \mathcal{S}_{n,C} \to 0, \tag{2.8}$$

where we identify a vector space E with the vector bundle π^*E, where π is the structure map to the point. The vector bundle $\mathcal{S}_{n,C}$ is called the Schwarzenberger bundle associated to the conic C. Its dual bundle has the fibre over a point $x = [q]$ equal to the dual space

$$(S^nU/qS^{n-2}U)^\vee = \{f \in S^nU^\vee : D_q(f) = 0\} = \mathrm{Ker}(\mathrm{ap}_q). \tag{2.9}$$

If we embed the dual projective line $|U^\vee|)$ in $|S^nU^\vee|$ by means of the Veronese map, then the divisor of zeros of q can be considered as a divisor $V(q)$ of degree 2 on the Veronese curve

$$R_n \subset |S^nU^\vee|,$$

or, equivalently, as a 1-secant of R_n. A hyperplane containing this divisor is equal to $V(qg)$ for some $g \in S^{n-2}U$. Thus the space $|\mathrm{Ker}(\mathrm{ap}_q)|$ can be identified with the projective span of $V(q)$. In other words, the fibres of the dual projective bundle $\mathcal{S}^*_{n,C}$ are equal to the secants of the Veronese curve R_n.

It follows from (2.8) that the vector bundle $\mathcal{S}_{n,C}$ has the first Chern class of degree $n-1$ and the second Chern class is equal to $n(n-1)/2$. Thus we expect that a general section of $\mathcal{S}_{n,C}$ has $n(n-1)/2$ zeroes. We identify the space of sections of $\mathcal{S}_{n,C}$ with the vector space S^nU. A point $[s] \in |S^nU|$ can be viewed as a hyperplane H_s in $|S^nU^\vee|$. Its zeros are the secants of R_n contained in H_s. Since H_s intersects R_n at n-points $p_1, \ldots, p_n$, any secant $\overline{p_i, p_j}$ is a secant contained in H_s. The number of such secants is equal to $n(n-1)/2$.

Recall that we can identify the conic with $|U|$ by means of the Veronese map $\nu_2 : |U| \to |S^2U|$. Similarly, the dual conic $C^\vee$ is identified with $|U^\vee|$. By using the Veronese map $\nu_n : |U^\vee| \to |S^nU^\vee$ we can identify $C^\vee$ with R_n. Now a point on R_n is a tangent line on the original conic C, hence n points $p_1, \ldots, p_n$ from above are the sides ℓ_i of an n-polygon circumscribing C. A secant $\overline{p_i, p_j}$ from above is a point in $\mathbb{P}^2$ equal to the intersection point $q_{ij} = \ell_i \cap \ell_j$. And the $n(n-1)/2$ points q_{ij} represent the zeros of a section s of the Schwarzenberger bundle $\mathcal{S}_{n,C}$.

For any two linearly independent sections s_1, s_2, their determinant $s_1 \wedge s_2$ is a section of $\bigwedge^2 \mathcal{S}_{n,C}$ and hence its divisor of zeros belongs to $|\mathcal{O}_{\mathbb{P}^2}(n-1)|$. When we consider the pencil $\langle s_1, s_2\rangle$ spanned by the two sections, the determinant of each member $s = \lambda s_1 + \mu s_2$ has the zeros on the same curve $V(s_1 \wedge s_2)$ of degree $m-1$.

Let us summarize this discussion by stating and the proving the following generalization of Darboux's Theorem.

Theorem 2.2.5. *Let C be a nonsingular conic in $\mathbb{P}^2$ and $\mathcal{S}_{n,C}$ be the associated Scwarzenberger rank 2 vector bundle over $\mathbb{P}^2$. Then n-polygons circumscribing C*

are parameterized by $|\Gamma(\mathcal{S}_{n,C})|$. The vertices of the polygon Π_s defined by a section s correspond to the subscheme $Z(s)$ of zeros of the section s. A curve of degree $n-1$ passing through the vertices corresponds to a pencil of a sections of $\mathcal{S}_{n,C}$ containing s and is equal to the determinant of a basis of the pencil.

Proof. A section s with the subscheme of zeros $Z(s)$ with ideal sheaf $\mathcal{I}_{Z(s)}$ defines the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2} \xrightarrow{s} \mathcal{S}_{n,C} \to \mathcal{I}_Z(n-1) \to 0.$$

A section of $\mathcal{I}_Z(n-1)$ is a plane curve of degree $n-1$ passing through $Z(s)$. The image of a section t of $\mathcal{S}_{n,C}$ in $\Gamma(\mathcal{I}_Z(n-1))$ is the discriminant curve $s\wedge t$. Any curve defined by an element from $\Gamma(\mathcal{I}_Z(n-1))$ passes through the vertices of the n-polygon Π_s and is uniquely determined by a pencil of sections containing s. $\square$

One can explicitly write the equation of a Poncelet curve as follows. First we choose a basis ξ_0, ξ_1 of the space U and the corresponding basis $(\xi_0^d, \xi_0^{d-1}\xi_1, \ldots, \xi_1^d)$ of the space S^dU. The dual basis in $S^nU^\vee$ is $(\binom{d}{i}t_0^{d-i}t_1^i)_{0\le i\le d}$. Now the coordinates in the plane $|S^2U|$ are $t_0^2, 2t_0t_1, t_2^2$, so a point in the plane is a binary conic $Q = a\xi_0^2 + 2b\xi_0\xi_1 + c\xi_1^2$. For a fixed $x = [Q] \in |S^2U|$, the matrix of the multiplication map $S^{n-2}U \to S^nU, G \mapsto QG$ is

$$K(x) = \begin{pmatrix} a & & & \\ 2b & a & & \\ c & 2b & \ddots & \\ & c & \ddots & \ddots & \\ & & \ddots & \ddots & a \\ & & & \ddots & 2b \\ & & & & c \end{pmatrix}$$

A section of $\mathcal{S}_{n,C}$ is given by $f = \sum_{i=0}^n c_i\xi_0^{n-i}\xi_1^i \in S^nU$. Its zeros is the set of points x such that the vector **c** of the coefficients belongs to the column subspace of the matrix $K(x)$. Now we vary f in a pencil of binary forms whose coefficient vector **c** belongs to the nullspace of some matrix A of size $(n-1)\times(n+1)$ and rank $n-1$. The determinant of this pencil of sections is the curve in the plane defined by the degree $n-1$ polynomial equation in $x = [a, b, c]$

$$\det\big(K(x)\cdot A\big) = 0.$$

Note that the conic C corresponding to our choice of coordinates is $V(t_1^2 - t_0t_2)$.

Remark 2.2.1. Recall that a section of $\mathcal{S}_{n,C}$ defines a n-polygon in the plane $|S^2U|$ corresponding to the hyperplane section $H_s \cap R_n$. Its vertices is the scheme of zeros $Z(s)$ of the section s. Let $\pi : X(s) \to \mathbb{P}^2$ be the blow-up of $Z(s)$. For a general s, the linear system of Poncelet curves through $Z(s)$ embeds the surface $X(s)$ in $|S^nU^\vee|$ with the image equal to $H_s \cap \mathrm{Sec}_1(R_n)$. The exceptional curves of the blow-up are mapped onto the secants of R_n which are contained in H_s. These are the secants

$\overline{p_i, p_j}$, where $H_s \cap R_n = \{p_1, \ldots, p_n\}$. The linear system defining the embedding is the proper transform of the linear system of curves of degree $n-1$ passing through $\frac{1}{2}n(n-1)$ points of $Z(s)$. This implies that the embedded surface $X(s)$ has the degree equal to $(n-1)^2 - \frac{1}{2}n(n-1) = \frac{1}{2}(n-1)(n-2)$. This is also the degree of the secant variety $\mathrm{Sec}_1(R_n)$. For example, take $n = 4$ to get that the secant variety of R_4 is a cubic hypersurface in $\mathbb{P}^4$ whose hyperplane sections are cubic surfaces isomorphic to the blow-up of the six vertices of a complete quadrilateral.

2.2.3 Invariants of pairs of conics

The Poncelet Theorem is an example of a *porism* which can be loosely stated as follows. If one can find one object satisfying a certain special property then there are infinitely many such objects. In case of Darboux's Theorem this is the property of the existence of a polygon inscribed in one conic and circumscribing the other conic. Here we consider another example of a porism between two conics. This time the relation is the following.

Given two nonsingular conics C and S there exists a self-conjugate triangle with respect to C which is inscribed in S. We say that the two conics are *conjugate* or *apolar*.

Proposition 2.2.6. *Let S and C be two nonsingular conics defined by symmetric matrices A and B respectively. Then C admits a self-conjugate triangle which is inscribed in S if and only if*

$$\mathrm{Tr}(AB^{-1}) = 0.$$

Moreover, if this condition is satisfied, for any point $x \in S \setminus (S \cap C)$ there exists a self-conjugate triangle inscribed in S with vertex at x.

Proof. Let Q be an invertible 3×3 matrix. Replacing A with $A' = Q^T AQ$ and B with $B' = Q^T BQ$ we check that

$$\mathrm{Tr}(A'B'^{-1}) = \mathrm{Tr}(Q^T AB^{-1}(Q^T)^{-1}) = \mathrm{Tr}(AB^{-1}).$$

This shows that the trace condition is invariant with respect to a linear change of variables. Thus we may assume that $C = V(t_0^2 + t_1^2 + t_2^2)$. Suppose there is a self-conjugate triangle with respect to C which is inscribed in S. Since the orthogonal group of C acts transitively on the set of self-conjugate triangles, we may assume that the triangle is the coordinate triangle. Then the points $[1, 0, 0]$, $[0, 1, 0]$, and $[0, 0, 1]$ must be on S. Hence

$$S = V(at_0t_1 + bt_0t_2 + ct_1t_2),$$

and the condition $\mathrm{Tr}(AB^{-1})$ is verified.

Let us show the sufficiency of the trace condition. Choose coordinates as above. Let

$$S = V(at_0^2 + bt_1^2 + ct_2^2 + 2dt_0t_1 + 2et_0t_2 + 2ft_1t_2). \tag{2.10}$$

The trace condition is

$$a + b + c = 0.$$

Let $x = [x_0, x_1, x_2]$ be any point on S and $\ell = V(x_0t_0 + x_1t_1 + x_2t_2)$ be the polar line $P_x(C)$. Without loss of generality, we may assume that $x_2 = -1$ so that we can write $t_2 = x_0t_0 + x_1t_1$ and take t_0, t_1 as coordinates on ℓ. The line ℓ intersects S at two points $[c_0, c_1]$ and $[d_0, d_1]$ which are the zeros of the binary form

$$q = at_0^2 + bt_1^2 + c(x_0t_0 + x_1t_1)^2 + 2dt_0t_1 + 2(et_0 + ft_1)(x_0t_0 + x_1t_1)$$

$$= (a + 2ex_0 + cx_0^2)t_0^2 + (b + 2fx_1 + cx_1^2)t_1^2 + 2(d + ex_1 + fx_0 + cx_0x_1)t_0t_1.$$

The line ℓ intersects C at the points

$$y = (a_0, a_1, x_0a_0 + x_1a_1),\ z = (b_0, b_1, b_0x_0 + b_1x_1).$$

Their coordinates on the line ℓ are the zeros of the binary form

$$q' = t_0^2 + t_1^2 + (x_0t_0 + x_1t_1)^2 = (1 + x_0^2)t_0^2 + 2x_0x_1t_0t_1 + (1 + x_1^2)t_1^2.$$

It follows from Proposition 2.1.1 that the points x, y, z are the vertices of a self-polar triangle if and only if (2.1) holds. To check this condition we will use that $a+b+c = 0$ and $ax_0^2 + bx_1^2 + c + dx_0x_1 - ex_0 - fx_1 = 0$. We have

$$(a+2ex_0+cx_0^2)(1+x_1^2)+(b+2fx_1+cx_1^2)(1+x_0^2)-2(d+ex_1+fx_0+cx_0x_1)x_0x_1$$

$$= a + b + 2ex_0 + cx_0^2 + 2fx_1 + cx_1^2 + (a + 2ex_0 + cx_0^2)x_1^2 + (b + 2fx_1 + cx_1^2)x_0^2$$
$$-2(d + ex_1 + fx_0 + cx_0x_1)x_0x_1$$

Replacing $a + b$ with $-c = -ax_0^2 - bx_1^2 - 2dx_0x_1 + 2ex_0 + 2fx_1$ we check that the sum is equal to zero. Thus starting from any point x on S we find that the triangle with vertices x, y, z is self-conjugate with respect to C. □

Remark 2.2.2. Let $\mathbb{P}^2 = |E|$ and $C = V(q), S = V(f)$, where $q, f \in S^2E^\vee$. Let $\check{C} = V(\psi)$, where $\psi \in S^2(E)$. Then the trace condition from Proposition 2.2.6 is

$$\langle \psi, f\rangle = 0, \tag{2.11}$$

where the pairing is the polarity pairing (1.2). In other words, C is conjugate to S if and only if the dual conic of S is apolar to C.

Consider the set of self-polar triangles with respect to C inscribed in S. We know that this set is either empty or of dimension ≥ 1. We consider each triangle as a set of its 3 vertices, i.e. as an effective divisor of degree 3 on S.

Proposition 2.2.7. *The closure X of the set of self-polar triangles with respect to C which are inscribed in S, if not empty, is a g_3^1, i.e. a linear system of divisors of degree 3.*

Proof. First we use that two self-polar triangles with respect to C and inscribed in S which share a common vertex must coincide. In fact, the polar line of the vertex must intersect S at the vertices of the triangle. Then the assertion is proved using the argument from the proof of Proposition 2.2.4. □

Note that a general g_3^1 contains 4 singular divisors corresponding to ramification points of the corresponding map $\mathbb{P}^1 \to \mathbb{P}^1$. In our case these divisors correspond to 4 intersection points of C and S.

Another example of a poristic statement is the following.

Theorem 2.2.8. *Let T and T' be two different triangles. The following assertions are equivalent:*

(i) there exists a conic S containing the vertices of the two triangles;

(ii) there exists a conic Σ touching the sides of the two triangles;

(iii) there exists a conic C with respect to which each of the triangles is self-polar.

Moreover, when one of the conditions is satisfied, there is an infinite number of triangles inscribed in S, circumscribed around Σ, and all of these triangles are self-polar with respect to C.

Proof. (iii)⇔ (ii) Let $[l_1], [l_2], [l_3]$ and $[m_1], [m_2], [m_3]$ be the sides of the two triangles considered as points in the dual plane $\check{\mathbb{P}}^2$. Consider the linear systems $V = |\mathcal{O}_{\check{\mathbb{P}}^2}(2) - [l_1] - [l_2[-[l_3]|$ and $W = |\mathcal{O}_{\check{\mathbb{P}}^2}(2) - [m_1] - [m_2] - [m_3]|$ of conics passing through the corresponding points. Let $C = V(f)$. We can write

$$f = a_1 l_1^2 + a_1 l_2^2 + a_3 l_3^2 = b_1 m_1^2 + b_2 m_2^2 + b_3 m_3^2$$

for some scalars a_i, b_i. For any $V(\psi) \in V \cup W$ we have $\langle \psi, f \rangle = 0$. This shows that the span of V and W in $|\mathcal{O}_{\check{\mathbb{P}}^2}(2)|$ is contained in a hyperplane orthogonal to f. Thus $V \cap W \neq \emptyset$ and a common conic vanishes at all l_i's and m_i's. Hence the dual conic Σ is touching the sides of the two triangles. Reversing the arguments, we find that condition (ii) implies that there exists a conic $V(f)$ such that $\langle \psi, f \rangle = 0$ for any $V(\psi) \in V \cup W$. Since, for any $V(\psi) \in V \cup W$, $\langle \psi, l_i^2 \rangle = \langle \psi, l_i^2 \rangle = 0$, we obtain that f belongs to the linear span of l_1^2, l_2^2, l_3^2, and also to the linear span of m_1^2, m_2^2, m_3^2. This proves the equivalence of (ii) and (iii). More details for this argument can be seen in the later chapter about the apolarity theory.

(iii)⇔ (i) This follows from Proposition 2.1.2.

Let us prove the last assertion. Suppose one of the conditions of the Theorem is satisfied. Then we have the conics C, S, Σ with the asserted properties with respect to the two triangles T, T'. By Proposition 2.2.7, the set of self-polar triangles with respect to C inscribed in S is a g_3^1. By Proposition 2.2.4, the set of triangles inscribed in S and circumscribing Σ is also a g_3^1. Two g_3^1's with 2 common divisors coincide. □

Let $C = V(f)$ and $S = V(g)$ be two conics (not necessary nonsingular). Consider the pencil $V(t_0 f + t_1 g)$ of conics spanned by C and S. The zeros of the discriminant equation $D = \text{discr}(t_0 f + t_1 g) = 0$ correspond to singular conics in the pencil. In coordinates, if f, g are defined by symmetric matrices $A = (a_{ij}), B = (b_{ij})$, respectively, then $D = \det(t_0 A + t_1 B)$ is a homogeneous polynomial of degree ≤ 3. Choosing different system of coordinates replaces A, B by $Q^T AQ, Q^T BQ$, where Q is an invertible matrix. This replaces D with $\det(Q)^2 D$. Thus the coefficients of D are invariants on the space of pairs of quadratic forms on $\mathbb{C}^3$ with respect to the action of the group $\mathrm{SL}(3)$. To compute D explicitly, we use the following formula for the determinant of the sum of two $n \times n$ matrices $X + Y$:

$$\det(X+Y) = \sum_{k=1}^{n} \sum_{1 \le i_1 < \ldots < i_k \le n} \Delta_{i_1,\ldots,i_k}, \tag{2.12}$$

where $\Delta_{i_1,\ldots,i_k}$ is the determinant of the matrix obtained from X by replacing the columns $X_{i_1}, \ldots, X_{i_k}$ with the columns $Y_{i_1}, \ldots, Y_{i_k}$. Applying this formula to our case, we get

$$D = \Delta t_0^3 + \Theta t_0^2 t_1 + \Theta' t_0 t_1^2 + \Delta' t_1^3, \tag{2.13}$$

where

$$\begin{aligned} \Delta &= \det A \qquad (2.14)\\ \Theta &= \det(A_1 A_2 B_3) + \det(A_1 B_2 A_3) + \det(B_1 A_2 A_3) = \mathrm{Tr}(B \cdot \mathrm{adj}(A))\\ \Theta' &= \det(B_1 B_2 A_3) + \det(B_1 A_2 B_3) + \det(A_1 B_2 B_3) = \mathrm{Tr}(A \cdot \mathrm{adj}(B))\\ \Delta' &= \det(B) \end{aligned}$$

where adj means the adjugate matrix of complementary minors. We immediately recognize the geometric meanings of vanishing of the coefficients of D.

The coefficient Δ (resp. Δ') vanishes if and only if C (resp. S) is a singular conic.

If Δ, Δ' are nonzero, then the coefficient Θ (resp. Θ') vanishes if and only if there exists a self-polar triangle of C inscribed in S (resp. a self-polar triangle of S inscribed in C). This follows from Proposition 2.2.6.

We can also express the condition that the two conics are Poncelet related.

Theorem 2.2.9. *Let C and S be two nonsingular conics. A triangle inscribed in C and circumscribing S exists if and only if*

$$\Theta'^2 - 4\Theta\Delta' = 0.$$

Proof. Choose a coordinate system such that $C = V(t_0t_1 + t_1t_2 + t_0t_2)$. Suppose there is a triangle inscribed in C and circumscribing S. Applying an orthogonal transformation, we may assume that the vertices of the triangle are the references points $[1,0,0], [0,1,0]$ and $[0,0,1]$. Let $S = V(g)$, where

$$g = at_0^2 + bt_1^2 + ct_2^2 + 2dt_0t_1 + 2et_0t_2 + 2ft_1t_2. \tag{2.15}$$

The condition that the triangle circumscribes S is that the points $[1,0,0], [0,1,0]$, and $[0,0,1]$ lie on the dual conic $\check{S}$. This implies that the diagonal entries $bc - f^2, ac - e^2, ab - d^2$ of the matrix $\mathrm{adj}(B)$ are equal to zero. Therefore, we may assume that

$$g = \alpha^2 t_0^2 + \beta^2 t_1^2 + \gamma^2 t_2^2 - 2\alpha\beta t_0t_1 - 2\alpha\gamma t_0t_2 - 2\beta\gamma t_1t_2. \tag{2.16}$$

We get

$$\Theta' = \mathrm{Tr}\Bigg(\begin{pmatrix} 0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 2\alpha\beta\gamma^2 & 2\alpha\gamma\beta^2\\ 2\alpha\beta\gamma^2 & 0 & 2\beta\gamma\alpha^2\\ 2\alpha\gamma\beta^2 & 2\beta\gamma\alpha^2 & 0 \end{pmatrix}\Bigg) = 4\alpha\beta\gamma(\alpha+\beta+\gamma),$$

$$\Theta = \mathrm{Tr}\Big(\begin{pmatrix} \alpha^2 & -\alpha\beta & -\alpha\gamma \\ -\alpha\beta & \beta^2 & -\beta\gamma \\ -\alpha\gamma & -\beta\gamma & \gamma^2 \end{pmatrix}\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}\Big) = -(\alpha+\beta+\gamma)^2,$$
$$\Delta' = -4(\alpha\beta\gamma)^2.$$

This checks that $\Theta'^2 - 4\Theta\Delta' = 0$.

Let is prove the sufficiency of the condition. Take a tangent line ℓ_1 to S intersecting C at two points x, y and consider tangent lines ℓ_2, ℓ_3 to S passing through x and y, respectively. The triangle with sides ℓ_1, ℓ_2, ℓ_3 circumscribes S and has two vertices on C. Choose the coordinates such that this triangle is the coordinate triangle. Then, we may assume that $C = V(at_0^2 + 2t_0t_1 + 2t_1t_2 + 2t_0t_2)$ and $S = V(g)$, where g is as in (2.16). Computing $\Theta'^2 - 4\Theta\Delta'$ we find that it is equal to zero if and only if $a = 0$. Thus the coordinate triangle is inscribed in C. $\square$

Remark 2.2.3. Choose a coordinate system such that $C = V(t_0^2 + t_1^2 + t_2^2)$. Then the condition that S is Poncelet related to C with respect to triangles is easily seen to be equal to

$$c_2^2 - c_1c_3 = 0,$$

where

$$\det(A - tI_3) = (-t)^3 + c_1(-t)^2 + c_2(-t) + c_3$$

is the characteristic polynomial of a symmetric matrix A defining S. This is a quartic hypersurface in the space of conics. The polynomials c_1, c_2, c_3 generate the algebra of invariants of $S^2(\mathbb{C}^3)^\vee$ with respect to the group $\mathrm{SL}(3)$.

2.2.4 The Salmon conic

One call also look for *covariants* or *contravariants* of a pair conics, that is, rational maps $|\mathcal{O}_{\mathbb{P}^2}(2)| \times |\mathcal{O}_{\mathbb{P}^2}(2)| \to |\mathcal{O}_{\mathbb{P}^2}(d)|$ or $|\mathcal{O}_{\mathbb{P}^2}(2)| \times |\mathcal{O}_{\mathbb{P}^2}(2)| \to |\mathcal{O}_{\mathbb{P}^2}(d)|^*$ which are defined geometrically, i.e. not depending on bases of the projective spaces involved.

Recall the definition of the *cross ratio* of four distinct ordered points $p_i = [a_i, b_i]$ on $\mathbb{P}^1$

$$R = [p_1p_2; p_3, p_4] = (p_1 - p_2)(p_3 - p_4)/(p_1 - p_3)(p_2 - p_4), \tag{2.17}$$

where

$$p_i - p_j = a_ib_j - a_jb_i.$$

It is immediately checked that the cross ratio does not take the values $0, 1, \infty$. It does not depend on the choice of projective coordinates. It is also invariant under a permutation of the four points equal to the product of two commuting transpositions. The permutation (12) changes R to $-R/(1-R)$ and the permutation (23) changes r to $1/r$. Thus there are at 6 possible cross ratios for an ordered set of 4 points

$$R,\ \frac{1}{R},\ 1-R,\ \frac{1}{1-R},\ \frac{R}{R-1},\ \frac{R-1}{R}.$$

The number of distinct cross ratios may be reduced to three or two. The first case happens if and only if one of them is equal to -1 (the other ones will be 2 and $1/2$).

The unordered set of four points in this case is called a *harmonic quadruple*. The second case happens when R satisfies $R^2+R+1=0$, i.e. R is one of two third roots of 1 not equal to 1. In this case we have *equianharmonic qudruple*.

Two pairs of points $\{p_1,p_2\}$ and $\{q_1,q_2\}$ are harmonic conjugate in sense of definition (2.1) if and only if $R(p_1q_1;q_2p_2) = -1$. To check this, we use that both definitions are projectively invariant , so we can assume that $\{p_1,p_2\} = \{0,\infty\}$. Then the other pair becomes a pair of complex numbers $z,-z$ and we check that the cross ratio $[0z;-z\infty] = -1$. We leave to prove the converse to the reader.

If we identify the projective space of binary forms of f degree 2 with the projective plane, the relation (2.1) can be viewed as a symmetric hypersurface H of bidegree $(1,1)$ in $\mathbb{P}^2\times\mathbb{P}^2$. In particular, it makes sense to speak about harmonic conjugate pairs of maybe coinciding points. We immediately check that a double point is harmonic conjugate to a pair of points if and only if it coincides with one of the roots of this form.

We can extend the definition of the cross ratio to any set of points no three of which coincide by considering the cross ratios as the point

$$\mathbf{R} = [(p_1-p_2)(p_3-p_4),(p_1-p_3)(p_2-p_4)] \in \mathbb{P}^1. \tag{2.18}$$

It is easy to see that two points coincide if and only if $\mathbf{R} = [0,1],[1,1],[1,0]$. This corresponds to $R = 0,1,\infty$.

The expression in the left-hand side of this formula is the invariant of a pair (f,g) of binary quadratic forms defined by taking the coefficient at t for the discriminant invariant of $f+tg$. It is analogous to the invariants Θ and Θ' for a pair of conics.

Now let $C = V(f)$ and $S = V(g)$ be a pair of conics. Consider the pencil of conics $C(\lambda,\mu) = V(\lambda f+\mu g)$. Write the equation of the dual conic $C(\lambda,\mu)^*$ in the form $A\lambda^2+\psi\lambda\mu+B\mu^2 = 0$. It is easy to see that $V(A) = C^\vee$ and $V(B) = S^*$ and $V(\psi)$ is the conic in the dual space defined by the symmetric matrix whose ij-entry is equal to the coefficient at $\lambda\mu$ in $\det(\lambda a_{ij}+\mu b_{ij})$, where $(a_{ij}),(b_{ij})$ are the matrices defining the dual conics.

Considering a pencil of lines as a $\mathbb{P}^1$ one can define a cross ratio of 4 ordered lines in a pencil. Four lines in a pencil define a *harmonic pencil* if the first two lines are harmonic conjugate to the last two lines. An example of a harmonic pencil is the set of lines $t_1=0,t_2=0,t_1-t_2=0,t_1+t_2=0$..

A *Salmon conic* associated to a pair of conics C and S is defined to be the locus of points x in $\mathcal{O}_{\mathbb{P}^2}$ such that the pairs of the tangents through x to C and to S are harmonic conjugate. Note that it makes sense even x lies on one of the conics, we consider the corresponding tangent as the double tangent.

Let us see that this locus is indeed a conic. The dual statement is that the locus of lines which intersect two conics at two pairs of harmonic conjugate pairs of points is a conic in the dual plane. We use the computations from the proof of Proposition 2.2.6. Without loss of generality, we assume that C is given by the equation $t_0^2+t_1^2+t_2^2=0$ and another one is given by a full equation (2.10). We work in the open subset $\alpha_2\neq 0$ to use t_0 and t_1 as the coordinates on ℓ. The condition that the line $\ell = V(\alpha_0t_0+\alpha_1t_1+\alpha_2t_2)$ intersects C and S at harmonic conjugate pairs of points is

$$(a\alpha_2^2+2e\alpha_2\alpha_0+c\alpha_0^2)(\alpha_2^2+\alpha_1^2)+(b\alpha_2^2+2f\alpha_2\alpha_1+c\alpha_1^2)(\alpha_2^2+\alpha_0^2)-$$

$$2(d\alpha_2^2 + e\alpha_1\alpha_2 + f\alpha_0\alpha_2 + c\alpha_0\alpha_1)\alpha_0\alpha_1 = 0.$$

It is easy to see that α_2^2 factors out from the left-hand-side leaving us with the equation of a conic

$$(b+c)\alpha_0^2 + (a+c)\alpha_1^2 + (a+b)\alpha_2^2 + 2e\alpha_0\alpha_1 + 2f\alpha_1\alpha_2 - 2d\alpha_0\alpha_1 = 0. \quad (2.19)$$

This is the equation of the dual of the Salmon conic.

The following is a remarkable property of the Salmon conic.

Theorem 2.2.10. *Let C and S be two conics such that the dual conics intersect at four distinct points representing the four common tangents of C and S. Then the eight tangency points lie on the Salmon conic associated with C and S.*

Proof. Let x be a point where the Salmon conic meets C. Then the tangent line ℓ through x to C represents a double line in the harmonic pencil formed by the four tangents through x to C and S. As we remarked before the conjugate pair of lines must contain ℓ. Thus ℓ is a common tangent to C and S and hence x is one of the eight tangency points. Conversely, the argument is reversible and shows that every tangency point lies on the Salmon conic. □

Remark 2.2.4. The Salmon conic F is obviously a covariant of a pair of conics C, S. Its dual conic $F^\vee$ is a contravariant of C, S. The *jacobian* $J(C, C', F)$ defined by the determinant of the jacobian matrix of the three quadratic polynomials is an example of a covariant of degree 3. Similarly, we get the contravariant of degree 3 equal to the jacobian of the dual conics. It is proven in [191] that any covariant of C, S is given by a polynomial in homogeneous forms defining $C, S, F, J(C, S, F)$. Similarly, any contravariant of C, S is given by a polynomial in homogeneous forms defining $C^\vee, S^\vee, F^\vee, J(C^\vee, S^\vee, F^\vee)$.

Exercises

2.1 Let E be a vector space of even dimension $n = 2k$ over a field K of characteristic 0 and $(e_1, \ldots, e_n)$ be a basis in E. Let $\omega = \sum_{i<j} a_{ij} e_i \wedge e_j \in \bigwedge^2 E^\vee$ and $A = (a_{ij})_{1 \le i \le j \le n}$ be the skew-symmetric matrix defined by the coefficients a_{ij}. Let $\bigwedge^k(\omega) = \omega \wedge \cdots \wedge \omega = ak! e_1 \wedge \cdots \wedge e_n$ for some $a \in F$. The element a is called the *pfaffian* of A and is denoted by $\mathrm{Pf}(A)$.

(i) Show that
$$\mathrm{Pf}(A) = \sum_{S \in \mathcal{S}} \epsilon(S) \prod_{(i,j) \in S} a_{ij},$$
where S is a set of pairs $(i_1, j_1), \ldots, (i_k, j_k)$ such that $1 \le i_s < j_s \le 2k, s = 1, \ldots, k$, $\{i_1, \ldots, i_k, j_1, \ldots, j_k\} = \{1, \ldots, n\}$, $\mathcal{S}$ is the set of such sets S, $\epsilon(S) = 1$ if the permutation $(i_1, j_1, \ldots, i_k, j_k)$ is even and -1 otherwise.

(ii) Compute $\mathrm{Pf}(A)$ when $n = 2, 4, 6$.

(iii) Show that, for any invertible matrix C,
$$\mathrm{Pf}({}^tC \cdot A \cdot C) = \det(C)\mathrm{Pf}(A).$$

(iv) Using (iii) prove that
$$\det(A) = \mathrm{Pf}(A)^2.$$

(iv) Show that
$$\mathrm{Pf}(A) = \sum_{i=1}^{n}(-1)^{i+j-1}\mathrm{Pf}(A_{ij})a_{ij},$$
where A_{ij} is the matrix of order $n-2$ obtained by deleting the i-th and j-th rows and columns of A.

(v) Let B be a skew-symmetric matrix of odd order $2k-1$ and B_i be the matrix of order $2k-2$ obtained from B by deleting the i-th row and i-th column. Show that the vector $(\mathrm{Pf}(B_1),\ldots,(-1)^{i+1}\mathrm{Pf}(B_i),\ldots,\mathrm{Pf}(B_{2k-1}))$ is a solution of the equation $B\cdot x = 0$.

(vi) Show that the rank of a skew-symmetric matrix A of any order n is equal to the largest m such that there exists $i_1 \ldots < i_m$ such that the matrix $A_{i_1\ldots i_m}$ obtained from A by deleting i_j-th rows and columns, $j = 1,\ldots,m$, has nonzero pfaffian .

2.2 Let P be a trisecant plane in the space of conics to the Veronese variety of double lines. Consider it as a point in the Grassmannian $G_1(|S^2E^\vee|) \cong G_1(\mathbb{P}^4)$. Show that the plane of hyperplanes through P, considered as a point in the dual Grassmannian $G_1(|S^2E|)$, is a 2-dimensional linear system of conics in the dual plane $E^\vee|$ with 3 base points corresponding to the double lines in P.

2.3 Let $V = \nu_2(\mathbb{P}^2)$ be a Veronese surface in $\mathbb{P}^5$.

(i) Show that a general 3-dimensional subspace L intersects V at 4 points.

(ii) Let P be a plane in $\mathbb{P}^5$ and L_P be the 2-dimensional linear system (a *net*) of conics $\nu_2^*(H)$ in $\mathbb{P}^2$, where H is a hyperplane in $\mathbb{P}^5$ containing P. Show that P is a trisecant plane if and only the set of base points of L_P consists of 3 points (counting with multiplicities). Conversely, the linear system of conics through 3 points defines a unique trisecant plane.

(iii) Show that the set of nets of conics with three base points (a subvariety of the Grassmannian of 2-planes in the space of conics) contains an irreducible divisor parameterizing nets with 3 distinct collinear points and an irreducible divisor parameterizing nets with 2 base points, one of them is infinitely near.

(iv) Using (iii) show that the anticanonical divisor of degenerate triangles is irreducible.

(v) Show that the trisecant planes intersecting the Veronese plane at one point (corresponding to net of conics with one base point of multiplicity 3) define a smooth rational curve in the boundary of the variety of self-polar triangles. Show that this curve is equal to the set of singular points of the boundary.

2.4 Let $U \subset (\mathbb{P}^2)^{(3)}$ be the subset of the symmetric product of $\mathbb{P}^2$ parameterizing the sets of three distinct points. For each set $Z \in U$ let L_Z be the linear system of conics containing Z. Consider the map $f : U \to G_1(\mathbb{P}^4)$, $Z \mapsto L_Z \in |\mathcal{O}_{\mathbb{P}^2}(2)|$.

(i) Consider the divisor D in U parameterizing sets of 3 distinct collinear points. Show that $f(D)$ is a closed subvariety of $G_1(\mathbb{P}^4)$ isomorphic to $\mathbb{P}^2$.

(ii) Show that the map f extends to the Hilbert scheme $(\mathbb{P}^2)^{[3]}$ of 0-cycles Z with $h^0(\mathcal{O}_Z) = 3$ (which admits a natural map $\pi : (\mathbb{P}^2)^{[3]} \to (\mathbb{P}^2)^{(3)}$ which is a resolution of singularities).

(iii) Show that the closure $\bar{D}$ of $\pi^{-1}(D)$ in the Hilbert scheme is isomorphic to a $\mathbb{P}^3$-bundle over $\mathbb{P}^2$ and the restriction of f to $\bar{D}$ is the projection map to its base. item[(iv)] Define the map $\tilde{f} : \mathcal{P} \to |\mathcal{O}_{\mathbb{P}^2}(2)|$ which assigns to a point in the fibre $p^{-1}(Z)$ the corresponding conic in the net of conics though Z. Show that the fibre of $\tilde{f}$ over a nonsingular conic C is isomorphic to the Fano variety of self-polar triangles of the dual conic $C^\vee$.

(v) Let $\mathcal{P}^s = \tilde{f}^{-1}(\mathcal{D}_2(2))$ be the preimage of the hypersurface of singular conics. Describe the fibres of the projections $p : \mathcal{P}^s \to (\mathbb{P}^2)^{[3]}$ and $\tilde{f} : \mathcal{P}^s \to \mathcal{D}_2(2)$.

2.5 Prove that the n-th symmetric product of $\mathbb{P}^n$ is a rational variety.

2.6 Two points x, y are called *conjugate* with respect to a nonsingular conic C if the line $\overline{x, y}$ intersects C at two points which are harmonic conjugate to x, y. Prove that x and y are conjugate if and only if $y \in P_x(C)$ and $x \in P_y(C)$.

2.7 Prove that two unordered pairs $\{a, b\}, \{c, d\}$ of points in $\mathbb{P}^1$ are harmonic conjugate if and only if there is an involution of $\mathbb{P}^1$ with fixed points a, b that switches c and d.

2.8 Prove the following *Hesse theorem*. If two pairs of opposite vertices of a quadrilateral are each conjugate for a conic, then the third pair is also conjugate. Such a quadrilateral is called a *Hesse quadrilateral*. Show that four lines form a polar quadrilateral for a conic if and only if it is a Hesse quadrilateral.

2.9 Show that any polar triangle of a conic can be extended to a polar quadrilateral.

2.9 Extend Darboux's Theorem to the case of two conics which do not intersect transversally.

2.10 Show that the secant lines of a Veronese curve R_m in $\mathbb{P}^m$ are parameterized by the surface in the Grassmannian $G_1(\mathbb{P}^m)$ isomorphic to $\mathbb{P}^2$. Show that the embedding of $\mathbb{P}^2$ into the Grassmannian is given by the Schwarzenberger bundle.

2.11 Let U be a 2-dimensional vector space. Use the construction of curves of degree $n-1$ Poncelet related to a conic to exhibit an isomorphism of linear representations $\bigwedge^2(S^nU)$ and $S^{n-1}(S^2U)$ of SL(U).

2.12 Assume that the pencil of sections of the Schwarzenberger bundle $\mathcal{S}_{n,C}$ has no base points. Show that the Poncelet curve associated to the pencil is nonsingular at a point x defined by a section s from the pencil if and only if the scheme of zeros $Z(s)$ is reduced.

2.13 Find the geometric interpretation of vanishing of the invariants Θ, Θ' from (2.13) in the case when C or S is a singular conic.

2.14 Express the condition that two conics are tangent in terms of the invariants $\Delta, \Delta', \Theta, \Theta'$.

2.15 Let p_1, p_2, p_3, p_4 be four distinct points on a nonsingular conic C. Let $\overline{p_i, p_j}$ denote the line through the points p_i, p_j. Show that the triangle with the vertices $A = \overline{p_1, p_3} \cap \overline{p_2, p_4}$, $B = \overline{p_1, p_2} \cap \overline{p_3, p_4}$ and $C = \overline{p_1, p_4} \cap \overline{p_2, p_3}$ is a self-conjugate triangle with respect to C.

2.16 Show that two pairs $\{a, b\}, \{c, d\}$ of points in $\mathbb{P}^1$ are harmonic conjugate if and only the cross ratio $[a, c; b, d]$ is equal to -1.

2.17 Let U be a linear space of dimension 2 and $V = S^2U$. Let C be the Veronese conic in $|V|$, the image of the Veronese map $v_2 : |U| \to |S^2U|$. Let us identify a nondegenerate quadratic from $q \in S^2U^\vee$ with a linear form $l_q \in V^\vee$ and let $P_q \in |V|$ be the polar point of the line $V(l_q)$ with respect to C. Consider the involution σ_q of C defined by the projection from P_q.

(i) Show that, under the isomorphism $v_2 : |U| \to C$, the involution s_q is equal to the involution defined by the polarity with respect to q, where U is identified with $U^\vee$ by means of am isomorphism $\bigwedge^2 U \cong \mathbb{C}$.

(ii Show that, for any two quadratic forms q, q' with no common zeros, σ_q and $\sigma_{q'}$ commute if and only if two pairs $V(q)$ and $V(q')$ of points in $|U|$ are harmonic conjugate.

(iii) Show that, for any three quadratic forms q, q', q'' with no two sharing a zero, $(s_q \circ s_{q'} \circ s_{q''})^2 = \mathrm{id}_{|U|}$ if and only if q, q', q'' are linearly dependent (the sufficiency condition extends to any odd number of q's).

(iv) Show that, given three distinct points a, b, c on C, the points $\mathbb{T}_a(C) \cap \overline{b, c}$, $\mathbb{T}_b(C) \cap \overline{a, c}$ and $\mathbb{T}_c(C) \cap \overline{a, b}$ are collinear.

2.18 Let $abcd$ be a quadrangle in $\mathbb{P}^2$, and p, q be the intersection points of two pairs of opposite sides $\overline{a,b}, \overline{c,d}$ and $\overline{b,c}, \overline{a,d}$. Let p', q' be the intersection points of the line $\overline{p,q}$ with the diagonals $\overline{a,c}$ and $\overline{b,d}$. Show that the pairs (p,q) and (p',q') are harmonic conjugate .

2.19 Show that the pair of points on a diagonal of a complete quadrilateral defined by its sides is harmonic conjugate to the pair of points defined by intersection with other two diagonals.

2.20 Find the condition on a pair of conics that the associate Salmon conic is degenerate.

Historical Notes

The Poncelet 's Closure Theorem which is the second part of Darboux's Theorem 2.2.2 was first discoverd by Poncelet himself [324]. We refer to the excellent account of the history of the Poncelet related conics to [34]. Other elementary and non-elementary treatments of the Poncelet properties and their generalizations can be found in [17],[18],
[195],[196]. The relationship between Poncelet curves and vector bundles is discussed in [414], [299], [415], [417]. Among many equivalent definitions of the Schwarzenberger bundles we chose one discussed in [133]. The papers of [289] and [216],[217] discuss the compactification of the variety of conjugate triangles. The latter two papers of N. Hitchin also discuss an interesting connection with Painleve equations.

The notion of the conjugacy of conics is due to Rosanes [342]. Reye called two conjugate conics apolar [333]. The condition (2.11) for conjugate conics was first discovered by O. Hesse in [212]. He also proved that this property is poristic. The condition for Poncelet relation given in terms of invariants of a pair of conics (Theorem 2.2.9) was first discovered by A. Cayley [52], [55]. The invariants of a pair of quadrics in any dimension were studied by C. Segre [375]. A good modern discussion of Poncelet's theorem and its applications can be found in [165].

The proof of Theorem 2.2.10 is due to J. Coolidge [92], Chapter VI, §3. The result was known to G. von Staudt [392] ((see [92], p. 66) and can be also found in Salmon's book on conics [355], p. 345. Although Salmon writes in the footnote on p. 345 that "I believe that I was the first to direct the attention to the importance of this conic in the theory of two conics", this conic was already known to Ph. La Hire [264] (see [92], p. 44).

Chapter 3

Plane cubics

3.1 Equations

3.1.1 Weierstrass equation

Let X be a nonsingular projective curve of genus 1. By Riemann-Roch, for any divisor D of degree $d > 0$, we have $\dim H^0(X, \mathcal{O}_X(D)) = d$. The complete linear system $\big|D\big| = |H^0(X, \mathcal{O}_X(D))|$ defines an isomorphism $X \cong C$, where C is a curve of degree d in $\mathbb{P}^{d-1}$ (see [206], Chapter IV, Corollary 3.2). We consider here the case $d = 3$, i.e., a plane cubic model $C = V(f)$ of X. By Theorem 1.1.8, C has an inflection point p_0. Without loss of generality, we may assume that $p_0 = [0, 0, 1]$ and the tangent line at this point has the equation $t_0 = 0$. This implies that $f = t_0 q(t_0, t_1, t_2) + a t_1^3$, where q is a quadratic polynomial. We may assume that $q = bt_2^2 + t_2 L(t_0, t_1) + q'(t_0, t_1)$ for some quadratic polynomial q' and a linear polynomial l. Notice that $b \neq 0$, otherwise, we can express t_2 as a rational function in t_0, t_1 and obtain that E is a rational curve. So, we may assume that $b = 1$. Replacing t_2 with $t_2 + \frac{1}{2} l(t_0, t_1)$ we may assume that $l = 0$. Now the equation looks as

$$f = t_0 t_2^2 + a t_1^3 + b t_1^2 t_0 + c t_1 t_0^2 + d t_0^3 = 0.$$

By scaling, we may assume that $a = 1$. Replacing t_1 with $t_1 + \frac{b}{3} t_0 \neq 0$, we may assume that $b = 0$. This gives us the *Weierstrass equation* of a nonsingular cubic:

$$t_0 t_2^2 + t_1^3 + \alpha t_1 t_0^2 + \beta t_0^3 = 0 \tag{3.1}$$

It is easy to see that C is nonsingular if and only if the polynomial $x^3 + \alpha x + \beta$ has no multiple roots, or, equivalently, its discriminant $\Delta = 4\alpha^3 + 27\beta^2$ is not equal to zero.

Two Weierstrass equations define isomorphic elliptic curves if and only if there exists a projective transformation transforming one equation to another. It is easy to see that it happens if and only if $(\alpha', \beta') = (\lambda^3 \alpha, \lambda^2 \beta)$ for some nonzero constant λ. This can be expressed in terms of the *absolute invariant*

$$j = \frac{4\alpha^3}{4\alpha^3 + 27\beta^2}. \tag{3.2}$$

Two elliptic curves are isomorphic if and only if their absolute invariants are equal.

The projection $[t_0,t_1,t_2] \mapsto [t_0,t_1]$ exhibits C as a double cover of $\mathbb{P}^1$ with the branch points $[1,x],[0,1]$, where $x^3+\alpha x+\beta = 0$. The corresponding points $[1,x,0]$, and $[0,1,0]$ on C are the ramification points. If we choose $p_0 = [0,1,0]$ to be the zero point in the group law on C, then $2p \sim 2p_0$ for any ramification point p implies that p is a 2-torsion point. Any 2-torsion point is obtained in this way. Here we use that the group law on a cubic curve with the distinguished point p_0 chosen as the zero point is given by the formula

$$p \oplus q \in |p+q-p_0|. \tag{3.3}$$

Note that, by Riemann-Roch, the complete linear system $|p+q-p_0|$ consists of one point.

It follows from the above computation that any nonsingular plane cubic $V(f)$ is projectively isomorphic to the plane cubic $V(t_2^2t_0+t_1^3+\alpha t_1t_0^2+\beta t_0^3)$. The functions $S: F \mapsto \alpha, T: F \mapsto \beta$ can be extended to invariants on the space of homogeneous cubic forms with respect to the groups of unimodular linear transformations. The explicit expresions of S and T in terms of the coefficients of f are rather long and can be found in many places (e.g. [136]).

Definition 3.1. *A nonsingular plane cubic $V(f)$ is called* harmonic *(resp.* equianharmonic*) if $S(f) = 0$ (resp. $T(f) = 0$).*

Theorem 3.1.1. *Let $C = V(f)$ be a nonsingular plane cubic and c be any point on C. The following conditions are equivalent.*

- *(i) C is a harmonic (resp. equianharmonic) cubic.*
- *(ii) The cross ratio of four roots of the polynomial $t_0(t_1^3+\alpha t_1t_0^2+\beta t_0^3)$, taken in any order, is equal to -1 (resp. a third root of -1);*
- *(iii) The group of automorphisms of C leaving the point c invariant is a cyclic group of order 4 (resp. 6).*

Proof. (i) $\Leftrightarrow$ (ii) Direct computation.

(ii) $\Leftrightarrow$ (iii) Let G be the group of automorphisms of C leaving c fixed. By choosing a projective embedding of C given by the linear system $|3c|$, we obtain that C is isomorphic to a plane cubic $V(f)$ given by a Weierstrass equation $f = 0$ and G is isomorphic to the group of projective transformations of $\mathbb{P}^2$ leaving the point $[0,0,1]$ invariant. By direct computation, it is easy to see that G consists of transformations $T: [t_0,t_1,t_2] \mapsto [\lambda^2t_0, \lambda^2t_1, \lambda^3t_2]$ leaving f invariant. Now the assertion is easily verified. $\square$

3.1.2 The Hesse equation

Since any flex tangent line intersects the curve with multiplicity 3, applying (1.23), we obtain that the curve has exactly 9 inflection points. Using the group law on an elliptic curve with an inflection point as the zero, we can interpret any inflection point p as a 3-torsion point. This follows from (3.3) since the divisor of the rational function l/l_0

$\mathrm{mod}\ (f)$, where $l = 0$ is the equation of the inflection tangent at p and $l_0 = 0$ is the equation of the inflection tangent at p_0, is equal to $3p - 3p_0$. This of course agrees with the fact the group $X[3]$ of 3-torsion points on an elliptic curve X is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^2$.

Let H be a subgroup of order 3 of X. Since the sum of elements of this group add up to 0, we see that the corresponding 3 inflection points p, q, r satisfy $p+q+r \sim 3p_0$. It is easy to see that the rational function on C with the divisor $p + q + r - 3p_0$ can be obtained as the restriction of the rational function $m(t_0, t_1, t_2)/l_0(t_0, t_1, t_2)$, where $m = 0$ defines the line containing the points p, q, r. There are 3 cosets with respect to each subgroup H. Since the sum of elements in each coset is again equal to zero, we get 12 lines, each containing three inflection points. Conversely, if a line contains three inflection points, the sum of these points is zero, and it is easy to see that the three points forms a conjugacy class with respect to some subgroup H. Each element of $(\mathbb{Z}/3\mathbb{Z})^3$ is contained in 4 cosets (it is enough to check this for the zero element). Thus we obtain a configuration of 12 lines and 9 points, each line contains 3 points, and each point is contained in 4 lines. This is the *Hesse line arrangement* $(12_3, 9_4)$.

Let ℓ_1, ℓ_2 be two inflection lines. Choose projective coordinates such that the equations of these lines are $t_0 = 0$ and $t_1 = 0$. Then it is easy to see that the equation of C can be written in the form

$$f(x_0, x_1, x_2) = t_0t_1(at_0 + bt_1 + ct_2) + dt_2^3, \tag{3.4}$$

where $at_0 + bt_1 + ct_2 = 0$ is a third inflection line. Suppose the three lines are concurrent. Then the equation can be further transformed to the form $t_0t_1(t_0 + t_1) + t_2^3 = 0$. Since the sets of three distinct points in $\mathbb{P}^1$ are projectively equivalent we can change the coordinates to assume that the equation is $t_0^3 + t_1^3 + t_2^3 = 0$. Obviously, it is in the Hesse form. So we may assume that three lines are non-concurrent. Consider the equation (3.4). By scaling the coordinate t_2 we may assume that $c = 3$. Let ϵ_3 be a primitive 3d root of 1. Define new coordinates u, v by the formula

$$at_0 + t_2 = \epsilon_3 u + \epsilon_3^2 v, \quad bt_0 + t_2 = \epsilon_3^2 u + \epsilon_3 v.$$

Then

$$\begin{aligned} abF(x_0, x_1, x_2) &= (\epsilon_3 u + \epsilon_3^2 v - t_2)(\epsilon_3^2 u + \epsilon_3 v - t_2)(-u - v + t_2) + dt_2^3 \\ &= -u^3 - v^3 + (d+1)t_2^3 - 3uvt_2 = 0. \end{aligned}$$

Since C is nonsingular, we have $d \neq 1$. After scaling the coordinate t_2 we arrive at the *Hesse canonical form* or *second canonical form* or *Hesse equation* of a plane cubic curve

$$t_0^3 + t_1^3 + t_2^3 + 6mt_0t_1t_2 = 0. \tag{3.5}$$

Here the expression for the last coefficient is given to simplify future computations. The condition that the curve is nonsingular is

$$1 + 8m^3 \neq 0. \tag{3.6}$$

The curve given given by this equation is singular if and if $8m^3 + 1 = 0$.

By reducing the Hesse equation to a Weiestrass forms one can express the absolute invariant (3.2) in terms of the parameter m:

$$j = \frac{64(m-m^4)^3}{(1+8m^3)^3}. \tag{3.7}$$

3.1.3 The Hesse pencil

Consider a pencil of plane cubics defined by the equation

$$\lambda(t_0^3+t_1^3+t_2^3)+\mu t_0t_1t_2 = 0. \tag{3.8}$$

It is called the *Hesse pencil*. Its base points are

$$\begin{aligned}
&[0,1,-1], \quad [0,1,-\epsilon], \quad [0,1,-\epsilon^2],\\
&[1,0,-1], \quad [1,0,-\epsilon^2], \quad [1,0,-\epsilon],\\
&[1,-1,0], \quad [1,-\epsilon,0], \quad [1,-\epsilon^2,0],
\end{aligned} \tag{3.9}$$

where $\epsilon = e^{2\pi i/3}$. As is easy to see they are the nine inflection points of any nonsingular member of the pencil. The singular members of the pencil correspond to the values of the parameters

$$(\lambda,\mu) = (0,1),\ (1,-3),\ (1,-3\epsilon),\ (1,-3\epsilon^2).$$

The last three values correspond to the three values of m for which the Hesse equation defines a singular curve.

Any triple of lines containing the nine base points belong to the pencil and define a singular member. Here they are:

$$\begin{gathered}
V(t_0), \quad V(t_1), \quad V(t_2),\\
V(t_0+t_1+t_2),\ V(t_0+\epsilon t_1+\epsilon^2t_2),\ V(t_0+\epsilon^2t_1+\epsilon t_2)\\
V(t_0+\epsilon t_1+t_2),\ V(t_0+\epsilon^2t_1+\epsilon^2t_2),\ V(t_0+t_1+\epsilon t_2)\\
V(t_0+\epsilon^2t_1+t_2),\ V(t_0+\epsilon t_1+\epsilon t_2),\ V(t_0+t_1+\epsilon^2t_2)
\end{gathered} \tag{3.10}$$

We leave to a suspicious reader to check that

$$(t_0+t_1+t_2)(t_0+\epsilon t_1+\epsilon^2t_2)(t_0+\epsilon^2t_1+\epsilon t_2) = t_0^3+t_1^3+t_2^3-3t_0t_1t_2,$$

$$(t_0+\epsilon t_1+t_2)(t_0+\epsilon^2t_1+\epsilon^2t_2)(t_0+t_1+\epsilon t_2) = t_0^3+t_1^3+t_2^3-3\epsilon t_0t_1t_2,$$

$$(t_0+\epsilon^2t_1+t_2)(t_0+\epsilon t_1+\epsilon t_2)(t_0+t_1+\epsilon^2t_2) = t_0^3+t_1^3+t_2^3-3\epsilon^2t_0t_1t_2.$$

The 12 lines (3.10) and 9 inflection points (3.9) form the Hesse configuration corresponding to any nonsingular member of the pencil.

Choose $[0,1,-1]$ to be the zero point in the group law on C. Then we can define an isomorphism of groups $\phi : (\mathbb{Z}/3\mathbb{Z})^2 \to X[3]$ by sending $[1,0]$ to $[0,1,-\epsilon]$, $[0,1]$ to $[1,0,-1]$. The points of the first row is the subgroup H generated by $\phi([1,0])$. The points of the second row is the coset of H containing $\phi((0,1))$.

Remark 3.1.1. Note that varying m in $\mathbb{P}^1 \setminus \{-\frac{1}{2}, -\frac{\epsilon}{2}, -\frac{\epsilon^2}{2}, \infty\}$ we obtain a family of elliptic curves X_m with a fixed isomorphism $\phi_m : (\mathbb{Z}/3\mathbb{Z})^2 \to X_m[3]$. By blowing up the 9 base points we obtain a rational surface $S(3)$ together with a morphism

$$f : S(3) \to \mathbb{P}^1$$

obtained from the rational map $\mathbb{P}^2 - \to \mathbb{P}^1, [t_0, t_1, t_2] \mapsto [t_0t_1t_2, t_0^3 + t_1^3 + t_2^3]$ by resolving (minimally) the indeterminacy points. The fibre of f over a point $(a, b) \in \mathbb{P}^2$ is isomorphic to the member of the Hesse pencil corresponding to $(\lambda, \mu) = (-b, a)$. One can show that this is a *modular family* of elliptic curves with 3-level, i.e. the universal object for the fine moduli space of pairs (X, ϕ), where X is an elliptic curve and $\phi : (\mathbb{Z}/3\mathbb{Z})^2 \to X[3]$ is an isomorphism of groups. There is a canonical isomorphism $\mathbb{P}^1 \cong Y$, where Y is the modular curve of level 3, i.e. a nonsingular compactification of the quotient of the upper half-plane $\mathcal{H} = \{a + bi \in \mathbb{C} : b > 0\}$ by the group

$$\Gamma(3) = \{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) : A \equiv I_3 \mod 3\}$$

which acts on $\mathcal{H}$ by Möbius transformations $z \mapsto \frac{az+b}{cz+d}$. The boundary of $H/\Gamma(3)$ in Y consists of 4 points (the cusps). They correspond to the singular members of the Hesse pencil.

3.1.4 The Hesse group

The *Hesse group* G_{216} is the group of projective transformations which preserve the Hesse pencil of cubic curves. First we see the obvious symmetries generated by the transformations

$$\tau : [t_0, , t_1, t_2] \mapsto [t_0, \epsilon_3 t_1, \epsilon_3^2 t_2]. \tag{3.11}$$

$$\sigma : [t_0, t_1, t_2] \mapsto [t_2, t_0, t_1]. \tag{3.12}$$

They define a projective representation of the group $(\mathbb{Z}/3\mathbb{Z})^2$, called the Schrödinger representation.

If we fix the group law by taking the origin to be $[0, 1, -1]$, then the transformation (3.11) induces on each nonsingular fibre the translation automorphism by the point $[0, 1, -\epsilon]$. The transformation (3.11) is the translation by the point $[1, 0, -1]$ and the transformation (3.12) is the translation by the point $[1, 0, -1]$.

Theorem 3.1.2. *The Hesse group G_{216} is a group of order 216 isomorphic to the semi-direct product*

$$(\mathbb{Z}/3\mathbb{Z})^2 \rtimes \mathrm{SL}(2, \mathbb{F}_3),$$

where the action of $\mathrm{SL}(2, \mathbb{F}_3)$ *on* $(\mathbb{Z}/3\mathbb{Z})^2$ *is the natural linear representation.*

Proof. Let $\sigma \in G_{216}$. It transforms a member of the Hesse pencil to another member. This defines a homomorphism $G_{216} \to \mathrm{Aut}(\mathbb{P}^1)$. An element of the kernel K leaves each member of the pencil invariant. In particular, it leaves invariant the curve $V(t_0t_1t_2)$. The group of automorphisms of this curve is generated by homotheties

$[t_0, t_1, t_2] \mapsto [t_0, at_1, bt_2]$ and permutation of coordinates. Suppose σ induces a homothety. Since it also leaves invariant the curve $V(t_0^3+t_1^3+t_2^3)$, we must have $1 = a^3 = b^3$. To leave invariant a general member we also need that $a^3 = b^3 = bc$. This implies that σ belongs to the subgroup generated by transformation (3.11). An even permutation of coordinates belongs to a subgroup generated by transformation (3.12). The odd permutation $\sigma_0 : [t_0, t_1, t_2] \mapsto [t_0, t_2, t_1]$ acts on the group of 3-torsion points of each nonsingular fibre as the inversion automorphism. Thus we see that

$$K \cong (\mathbb{Z}/3\mathbb{Z})^2 \rtimes \langle \sigma_0 \rangle.$$

Now let I be the image of Hes in Aut$(\mathbb{P}^1)$. It acts by permuting the four singular members of the pencil and thus leaves the set of zeros of the binary form

$$\Delta = (8t_1^3 + t_0^3)t_0$$

invariant. It follows from the invariant theory that this implies that H is a subgroup of $\mathfrak{A}_4$. We claim that $H = \mathfrak{A}_4$. Consider the projective transformations given by the matrices

$$\sigma_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \varepsilon & \varepsilon^2 \\ 1 & \varepsilon^2 & \varepsilon \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & \varepsilon & \varepsilon \\ \varepsilon^2 & \varepsilon & \varepsilon^2 \\ \varepsilon^2 & \varepsilon^2 & \varepsilon \end{pmatrix}$$

The transformations $\sigma_0, \sigma_1, \sigma_2$ generate a subgroup isomorphic to the quaternion group Q_8 with center generated by σ_0. The transformation $\sigma_3 : [t_0, t_1, t_2] \mapsto [\varepsilon t_0, t_2, t_1]$ satisfies $\sigma_3^3 = \sigma_0$. It acts by sending a curve X_m to $X_{\varepsilon m}$. It is easy to see that the transformations $\sigma_1, \sigma_2, \sigma_3, \tau$ generate the group isomorphic to SL$(2, \mathbb{F}_3)$. Its center is (σ_0) and the quotient by the center is isomorphic to $\mathfrak{A}_4$. In other words, this group is the binary tetrahedral group. Note that the whole group can be generated by transformations $\sigma, \tau, \sigma_0, \sigma_1$. $\square$

Recall that a linear operator $\sigma \in \mathrm{GL}(V)$ of a complex vector space of dimension n is called a *complex reflection* if it is of finite order and the rank of $\sigma - \mathrm{id}_V$ is equal to 1. The kernel of $\sigma - \mathrm{id}_V$ is a hyperplane H_v in V, called the *reflection hyperplane* of σ. It is an invariant with respect to σ and its stabilizer is a cyclic group. A *complex reflection group* is a finite subgroup G of $\mathrm{GL}(V)$ generated by complex reflections. One can choose a unitary inner product on V such that any complex reflection σ from V can be written in the form

$$s_{\alpha,\eta} : v \mapsto v + (\eta - 1)(v, \alpha)v,$$

where α is a vector of norm 1 perpendicular to the reflection hyperplane H_σ of σ, and η is a non-trivial root of unity of order equal to the order of σ.

Recall the basic facts about complex reflection groups (see, for example, [391]):

- The algebra of invariants $(S^\bullet V)^G \cong \mathbb{C}[t_1, \ldots, t_n]^G$ is freely generated by n invariant polynomials $f_1, \ldots, f_n$ (geometrically $V/G \cong \mathbb{C}^n$)).
- The product of degrees d_i of the polynomials $f_1, \ldots, f_n$ is equal to the order of G.

- The number of complex reflections in G is equal to $\sum(d_i - 1)$.

All complex reflections group were classified by G. Shephard and J. Todd [387]. There are 5 conjugacy classes of complex reflection subgroups of $\mathrm{GL}(3,\mathbb{C})$. Among them is the group isomorphic to a central extension of degree 3 of the Hesse group. It is generated by complex reflections $s_{\alpha,\eta}$, where $V(\alpha)$ is one of the 12 hyperplanes (3.10) in $|V|$ and α is the unit normal vector (a,b,c) to the hyperplane $V(at_0+bt_1+ct_2)$ and $\eta^3 = 1$. Note that each reflection $s_{\alpha,\eta}$ leaves invariant the hyperplanes with normal vector orthogonal to α. For example, $s_{[1,0,0],\varepsilon}$ leaves invariant the hyperplanes $V(t_i), i = 0,1,2$. This implies that each of the 12 complex reflections leave the Hesse pencil invariant. Thus the image of G in $\mathrm{PGL}(3,\mathbb{C})$ is contained in the Hesse group. It follows from the classification of complex reflection groups (or could be checked directly, see [391]) that it is equal to the Hesse group and the subgroup of scalar matrices from G is a cyclic group of order 3.

Each of the 12 reflection hyperplanes defines 2 complex reflections. This gives 24 complex reflections in G. This number coincides with the number of elements of order 3 in Hes and so there are no more complex reflections ion G. Let $d_1 \le d_2 \le d_3$ be the degrees of the invariants generating the algebra of invariants of G. We have $d_1 + d_2 + d_3 = 27, d_1d_2d_3 = 648$. This easily gives $d_1 = 6, d_2 = 9, d_3 = 12$. There are obvious reducible curves of degree 9 and 12 in $\mathbb{P}^2$ invariant with respect to G. The curve of degree 9 is the union of the lines whose normal vectors are the coordinate vectors of the base points of the Hesse pencil. One can check that each such line intersects a nonsingular member of the pencil at nontrivial 2-torsion points with respect to the group law defined by the corresponding base point. For each nonsingular member of the Hesse pencil this line is classically called the *harmonic line* of the corresponding inflection point. The equation of the union of 9 harmonic lines is

$$f_9 = (t_0^3 - t_1^3)(t_0^3 - t_2^3)(t_1^3 - t_2^3) = 0. \tag{3.13}$$

The curve of degree 12 is the union of the 12 lines (3.10). Its equation is

$$f_{12} = t_0t_1t_2[27t_0^3t_1^3t_2^3 - (t_0^3 + t_1^3 + t_2^3)^3] = 0 \tag{3.14}$$

A polynomial defining an invariant curve is a *relative invariant* of G (it is an invariant with respect to the group $G' = G \cap \mathrm{SL}(3,\mathbb{C})$). One checks that the polynomials f_9 is indeed an invariant, but the polynomial f_{12} is only a relative invariant. So, there exists another curve of degree 12 whose equation defines an invariant of degree 12. What is this curve? Recall that the Hesse group acts on the base of the Hesse pencil via the action of the tetrahedron group $\mathfrak{A}_4$. It has 3 special orbits with stabilizers of order 2,3 and 3. The first orbit consists of 6 points such that the fibres over these points are harmonic cubics. The second orbit consists of 4 points such that the fibres over these points are equiequianharmoniccubics. The third orbit consists of 4 points corresponding to singular members of the pencil. It is not difficult to check that the product of the equations of the equiequianharmoniccubics defines an invariant of degree 12. Its equation is

$$f'_{12} = (t_0^3 + t_1^3 + t_2^3)[(t_0^3 + t_1^3 + t_2^3)^3 + 216t_0^3t_1^3t_2^3] = 0 \tag{3.15}$$

An invariant of degree 6 is

$$f_6 = 7(t_0^6 + t_1^6 + t_2^6) - 6(t_0^3 + t_2^3 + t_3^3)^2. \tag{3.16}$$

The product of the equations defining 6 harmonic cubics is an invariant of degree 18

$$f_{18} = (t_0^3+t_1^3+t_2^3)^6 - 540t_0^3t_1^3t_2^3(t_0^3+t_1^3+t_2^3)^3 - 5832t_0^6t_1^6t_2^6 = 0 \tag{3.17}$$

3.2 Polars of a plane cubic

3.2.1 The Hessian of a cubic hypersurface

Let $X = V(f)$ be a cubic hypersurface in $\mathbb{P}^n$. We know that the Hessian $\mathrm{He}(X)$ is the locus of points $a \in \mathbb{P}^n$ such that the polar quadric $P_a(V))$ is singular. Also we know that, for any $a \in \mathrm{He}(X)$,

$$\mathrm{Sing}(P_a(X)) = \{b \in \mathbb{P}^2 : D_b(D_a(f)) = 0\}.$$

Since $P_b(P_a(X)) = P_a(P_b(X))$ we obtain that $b \in \mathrm{He}(X)$.

Theorem 3.2.1. *The Hessian* $\mathrm{He}(X)$ *of a cubic hypersurface* X *contains the Steinerian* $\mathrm{St}(X)$. *If* $\mathrm{He}(X) \neq \mathbb{P}^n$, *then*

$$\mathrm{He}(X) = \mathrm{St}(X).$$

For the last assertion one only needs to compare the degrees of the hypersurfaces. They are equal to $n+3$.

In particular, the rational map, if defined,

$$\mathrm{st}_X^{-1} : \mathrm{St}(X) \to \mathrm{He}(X), a \mapsto \mathrm{Sing}(P_a(X)) \tag{3.18}$$

is a birational automorphism of the Hessian hypersurface. We have noticed this already in Chapter 1.

Proposition 3.2.2. *Assume* X *has only isolated singularities. Then* $\mathrm{He}(X) = \mathbb{P}^n$ *if and only if* X *is a cone over a cubic hypersurface in* $\mathbb{P}^{n-1}$.

Proof. Let $W = \{(a,b) \in \mathbb{P}^n \times \mathbb{P}^n : P_{a,b^2}(X) = 0\}$. For each $a \in \mathbb{P}^n$, the fibre of the first projection over the point a is equal to the first polar $P_a(X)$. For any $b \in \mathbb{P}^n$, the fibre of the second projection over the point b is equal to the second polar $P_{b^2}(X) = V(\sum \partial_i f(b)t_i)$. Let $U = \mathbb{P}^n \setminus \mathrm{Sing}(X)$. For any $b \in U$, the fibre of the second projection is a hyperplane in $\mathbb{P}^n$. This shows that $p_2^{-1}(U)$ is nonsingular. The restriction of the first projection to U is a morphism of nonsingular varieties. The general fibre of this morphism is a regular scheme over the general point of $\mathbb{P}^n$. Since we are in characteristic 0, it is a smooth scheme. Thus there exists an open subset $W \subset \mathbb{P}^n$ such that $p_1^{-1}(W) \cap U$ is nonsingular. If $\mathrm{He}(X) = 0$, all polar quadrics $P_a(X)$ are singular, and a general polar must have singularities inside of $p_2^{-1}(\mathrm{Sing}(X))$. This means that $p_1(p_2^{-1}(\mathrm{Sing}(X))) = \mathbb{P}^n$. For any $x \in \mathrm{Sing}(X)$, all polar quadrics contain x and either all of them are singular at x or there exists an open subset $U_x \subset \mathbb{P}^n$ such all quadrics $P_a(X)$ are nonsingular at x for $a \in U_x$. Suppose that for any $x \in \mathrm{Sing}(X)$ there exists a polar quadric which is nonsingular at x. Since the number of isolated singular points is finite, there will be an open set of points $a \in \mathbb{P}^n$ such that the fibre $p_1^{-1}(a)$ is nonsingular in $p_2^{-1}(\mathrm{Sing}(X))$. This is a contradiction. Thus, there exists a point $c \in \mathrm{Sing}(X)$ such that all polar quadrics are singular at x. This implies that c is a common solution of the systems of linear equations $\mathrm{He}(f_3)(a) \cdot X = 0, a \in \mathbb{P}^n$. Thus the first partials of f_3 are linearly dependent. Now we apply Proposition 1.1.2 to obtain that X is a cone. □

Remark 3.2.1. The example of a cubic hypersurface in $\mathbb{P}^4$ which we considered in Remark 1.1.1 shows that the assumption of the Theorem cannot be weakened. Its singular locus is the plane $t_0 = t_1 = 0$.

3.2.2 The Hessian of a plane cubic

Consider a plane cubic $C = V(f)$ with equation in the Hesse canonical form (3.5). The partials of $\frac{1}{3}f$ are

$$t_0^2 + 2mt_1t_2, \quad t_1^2 + 2mt_0t_2, \quad t_2^2 + 2mt_0t_1 \tag{3.19}$$

Thus the Hessian of C has the following equation:

$$\mathrm{He}(C) = \begin{vmatrix} t_0 & mt_2 & mt_1 \\ mt_2 & t_1 & mt_0 \\ mt_1 & mt_0 & t_2 \end{vmatrix} = (1+2m^3)t_0t_1t_2 - m^2(t_0^3 + t_1^3 + t_2^3). \tag{3.20}$$

In particular, the Hessian of the member of the Hesse pencil corresponding to the parameter $(\lambda, \mu) = (1, 6m), m \neq 0$, is equal to

$$t_0^3 + t_1^3 + t_2^3 - \frac{1+2m^3}{m^2}t_0t_1t_2 = 0, \quad m \neq 0, \tag{3.21}$$

or, if $(\lambda, \mu) = (1, 0)$ or $(0, 1)$, then the Hessian is equal to $V(t_0t_1t_2)$.

Lemma 3.2.3. *Let C be a nonsingular cubic. The following assertions are equivalent:*

(i) $\dim \mathrm{Sing}(P_a(C)) > 0$*;*

(ii) $a \in \mathrm{Sing}(\mathrm{He}(C))$*;*

(iii) $\mathrm{He}(C)$ *is the union of three nonconcurrent lines;*

(iv) C *is isomorphic to a Fermat cubic* $t_0^3 + t_1^3 + t_1^3 = 0$*;*

(v) $\mathrm{He}(C)$ *is a singular cubic;*

(vi) C *is an equianharmonic cubic.*

Proof. Use the Hesse equation for a cubic and for its Hessian. We see that $\mathrm{He}(C)$ is singular if and only if either $m = 0$ or $1 + 8(-\frac{1+2m^3}{6m^2})^3 = 0$. Obviously, $m = 1$ is a solution of the second equation. Other solutions are ϵ, ϵ^2. This corresponds to $\mathrm{He}(C)$, where C os of the form $V(t_0^3 + t_1^3 + t_1^3)$, or is given by the equation

$$t_0^3 + t_1^3 + t_1^3 + 6\epsilon^i t_0t_1t_2 = (\epsilon^i t_0 + \epsilon t_1 + t_2)^3 + (t_0 + \epsilon^i t_1 + t_2)^3$$

$$+(t_0+t_1+\epsilon^i t_2)^3 = 0,$$

where $i = 1, 2$, or

$$t_0^3+t_1^3+t_1^3+6t_0t_1t_2 = (t_0+t_1+t_2)^3+(t_0+\epsilon t_1+\epsilon^2 t_2)^3$$
$$+(t_0+\epsilon^2 t_1+\epsilon t_2)^3 = 0.$$

This computation proves the equivalence of (iii), (iv), (v).

Assume (i) holds. Then the rank of the Hessian matrix He is equal to 1. It is easy to see that the first two rows are proportional if and only if $m(m^3-1) = 0$. It follows from the previous computation that this implies (iv). The corresponding point a is one of the three intersection points of the lines such that the cubic is equal to the sum of the cubes of linear forms defining these lines. Direct computation shows that (ii) holds. This shows the implication (i) ⇒ (ii).

Assume (ii) holds. Again the previous computations show that $m(m^3-1) = 0$ and the Hessian curve is the union of three lines. Again (i) is directly verified.

The equivalence of (iv) and (vi) follows from Theorem 3.1.1 since the transformation $[t_0, t_1, t_2] \to [t_1, t_0, e^{2\pi i/3}t_2]$ generates a cyclic group of order 6 of automorphisms of C leaving the point $[1, -1, 0]$ fixed. □

Corollary 3.2.4. *Assume that $C = V(f)$ is not isomorphic to a Fermat cubic. Then the Hessian cubic is not singular, and the map $a \mapsto \mathrm{Sing}(P_a(C))$ is an involution on C without fixed points.*

Proof. The only unproved assertion is that the involution does not have fixed points. A fixed point a has the property that $D_a(D_a(f)) = D_{a^2}(f) = 0$. It follows from Theorem 1.1.1 that this implies that $a \in \mathrm{Sing}(C)$. □

Remark 3.2.2. Consider the Hesse pencil of cubics with parameters $(\lambda, \mu) = (m_0, 6m)$

$$C(m_0, m) = V(m_0(t_0^3+t_1^3+t_2^3)+6m_1t_0t_1t_2).$$

Taking the Hessian of each curve from the pencil we get the pencil

$$H(\lambda) = V(\lambda_0t_0^3+t_1^3+t_2^3+6\lambda_1t_0t_1t_2).$$

The map $C(m_0, m) \to \mathrm{He}(C(m_0, m))$ defines a regular map

$$H : \mathbb{P}^1 \to \mathbb{P}^1, \quad [m_0, m_1] \mapsto [t_0, t_1] = [-m_0m_1^2, m_0^3+2m_1^3] \tag{3.22}$$

This map is of degree 3. For a general value of the inhomogeneous parameter $\lambda = t_1/t_0$, the preimage consists of three points with inhomogeneous coordinate $m = m_1/m_0$ satisyfing the cubic equation

$$6\lambda m^3 - 2m^2 + 1 = 0. \tag{3.23}$$

We know that the points

$$[\lambda_0, \lambda_1] = [0, 1], [1, -\tfrac{1}{2}], [1, -\tfrac{\epsilon}{2}], [1, -\tfrac{\epsilon^2}{2}]$$

correspond to singular members of the λ-pencil. These are the branch points of the map H. Over each branch point we have two points in the preimage. The points

$$(m_0, m_1) = [1,0], [1,1], [1,\epsilon], [1,\epsilon^2].$$

are the ramification points corresponding to cubics isomorphic to the Fermat cubic. A non-ramication point in the preimage corresponds to a singular member.

Let $C(m) = C(1,m)$. If we fix a group law on a $H(m) = \mathrm{He}(C(m))$, we can identify the involution described in Corollary 3.2.4 with the translation with respect to a non-trivial 2-torsion point η (see Exercises). Given a nonsingular cubic curve $H(m)$ together with a fixed-point-free involution τ there exists a unique nonsingular cubic $C(m)$ such that $H(m) = \mathrm{He}(C(m))$ and the involution τ is the involution described in the corollary. Thus the 3 roots of the equation (3.23) can be identified with 3 non-trivial torsion points on $H(m)$. We refer to Exercises for a reconstruction of $C(m)$ from the pair $(H(m), \eta)$.

Recall that the Cayleyan curve of a plane cubic C is the locus of lines $\overline{p,q}$ in the dual plane such that $a \in \mathrm{He}(C)$ and b is the singular point of $P_a(C)$. Each such line intersects $\mathrm{He}(C)$ at three points a, b, c. The following gives the geometric meaning of the third intersection point.

Proposition 3.2.5. *Let c be the third intersection point of a line $\ell \in \mathrm{Cay}(C)$ and $\mathrm{He}(C)$. Then ℓ is a component of the polar $P_d(C)$ whose singular point is c. The point d is the intersection point of the tangents of $\mathrm{He}(C)$ at the points a and b.*

Proof. Since $b \in \mathrm{Sing}(P_a(C))$, we have $D_b(D_a(f)) = 0$. Similarly, we obtain that $D_b(D_a(f)) = 0$. This implies that $D_x(D_{ab}(f)) = 0$ for any $x \in \mathbb{P}^2$. This means that the points $a, b \in \ell$ are conjugate with respect to all polar quadrics. Let U be the 2-dimensional subspace of $\mathbb{C}^3$ defining the line ℓ. The restriction of the quadrics $P_x(C)$ to ℓ is defined by a quadratic form q_x on U. Let b_x be the corresponding polar bilinear form. Let $\mathbf{a}, \mathbf{b} \in U$ be vectors spanning the lines $a, b \in \ell = |U|$. For all $x \in \mathbb{P}^2$, we have $b_x(\mathbf{a}, \mathbf{b}) = 0$. Consider the unique polar conic $Q_d = V(q_d)$ passing through the points a, b. We have

$$0 = 2b_d(\mathbf{a}+\mathbf{b}) = q_d(\mathbf{a}+\mathbf{b}) - q_d(\mathbf{a}) - q_d(\mathbf{b}) = q_d(\mathbf{a}+\mathbf{b}).$$

This means that the conic Q_d intersects the line ℓ at three points corresponding to the vectors $\mathbf{a}, \mathbf{b}, \mathbf{a}+\mathbf{b}$. Thus ℓ is contained in Q_d. Also this implies that Q_d is a singular quadric, and hence $d \in \mathrm{He}(C)$ and its singular point c belongs to ℓ. Thus c is the third intersection point of ℓ with C.

It remains to prove the last assertion. Chose a group law on the curve $\mathrm{He}(C)$ by fixing an inflection point as the zero point. We know that the Steiner involution is defined by the translation $x \mapsto x \oplus \eta$, where η is a fixed 2-torsion point. Thus $b = a \oplus \eta$. It follows from the definition of the group law on a nonsingular cubic that the tangents $\mathbb{T}_a(\mathrm{He}(C))$ and $\mathbb{T}_b(\mathrm{He}(C))$ intersect at a point d on $\mathrm{He}(C)$. In the group law $d + 2a = 0$, hence $d = -2a$. Since a, b, c lie on a line, we get $c = -a - b$ in the group law. After subtracting, we get $d - c = b - a = \eta$. Thus the points x and c is an orbit of the Steiner involution. This shows that c is the singular point of $P_d(C)$. By Proposition 1.2.4, $P_d(C)$ contains the points a, b. Thus $\overline{a,b}$ is a component of $P_d(C)$. $\square$

It follows from the above proposition that the Cayleyan curve of a nonsingular cubic C parametrizes the line components of singular polar conics of C. It is also isomorphic to the quotient of $\mathrm{He}(C)$ by the *Steinerian involution* from Corollary 3.2.4 . Since this involution does not have fixed point the quotient map $\mathrm{He}(C) \to \mathrm{Cay}(C)$ is a unramified cover of degree 2. In particular, $\mathrm{Cay}(C)$ is a nonsingular curve of genus 1.

Let us find the equation of the Cayleyan curve. A line ℓ belongs to $\mathrm{Cay}(X)$ of and only the restriction of the linear system of polar conics of X to ℓ is of dimension 1. This translates into the condition that the restriction of the partials of X to ℓ is a linearly dependent set of three binary forms. So, write ℓ in the parametric form as the image of the map $\mathbb{P}^1 \to \mathbb{P}^2$ given by $[u,v] \mapsto [a_0u+b_0v, a_1u+b_1v, a_2u+b_2v]$. The the pull-backs of the partials from (3.19) define 3 binary forms in u,v $(a_0u+b_0v)^2+2m(a_1u+b_1v)(a_2u+b_2v)$ and so on. The condition of linear dependence is given by the vanishing of the determinant

$$\det\begin{pmatrix} a_0^2+2ma_1a_2 & 2a_0b_0+2m(a_1b_2+a_2b_1) & b_0^2+2mb_1b_2 \\ a_1^2+2ma_0a_2 & 2a_1b_1+2m(a_0b_2+a_2b_0) & b_1^2+2mb_0b_2 \\ a_2^2+2ma_0a_1 & 2a_2b_2+2m(a_0b_1+a_1b_0) & b_2^2+2mb_0b_2 \end{pmatrix}$$

The coordinates of ℓ in the dual plane are

$$[\xi_0,\xi_1,\xi_2] = [a_1b_2-a_2b_1, a_2b_0-a_0b_2, a_0b_1-a_1b_0].$$

Computing the determinant we find that the equation of $\mathrm{Cay}(X)$ in the coordinates ξ_0,ξ_1,ξ_2 is

$$\xi_0^3+\xi_1^3+\xi_2^3+6m'\xi_0,\xi_1,\xi_2 = 0, \tag{3.24}$$

where $m' = (1-4m^3)/6m$. Using the formula (3.7) for the absolute invariant of the curve, this can be translated into an explicit relationship between the absolute invariant of an elliptic curve E and the isogeneous elliptic curve $E/(t_e)$, where t_e is the translation automorphism by a non-trivial 2-torsion point.

Note that this agrees with the degree of the Cayleyan curve found in Proposition 1.1.14.

3.2.3 The dual curve

Write the equation of a general line in the form $t_2 = \xi_0t_0+\xi_1t_1$ and plug in the Hesse equation (3.21). The corresponding cubic equation has a multiple root if and only if the line is a tangent. We have

$$(\xi_0t_0+\xi_1t_1)^3+t_0^3+t_1^3+6mt_0t_1(\xi_0t_0+\xi_1t_1)$$

$$= (\xi_0^3+1)t_0^3+(\xi_1^3+1)t_1^3+(3\xi_0^2\xi_1+6m\xi_0)t_0^2t_1+(3\xi_0\xi_1^2+6m\xi_1)t_0t_1^2 = 0.$$

The condition that there is a multiple root is that the discriminant of the homogeneous cubic form in t_0,t_1 is zero. The discriminant of the cubic form $at_0^3+bt_0^2t_1+ct_0t_1^2+dt_1^3$ is equal to

$$D = b^2c^2+18abcd-4ac^3-4b^3d-27a^2d^2.$$

After plugging in, we obtain

$$(3\xi_0^2\xi_1+6m\xi_0)^2(3\xi_0\xi_1^2+6m\xi_1)^2+18(3\xi_0^2\xi_1+6m\xi_0)(3\xi_0\xi_1^2+6m\xi_1)(\xi_0^3+1)(\xi_1^3+1)$$
$$-4(\xi_0^3+1)(3\xi_0\xi_1^2+6m\xi_1)-4(\xi_1^3+1)(3\xi_1\xi_0^2+6m\xi_0)-[27(\xi_0^3+1)^2(\xi_1^3+1)^2$$
$$=-27+864\xi_0^3\xi_1^3m^3+648\xi_0^2\xi_1^2m-648m^2\xi_0\xi_1^4-648m^2\xi_0^4\xi_1+648m^2\xi_0\xi_1$$
$$+1296m^4\xi_0^2\xi_1^2-27\xi_1^6-27\xi_0^6+54\xi_0^3\xi_1^3-864\xi_1^3m^3-864\xi_0^3m^3-54\xi_1^3-54\xi_0^3.$$

It remains to homogenize the equation and divide by (-27) to obtain the equation of the dual curve

$$\xi_0^6+\xi_1^6+\xi_2^6-(2+32m^3)(\xi_0^3\xi_1^3+\xi_0^3\xi_2^3+\xi_2^3\xi_1^3)$$
$$-24m^2\xi_0\xi_1\xi_2(\xi_0^3+\xi_1^3+\xi_2^3)-(24m+48m^4)\xi_0^2\xi_1^2\xi_2^2=0. \tag{3.25}$$

According to the Plücker formula (9.49) the dual curve of a nonsingular member, being of geometric genus 1, must have 9 cusps. They correspond to the flex tangent of the original curve. The inflection points are given in (3.8). Computing the equations of the tangents we find the following singular points of the dual curve:

$$[-2m,1,1],\ [1,-2m,1],\ [1,1,-2m],\ [-2m\varepsilon,\varepsilon^2,1],\ [-2m\varepsilon,1,\varepsilon^2],$$
$$[\varepsilon^2,-2m\varepsilon,1],\ [1,-2m\varepsilon,1,\varepsilon^2],\ [1,\varepsilon^2,-2m\varepsilon],\ [\varepsilon^2,1,-2m].$$

One easily checks that the polar $P_a(C)$ with pole at a base point of the Hesse pencil contains the flex tangent at a as a line component. This shows that the Caylean curve $\mathrm{Cay}(C)$ passes through the singular points of the dual cubic. The pencil spanned by the dual cubic and the Cayleyan cubic taken with multiplicity 2 is a pencil of sextic curves with 9 double points (an *Halphen pencil* of index 2). But this is another story.

3.2.4 Polar polygons

Since for any three general points in $\mathbb{P}^2$ there exists a plane cubic singular at these points (the union of three lines), a general ternary cubic form does not admit polar triangles. Of course this is easy to see by counting constants.

A plane cubic curve projectively isomorphic to the cubic $C=V(t_0^3+t_1^3+t_2^3)$ will be called a *Fermat cubic*. Obviously, such a curve admits a polar 3-polyhedron (*polar triangle*).

Proposition 3.2.6. *A plane cubic admits a polar triangle if and only if either it is a Fermat cubic or it is equal to the union of three distinct concurrent lines.*

Proof. Suppose $C=V(l_1^3+l_2^3+l_3^3)$. Without loss of generality, we may assume that l_1^3 is not proportional to l_2^3. Thus, after coordinate change $C=V(t_0^3+t_1^3+l^3)$. If $l[t_0,t_1,t_2]$ does not depend on t_2, the curve C is the union of three distinct concurrent lines. Otherwise, we can change coordinates to assume that $l=t_2$ and get a Fermat cubic. □

Remark 3.2.3. If C is a Fermat cubic, then its polar triangle is unique. Its sides are the three first polars of C which are double lines.

By counting constants, we see that a general cubic admits a polar 4-polyhedron (*polar quadrangle*). We call a polar quadrangle $\{l_1,\ldots,l_4\}$ *nondegenerate* if it is defined by 4 points in $|E|$ no three of which are collinear. It is clear that a polar quadrangle is nondegenerate if and only if the linear system of conics in $|E|$ through the points $\ell_1,\ldots,\ell_4$ is an irreducible pencil (i.e. a linear system of dimension 1 whose general member is irreducible). This allows us to define a *nondegenerate generalized polar quadrangle* of C as a generalized polyhedron Z of C such that $|\mathcal{I}_Z(2)|$ is an irreducible pencil.

Lemma 3.2.7. *C admits a degenerate polar quadrangle if and only if it is one of the following curves:*

(i) a Fermat cubic;

(ii) a cuspidal cubic;

(ii) the union of three concurrent lines (not necessary distinct);

Proof. We have

$$t_0^3+t_1^3+t_2^3=\frac{1}{3}(t_0+t_1)^3+\frac{1}{3}(t_0+at_1)^3+\frac{1}{3}(t_0+a^2t_1)^3+t_2^3,$$

where $a=e^{2\pi i/3}$.

We also have

$$t_0t_1(9t_0+15t_1)=(t_0+t_1)^3+(t_0+2t_1)^3-2t_0^3-5t_1^3,$$
$$t_0t_1^2=(2t_0+t_1)^3+(t_0-4t_1)^3-9t_0^3+15t_1^3,$$
$$(2-c^3)t_0^3=(t_0+at_1)^3+(t_0+bt_1)^3-(ct_0+dt_1)^3-(a^3+b^3-d^3)t_1^3,$$

where $a^2+b^2=cd^2, a+b=c^2d, c^3+2\neq 0$.

All cuspidal cubics are projectively equivalently. So it is enough to demonstrate a degenerate polar quadrangle for $V(t_0^3+6t_1^2t_2)$. We have

$$t_0^3+6t_1^2t_2=(t_1+t_2)^3+(t_2-t_1)^3-2t_2^3+t_0^3.$$

Now let us prove the converse. Suppose

$$f=l_1^3+l_2^3+l_3^3+l_4^3,$$

where l_1,l_2,l_3 vanish at a common point a which we identify with a vector in E. We have

$$\frac{1}{3}D_a(f)=l_1(a)l_1^2+l_2(a)l_2^2+l_3(a)l_3^2+l_4(a)l_4=l_4(a)l_4^2.$$

This shows that the first polar $P_a(V(f))$ is either the whole $\mathbb{P}^2$ or a double line $V(l_4^2)$. In the first case C is the union of three concurrent lines. Assume that the second case occurs. We can choose coordinates such that $a=[1,0,0]$ and $l=V(t_0)$. Write

$$f=f_0t_0^3+f_1t_0^2+f_2t_0+f_3,$$

where f_i are homogeneous forms of degree i in variables t_1, t_2. Then $D_a(f) = \partial_0(f) = 3t_0^2 f_0 + 2t_0 f_1 + f_2$. This can be proportional to t_0^2 only if $f_1 = f_2 = 0$. Thus $V(f) = V(f_0 t_0^3 + f_3(t_1, t_2))$. If f_3 has no multiple linear factors, we can choose coordinates such that $f_3 = t_1^3 + t_2^3$, and get the cubic. If f_3 has a linear factor with multiplicity 2, we reduce f_3 to the form $t_1^2 t_2$. This is the case of a cuspidal cubic. Finally, if f_3 is a cube of a linear form, we reduce the latter to the form t_1^3 and get three concurrent lines.

□

Remark 3.2.4. The locus of Fermat cubics is isomorphic to the homogeneous space $\mathrm{PSL}(3)/3^2 \rtimes \mathfrak{S}_3$. Its closure in $|S^3 E^\vee|$ is a hypersurface F and consists of curves listed in the assertion of the previous Lemma and also reducible cubics equal to the unions of irreducible conics with its tangent lines. The explicit equation of the hypersurface F is given by the *Aronhold invariant* I_4 of degree 4 in the coefficients of the cubic equation. Its formula can be found in many text-books in invariant theory (e.g. [136]). If the cubic is written in a Weierstrass form $f = t_0 t_2^2 + t_1^3 + a t_0^2 t_1 + b t_0^3 = 0$, then $I_4(f) = \lambda a$, for some nonzero constant λ independent of f. A nice expression forI_4 in terms of a pfaffian of a skew-symmetric matrix was given by G. Ottaviani [304].

Lemma 3.2.8. *The following properties equivalent:*

(i) $\mathrm{AP}_1(f) \neq \{0\}$*;*

(ii) $\dim \mathrm{AP}_2(f) > 2$*;*

(iii) $V(f)$ *is equal to the union of three concurrent lines.*

Proof. By the apolarity duality

$$(A_f)_1 \times (A_f)_2 \to (A_f)_3 \cong \mathbb{C},$$

we have

$$\dim(A_f)_1 = 3 - \dim AP_1(f) = \dim(A_f)_2 = 6 - \dim \mathrm{AP}_2(f).$$

Thus $\dim \mathrm{AP}_2(f) = 3 + \dim AP_1(f)$. This proves the equivalence of (i) and (ii). By definition, $\mathrm{AP}_1(f) \neq \{0\}$ if and only if $D_\psi(f) = 0$ for some nonzero linear operator $\psi = \sum a_i \partial_i$. After a linear change of variables, we may assume that $\psi = \partial_0$, and then $\partial_0(f) = 0$ if and only if C does not depend on t_0, i.e. C is the union of three concurrent lines. □

Lemma 3.2.9. *Let Z be a nondegenerate generalized polar quadrangle of f. Then $|\mathcal{I}_Z(2)|$ is a pencil of conics in $|E^\vee|$ contained in the linear system $|\mathrm{AP}_2(f)|$. Conversely, let Z be a 0-dimensional cycle of length 4 in $|E|$. Assume that $|\mathcal{I}_Z(2)|$ is an irreducible pencil contained in $|\mathrm{AP}_2(f)|$. Then Z is a nondegenerate generalized polar quadrangle of f.*

Proof. The first assertion follows from the definition of nondegeneracy and Proposition 1.3.6. Let us prove the converse. Let $V(\lambda q_1 + \mu q_2)$ be the pencil of conics $|\mathcal{I}_Z(2)|$.

Since $\mathrm{AP}(f)$ is an ideal, the linear system L of cubics of the form $V(q_1l_1 + q_2l_2)$, where l_1, l_2 are linear forms, is contained in $|\mathrm{AP}_3(f)|$. Obviously, it is contained in $|\mathcal{I}_Z(3)|$. Since $|\mathcal{I}_Z(2)|$ has no fixed part we may choose q_1 and q_2 with no common factors. Then the map $E^\vee \oplus E^\vee \to I_Z(3)$ defined by $(l_1, l_2) \to q_1l_1 + q_2l_2$ is injective hence $\dim L = 5$. Assume $\dim |\mathcal{I}_Z(3)| \geq 6$. Choose 3 points in general position on an irreducible member C of $|\mathcal{I}_Z(2)|$ and 3 non-collinear points outside C. Then find a cubic K from $|\mathcal{I}_Z(3)|$ which passes through these points. Then K intersects C with total multiplicity $4 + 3 = 7$, hence contains C. The other component of K must be a line passing through 3 non-collinear points which is absurd. So, $\dim |\mathcal{I}_Z(3)| = 5$ and we have $L = |\mathcal{I}_Z(3)|$. Thus $|\mathcal{I}_Z(3)| \subset |\mathrm{AP}_3(f)|$ and, by Proposition 1.3.6, Z is a generalized polar quadrangle of C.

□

Corollary 3.2.10. *Suppose $C = V(f)$ is not the union of three concurrent lines. The subset of* VSP$(C; 4)$ *consisting of nondegenerated generalized polar quadrangles is isomorphic to an open subset of the plane* $|\mathrm{AP}_2(f)^\vee|$.

Example 3.2.1. Let $V(f)$ be the union of an irreducible conic and its tangent line. After a linear change of variables we may assume that $f = t_0(t_0t_1 + t_2^2)$. It is easy to check that $\mathrm{AP}_2(f)$ is spanned by $\xi_1^2, \xi_1\xi_2, \xi_2^2 - \xi_0\xi_1$. It follows from Lemma 3.2.7 that f does not admit degenerate polar quadrangle. Thus any polar quadrangles of C is the base locus of an irreducible pencil in $|\mathrm{AP}_2(f)|$. However, it is easy to see that all nonsingular conics in $|\mathrm{AP}_2(f)|$ are tangent at the point $[0, 1, 0]$. Thus no pencil has 4 distinct base points. This shows that

$$\mathrm{VSP}(f; 4)^o = \emptyset.$$

Of course, $\mathrm{VSP}(f; 4) \neq \emptyset$. Any irreducible pencil in $|\mathrm{AP}_2(f)|$ defines a generalized polar quadrangle. It is easy to see that the only reducible pencil is $V(\lambda\partial_1^2 + \mu\partial_1\partial_2)$. Thus $\mathrm{VSP}(f; 4)$ contains a subvariety isomorphic to a complement of one point in $\mathbb{P}^2 = |\mathrm{AP}_2(f)^\vee|$. To compactify it by $\mathbb{P}^2$ we need to find one more generalized polar quadrangle. Consider the subscheme Z of degree 4 concentrated at the point $[1, 0, 0]$ with ideal at this point generated by (x^2, xy, y^3), where we use inhomogeneous coordinates $x = \xi_1/\xi_0, y = \xi_2/\xi_0$. The linear system $|\mathcal{I}_Z(3)|$ is of dimension 5 and consists of cubics of the form $V(\xi_0\xi_1(a\xi_1 + b\xi_2) + g_3(\xi_1, \xi_2))$. Thus Z is linearly 3-independent. One easily computes $\mathrm{AP}_3(f)$. It is generated by all monomials except $\xi_0^2\xi_1$ and $\xi_0\xi_2^2$ and also the polynomial $\xi_0\xi_2^2 - \xi_0^2\xi_1$. We see that $|\mathcal{I}_Z(3)| \subset |\mathrm{AP}_3(f)|$. Thus Z is a generalized polar quadrangle of C. It is nondegenerate since $|\mathcal{I}_Z(2)|$ is the pencil $V(\lambda\xi_1^2 + \mu\xi_1\xi_2)$. So, we see that $\mathrm{VSP}(f; 4)$ is isomorphic to the plane $|\mathrm{AP}_2(f)|^*$.

Example 3.2.2. Let $V(f)$ be an irreducible nodal cubic. Without loss of generality, we may assume that $f = t_2^2t_0 + t_1^3 + t_1^2t_0$. The space of apolar quadratic forms is spanned by $\partial_0^2, \partial_1\partial_2, \partial_2^2 - \partial_1^2$. The net $|\mathrm{AP}_2(f)|$ is base point-free. It is easy to see that its discriminant curve is the union of three distinct non-concurrent lines. Each line defines a pencil with singular general member but without fixed part. So, $\mathrm{VSP}(f; 4) = |\mathrm{AP}_2(f)|^*$.

Example 3.2.3. Let $V(f)$ be the union of an irreducible conic and a line which intersects the conic transversally. Without loss of generality, we may assume that $f = t_0(t_0^2 + t_1t_2)$. The space of apolar quadratic forms is spanned by $\xi_1^2, \xi_2^2, 6\xi_1\xi_2 - \xi_0^2$. The net $|\mathrm{AP}_2(f)|$ is base point-free. It is easy to see that its discriminant curve is the union of a conic and a line intersecting the conic transversally. The line defines a pencil with singular general member but without fixed part. So, $\mathrm{VSP}(f;4) = |\mathrm{AP}_2(f)^\vee|$.

Example 3.2.4. Let $V(f)$ be a cuspidal cubic. Without loss of generality, we may assume that $f = t_1^2t_0 + t_2^3$. The space of apolar quadratic forms is spanned by $\xi_0^2, \xi_0\xi_2, \xi_2\xi_1$. The net $|\mathrm{AP}_2(f)|$ has 2 base points $[0,1,0]$ and $[0,0,1]$. The point $[0,0,1]$ is a simple base point. The point $[0,1,0]$ is of multiplicity 2 with the ideal locally defined by (x^2, y). Thus base point scheme of any irreducible pencil is not reduced. There are no polar 4-polyhedra defined by the base-locus of a pencil of conics in $|\mathrm{AP}_2(f)|$. The discriminant curve is the union of two lines, each defining a pencil with a fixed line component. So $|\mathrm{AP}_2(f)^\vee|$ minus 2 points parametrizes generalized polar 4-polyhedra. We know that $V(f)$ admits degenerate polar 4-polyhedra. Thus $\mathrm{VSP}(f;4)^o$ is not empty and consists of degenerate polar 4-polyhedra.

Example 3.2.5. Let $V(f)$ be a nonsingular cubic curve. We know that its equation can be reduced to a Hesse form $V(t_0^3 + t_1^3 + t_2^3 + 6at_0t_1t_2)$, where $1 + 8a^3 \neq 0$. The space of apolar quadratic forms is spanned by $a\xi_0\xi_1 - \xi_2^2,\ a\xi_1\xi_2 - \xi_0^2,\ a\xi_0\xi_2 - \xi_1^2$. The curve $V(f)$ is a Fermat cubic if and only if $a(a^3 - 1) = 0$. In this case the net has 3 ordinary base points and the discriminant curve is the union of 3 non-concurrent lines. The net has 3 pencils with fixed part defined by these lines. Thus the set of nondegenerate generalized polyhedrons is equal to the complement of 3 points in $|\mathrm{AP}_2(f)^\vee|$. We know that a Fermat cubic admits degenerate polar 4-polyhedra.

Suppose $V(f)$ is not a Fermat cubic. Then the net $|\mathrm{AP}_2(f)|$ is base point-free. Its discriminant curve is a nonsingular cubic. All pencils are irreducible. There are no degenerate generalized polygons. So, $\mathrm{VSP}(f;4) = |\mathrm{AP}_2(f)^\vee|$.

Example 3.2.6. Assume that $V(f) = V(t_0t_1t_2)$ is the union of 3 non-concurrent lines. Then $\mathrm{AP}_2(f)$ is spanned by $\xi_0^2, \xi_1^2, \xi_2^2$. The net $|\mathrm{AP}_2(f)|$ is base point-free. The discriminant curve is the union of three non-concurrent lines representing pencils without fixed point but with singular general member. Thus $\mathrm{VSP}(f;4) = |\mathrm{AP}_2(f)^\vee|$.

It follows from the previous examples that $|\mathrm{AP}_2(f)|$ is base point-free net of conics if and only if C does not belong to the closure of the orbit of Fermat cubics.

Theorem 3.2.11. *Assume that C does not belong to the closure of the orbit of Fermat cubics. Then $|\mathrm{AP}_2(f)|$ is a base point-free net of conics and*

$$\mathrm{VSP}(f;4) \cong |\mathrm{AP}_2(f)^\vee| \cong \mathbb{P}^2.$$

The variety $\mathrm{VSP}(f;4)^o$ is isomorphic to the open subset of $|\mathrm{AP}_2(f)^\vee|$ whose complement is the curve B of pencils with non-reduced base-locus. The curve B is a plane sextic with 9 cusps if $V(f)$ is a nonsingular curve, the union of three non-concurrent lines if $V(f)$ is an irreducible nodal curve or the union of three lines, and the union of a conic and its two tangent lines if $V(f)$ is the union of a conic and a line.

Proof. The first assertion follows from the Examples 3.2.2-3.2.6. Since the linear system of conics $|\mathrm{AP}_2(f)|$ is base point-free, it defines a regular map

$$\phi : |E| \to |\mathrm{AP}_2(f)^\vee|.$$

The preimage of a line is a conic from $|\mathrm{AP}_2(f)|$. The lines through a point q in $|\mathrm{AP}_2(f)^\vee|$ define a pencil with base locus $\phi^{-1}(q)$. Thus pencils with non-reduced locus are parametrized by the branch curve B of the map ϕ.

If C is a nonsingular cubic, we know from Example 3.2.5 that the discriminant curve Δ is a nonsingular cubic. A line in $|\mathrm{AP}_2(f)|$ defines a pencil of conics. Its singular members are the intersection points of the line and Δ. It is easy to see that the pencil has exactly 3 singular members if and only if its base point locus consists of 4 distinct points. Thus the curve B is the dual curve of Δ. By the duality, Δ is dual of B. We know that the dual of a nonsingular plane cubic is a plane sextic with 9 cusps.

If C is an irreducible nodal curve, we know from Example 3.2.2 that Δ is the union of three non-concurrent lines. The locus of lines intersecting Δ not transversally is the union of three pencils of lines. As above B must be the union of three non-concurrent lines.

If C is the union of a conic and a line, we know from Example 3.2.3 that Δ is the union of a conic and a line intersecting the conic transversally. Obviously, B must contain an irreducible component dual to the conic. Other irreducible components must be two tangent lines to the conic.

Finally, if C is the union of three lines, the map ϕ is given by $[t_0, t_1, t_2] \to [t_0^2, t_1^2, t_2^2]$ and as is easy to see its branch locus is the union of the coordinate lines. $\square$

Let $C \subset |S^3E^\vee| \cong \mathbb{P}^9$ be the locus of three concurrent lines. For each $V(f) \in |S^3E^\vee|\setminus C$, the space $\mathrm{AP}_2(f)$ is 3-dimensional. This defines a regular map $a : |S^3E^\vee|\setminus C \to G(3, S^2E)$. Both the varieties are 9-dimensional. Fix a 3-dimensional subspace L of S^2E and consider the linear map

$$\tilde{a} : S^3E^\vee \to \mathrm{Hom}(L, E^\vee), \quad \tilde{a}(f)(\psi) = D_\psi(f).$$

Its kernel consists of cubic forms C such that $L \subset \mathrm{AP}_2(f)$. Note that the map $\tilde{a}$ is a linear map from a 10-dimensional space to a 9-dimensional space. One expects that its kernel is 1-dimensional. This shows that, for a general point $L \in G(3, S^2E)$ the preimage a^{-1} is a one-point. Thus the map a is birational.

Exercises

3.1 Find the Hessian form of a nonsingular cubic given by the Weierstrass equation.

3.2 Show that a cubic curve given by the Hessian equation (3.5) is a harmonic (resp. equianharmonic) cubic if and only if $1 - 20m^3 - 8m^6 = 0$ (resp. $m(m^3 - 1) = 0$).

3.3 Let $H = \mathrm{He}(C)$ be the Hessian cubic of a nonsingular plane cubic curve C not isomorphic to a Fermat cubic. Let $\tau : H \to H$ be the Steinerian automorphism of H which assigns to $a \in H$ the unique singular point of $P_a(C)$.

(i) Let $\tilde{H} = \{(a, \ell) \in H \times \check{\mathbb{P}}^2 : \ell \subset P_a(C)\}$. Show that the projection $p_1 : \tilde{H} \to H$ is an unramified double cover.

(ii) Show that $\tilde{H} \cong H/\langle\tau\rangle$. [Hint: for any $(x, \ell) \in \tilde{H}$, $\ell = \langle a, \tau(a)\rangle$ for a unique point $a \in C$].

3.4 Let $C = V(f) \subset \mathbb{P}^2$ be a nonsingular cubic.

(i) Show that the set of second polars of C with respect to points on a fixed line ℓ is a conic in the dual plane. Its dual conic $C(\ell)$ in $\mathbb{P}^2$ is called the *polar conic* of the line.

(ii) Show that $C(\ell)$ is equal to the set of poles of ℓ with respect to polar conics $P_x(C)$, where $x \in \ell$.

(iii) What happens to the conic $C(\ell)$ when the line ℓ is tangent to C?

(iv) Show that the polar conic $C(\ell)$ of a nonsingular cubic C coincides with the locus of points x such that $P_x(C)$ is tangent to ℓ.

(v) Show that the set of lines ℓ such that $C(\ell)$ is tangent to ℓ is the dual curve of C.

(vi) Let $\ell = V(a_0t_0 + a_1t_1 + a_2t_2)$. Show that $C(\ell)$ can be given by the equation

$$g(a,t) = \det\begin{pmatrix} 0 & a_0 & a_1 & a_2 \\ a_0 & \frac{\partial^2 f}{\partial t_0^2} & \frac{\partial^2 f}{\partial t_0 \partial t_1} & \frac{\partial^2 f}{\partial t_0 \partial t_2} \\ a_1 & \frac{\partial^2 f}{\partial t_1 \partial t_0} & \frac{\partial^2 f}{\partial t_1^2} & \frac{\partial^2 f}{\partial t_1 \partial t_2} \\ a_2 & \frac{\partial^2 f}{\partial t_2 \partial t_0} & \frac{\partial^2 f}{\partial t_2 \partial t_1} & \frac{\partial^2 f}{\partial t_2^2} \end{pmatrix} = 0.$$

(vii) Show that the dual curve $C^\vee$ of C can be given by the equation (the *Schläfli equation*)

$$\det\begin{pmatrix} 0 & \xi_0 & \xi_1 & \xi_2 \\ \xi_0 & \frac{\partial^2 g(\xi,T)}{\partial t_0^2}(\xi) & \frac{\partial^2 g(\xi,T)}{\partial t_0 \partial t_1}(\xi) & \frac{\partial^2 g(\xi,T)}{\partial t_0 \partial t_2}(\xi) \\ \xi_1 & \frac{\partial^2 g(\xi,T)}{\partial t_1 \partial t_0}(\xi) & \frac{\partial^2 g(\xi,T)}{\partial t_1^2}(\xi) & \frac{\partial^2 g(\xi,T)}{\partial t_1 \partial t_2}(\xi) \\ \xi_2 & \frac{\partial^2 g(\xi,T)}{\partial t_2 \partial t_0}(\xi) & \frac{\partial^2 g(\xi,T)}{\partial t_2 \partial t_1}(\xi) & \frac{\partial^2 g(\xi,T)}{\partial t_2^2}(\xi) \end{pmatrix}.$$

3.5 Let $C \subset \mathbb{P}^{d-1}$ be an elliptic curve embedded by the linear system $|\mathcal{O}_C(dp_0)|$, where p_0 is a point in C. Assume $d = p$ is prime.

(i) Show that the image of any p-torsion point is an osculating point of C, i.e., a point such that there exists a hyperplane (an *osculating hyperplane*) which intersects the curve only at this point.

(ii) Show that there is a bijective correspondence between the sets of cosets of $(\mathbb{Z}/p\mathbb{Z})^2$ with respect to subgroups of order p and hyperplanes in $\mathbb{P}^{p-1}$ which cut out in C the set of p osculating points.

(iii) Show that the set of p-torsion points and the set of osculating hyperplanes define a $(p^2_{p+1}, p(p+1)_p)$-configuration of p^2 points and $p(p+1)$ hyperplanes (i.e. each point is contained in $p+1$ hyperplanes and each hyperplane contains p points).

(iv) Find a projective representaion of the group $(\mathbb{Z}/p\mathbb{Z})^2$ in $\mathbb{P}^{p-1}$ such that each osculating hyperplane is invariant with respect to some cyclic subgroup of order p of $(\mathbb{Z}/p\mathbb{Z})^2$.

3.6 A point on a nonsingular cubic is called a *sextuple point* if there exists an irreducible conic intersecting the cubic at this point with multiplicity 6. Show that there are 27 sextuple points.

3.7 The pencil of lines through a point on a nonsingular cubic curve C contains four tangent lines. Show that the twelve contact points of three pencils with collinear base points on C lie on 16 lines forming a configuration $(12_4, 16_3)$ (the *Hesse-Salmon configuration*).

3.8 Show that the polar of the cubic with pole at its inflection point is the union of the tangent line at this point and the harmonic line ℓ which intersects the cubic at three points which are the nonzero 2-torsion points with respect to the group law with the pole equal to the zero point. Show that the nine harmonic lines and 12 singular points of singular members of the pencil form a configuration $(9_4, 12_3)$ (the *dual Hesse configuration of lines and points*).

3.9 Prove that the second polar of a nonsingular cubic C with respect to the point a on the Hessian He(C) is equal to the tangent line $\mathbb{T}_a(\text{He}(C))$.

3.10 Let a, b be two points on the Hessian curve He(C) forming an orbit with the respect to the Steinerian involution. Show that the line $\overline{a,b}$ is tangent to Cay(C) at some point d. Let c be the third intersection point of He(C) with the line $\overline{a,b}$. Show that the pairs (a, b) and (c, d) are harmonically conjugate.

3.11 Show that from each point a on the He(C) one can pass three tangent lines to Cay(C). Let b be the singular point of $P_a(C)$. Show that the set of the three tangent lines consists of the line $\overline{a,b}$ and the components of the reducible polar conic $P_b(C)$.

3.12 Let $C = V(\sum_{0\le i\le j\le k\le 2} a_{ijk}t_it_jt_k)$. Show that the Cayleyan curve Cay(C) can be given by the equation

$$\det\begin{pmatrix} a_{000} & a_{001} & a_{002} & \xi_0 & 0 & 0\\ a_{110} & a_{111} & a_{112} & 0 & \xi_1 & 0\\ a_{220} & a_{221} & a_{222} & 0 & 0 & \xi_2\\ 2a_{120} & 2a_{121} & 2a_{122} & 0 & \xi_2 & \xi_1\\ 2a_{200} & 2a_{201} & 2a_{202} & \xi_2 & 0 & \xi_1\\ 2a_{010} & 2a_{011} & 2a_{012} & \xi_1 & \xi_0 & 0 \end{pmatrix} = 0$$

[82], p. 245. Generalize this formula to the case of a net of conics not necessary of polars to a cubic curve. The corresponding curve parameterizing line components of singular conics is called the *Hermite curve*.

3.13 Show that the group of projective transformations leaving a nonsingular plane cubic invariant is a finite group of order 18, 36 or 54. Determine these groups.

3.14 Let C be nonsingular projective curve C of genus 1.

(i) Show that C is isomorphic to a curve in the weighted projective plane $\mathbb{P}(1, 1, 2)$ given by the equation $t_2^2 + p_4(t_0, t_1) = 0$, where p_4 is a homogeneous polynomial of degree 4 (a binary quartic).

(ii) Show that a general binary quartic can be reduced by a linear change of variables to the form $t_0^4 + t_1^4 + 6at_0^2t_1^2$. (Hint: write p_4 as the product of two quadratic forms, and reduce them simultaneously to sum of squares).

(iii) Show that for C such reduction with $a \ne \pm\frac{1}{3}$ is always possible.

(iv) Show that the linear symmetries of the reduced quartic define a group of automorphisms of C which can be identified, after a choice of a group law on C, with the group of translations by 2-torsion points.

(v) Show that the absolute invariant of C is related to the coefficient a from part (i) via the formula $j = \frac{(1+3a^2)^3}{(9a^2-1)^2}$.

(vi) Show that a harmonic (resp. equianharmonic) cubic corresponds to the binary quartics $t_0^4 \pm 6t_0^2t_1^2 + t_1^4$ (resp. $t_0^4 \pm 2\sqrt{-3}t_0^2t_1^2 + t_1^4$).

3.15 Find all ternary cubics C such that VSP$(C;4)^o = \emptyset$.

3.16 Show that a plane cubic curve belongs to the closure of the Fermat locus if and only if it admits a first polar equal to a double line or the whole space.

3.17 Show that any plane cubic curve is equal to the set of intersection points of corresponding members of a pencil of lines and a pencil of conics.

Historical Notes

The discovery that any plane cubic can be written by a Weierstrass equation is due to Newton. It was Weierstrass who showed that the equation can be parametrized by elliptic functions, the Weierstrass functions $\wp(z)$ and $\wp(z)'$. The Hesse pencil was introduced and studied by O. Hesse [209],[210]. It has also known as the *syzygetic pencil* (see [82]). More facts about the Hesse pencils and its connection to other constructions in modern algebraic geometry can be found in [11].

The equations of the Cayleyan curve of a plane cubic given in the Hesse form can be found in [356]. The equation from Exercise 3.12 is taken from [82]. The equation of the dual cubic curve given in the Hesse form can be found in [356]. The Schläfli equation from Exericise 3.4 was given by L. Schläfli in [360]. Its modern proof is given in [183].

The polar polygons of a plane cubics were first studied by F. London [270] and G. Scorza [368]. A modern treatment of some of their results is given in [132] (see also [329] for related results).

As always we refer for more historical information and survey of many results which were omitted in our exposition to classical books [82], [308], [143], [364].

Chapter 4

Determinantal equations

4.1 Plane curves

4.1.1 The problem

Here we will try to solve the following problem. Given a homogeneous polynomial $f(t_0, \ldots, t_n)$ find a $d \times d$ matrix $A = (l_{ij}(t))$ with linear forms as its entries such that

$$f(t_0, \ldots, t_n) = \det(l_{ij}(t)). \tag{4.1}$$

We will also try to find in how many essentially different ways one can do it.

First let us reinterpret this problem geometrically and coordinate free. Let E be a vector space of dimension $n+1$ and let V, W be vector spaces of dimension d. A square matrix corresponds to a linear map $V \to W$, or an element of $V^\vee \otimes W$. A matrix with linear forms corresponds to an element of $E^\vee \otimes V^\vee \otimes W$, or a linear map $\phi' : E \to V^\vee \otimes W$.

We shall assume that the map ϕ is injective (otherwise the hypersurface $V(f)$ is a cone, so we can solve our problem by induction on the number of variables). Let

$$\phi : |E| \to |V^\vee \otimes W| \tag{4.2}$$

be the regular map of the associated projective spaces. Let $\mathcal{D}_d \subset |V^\vee \otimes W|$ be the hypersurface parameterizing non-invertible linear maps. If we choose bases in V, W, then $\mathcal{D}_d$ is given by the determinant of a square matrix (whose entries will be coordinates in $V^\vee \otimes W$). The preimage of $\mathcal{D}_d$ in $|E|$ is a hypersurface $V(f)$ of degree d. Our problem is to construct such a map ϕ in order that a given hypersurface is obtained in this way.

Note that the singular locus $\mathcal{D}_d^{\text{sing}}$ of the determinantal variety $\mathcal{D}_d$ corresponds to matrices of corank ≥ 2. It is easy to see that its codimension in $|V^\vee \otimes W|$ is equal to 4. If the image of $|E|$ intersects $\mathcal{D}_d^{\text{sing}}$, then $\phi^{-1}(\mathcal{D}_d)$ will be a singular hypersurface. So, a nonsingular hypersurface of dimension ≥ 3 cannot be given by a determinantal equation.

4.1.2 Plane curves

Let us first consider the case of nonsingular plane curves $C = V(f) \subset \mathbb{P}^2$. Assume that C admits a determinantal form. As we have explained, the image of the map ϕ does not intersect $\mathcal{D}_d^{\text{sing}}$. Thus, for any $x \in C$, the corank of the matrix $\phi(x)$ is equal to 1 (here we consider a matrix up to proportionality since we are in the projective space). The kernel of this matrix is a one-dimensional subspace of V, i.e., a point in $|V|$. This defines a regular map

$$r : C \to |V|, \quad x \mapsto \text{Ker}(\phi(x)).$$

Now let ${}^t\phi(x) : W^\vee \to V^\vee$ be the transpose map. In coordinates, it corresponds to the transpose matrix. Its kernel is isomorphic to $\text{Im}(\phi(x))^\perp$ and is also one-dimensional. So we have another regular map

$$l : C \to |W^\vee|, \quad x \mapsto \text{Ker}({}^t\phi(x)).$$

Let

$$\mathcal{L} = r^*\mathcal{O}_{|V|}(1), \quad \mathcal{M} = l^*\mathcal{O}_{|W^\vee|}(1).$$

These are invertible sheaves on the curve C. We can identify V with $H^0(C, \mathcal{L})^\vee$ and W with $H^0(C, \mathcal{M})$ (see Lemma 4.1.2 below). Consider the composition of regular maps

$$\psi : C \xrightarrow{r \times l} |V| \times |W^\vee| \xrightarrow{s_2} |V \otimes W^\vee|, \tag{4.3}$$

where s_2 is the Segre map. It follows from the definition of the Segre map, that the tensor $\psi(x)$ is equal to $r(x) \otimes l(x)$. It can be viewed as a linear map $V^\vee \to W^\vee$. In coordinates, the matrix of this map is the product of the column vector defined by $\text{Ker}(\phi(x))$ and the row vector defined by $\text{Ker}({}^t\phi(x))$. It is a rank 1 matrix equal to the adjugate matrix of the matrix $A = \phi(x)$ (up to proportionality). Recall that a square matrix of rank 1 has a solution defined by any column of the adjugate matrix (since we have $A \cdot \text{adj}(A) = 0$). Similarly, the kernel of the transpose of A is given by any row of the adjugate matrtix. Thus the entries of the matrix $\psi(x)$ are cofactors of the matrix $\phi(x)$. Consider the rational map

$$\text{Adj} : |V^\vee \otimes W| - \to |V \otimes W^\vee| \tag{4.4}$$

defined by taking the adjugate matrix. Recall that the adjugate matrix should be considered as a linear map $\bigwedge^{d-1} V \to \bigwedge^{d-1} W$ and we can identify $|\bigwedge^{d-1} V^\vee \otimes \bigwedge^{d-1} W|$ with $|V \otimes W^\vee|$. Although it is not well-defined on vector spaces, it is well-defined, as a rational map, on the projective spaces (see Example 1.1.2). Let $\Psi = \text{Adj} \circ \phi$, then ψ is equal to the restriction of Ψ to C. Since Adj is defined by polynomials of degree $d-1$ (after we choose bases in V, W), we have

$$\Psi^*\mathcal{O}_{|V \otimes W^\vee|}(1) = \mathcal{O}_{|E|}(d-1).$$

This gives

$$\psi^*\mathcal{O}_{|V \otimes W^\vee|}(1) = \mathcal{O}_{|E|}(d-1) \otimes \mathcal{O}_C = \mathcal{O}_C(d-1).$$

On the other hand, we get

$$\psi^*\mathcal{O}_{|V \otimes W^\vee|}(1) = (s_2 \circ (r \times l))^*\mathcal{O}_{|V \otimes W^\vee|}(1)$$

$$= (r\times l)^*\big(s_2^*\mathcal{O}_{|V\otimes W^\vee|}(1)\big) = (r\times l)^*\big(p_1^*\mathcal{O}_{|V|}(1)\otimes p_2^*\mathcal{O}_{|W^\vee|}(1)\big)$$
$$= r^*\mathcal{O}_{|V|}(1)\otimes l^*\mathcal{O}_{|W|}(1) = \mathcal{L}\otimes\mathcal{M}.$$

Here $p_1 : |V|\times|W^\vee| \to |V|,\ p_2 : |V|\times|W^\vee|\to|W^\vee|$ are the projection maps. Comparing the two isomorphisms, we obtain

Lemma 4.1.1.

$$\mathcal{L}\otimes\mathcal{M}\cong\mathcal{O}_C(d-1). \tag{4.5}$$

Remark 4.1.1. It follows from Example 1.1.2 that the rational map (4.4) is given by the polars of the determinantal hypersurface. In fact, if $A = (t_{ij})$ is a matrix with independent variables as entries, then $\frac{\partial\det(A)}{\partial t_{ij}} = M_{ij}$, where M_{ij} is the ij-th cofactor of the matrix A. The map Adj is a birational map since Adj$(A) = A^{-1}\det(A)$ and the map $A\to A^{-1}$ is obviously invertible. So, the determinantal equation is an example of a homogeneous polynomial such that the corresponding polar map is a birational map. Such a polynomial is called a *homaloidal polynomial* (see [134]).

Lemma 4.1.2. *Let* $g = \frac{1}{2}(d-1)(d-2)$ *be the genus of the curve* C. *Then*

(i) $\deg(\mathcal{L}) = \deg(\mathcal{M}) = \frac{1}{2}d(d-1) = g-1+d$*;*

(ii) $H^0(C,\mathcal{L})\cong V^\vee,\ H^0(C,\mathcal{M}) = W$*;*

(iii) $H^i(C,\mathcal{L}(-1))\cong H^i(C,\mathcal{M}(-1)) = \{0\},\ i = 0,1.$

Proof. Let us first first prove (iii). A nonzero section of $H^0(C,\mathcal{L}(-1))$ is a section of $\mathcal{L}$ which defines a hyperplane in $|V|$ which intersects the image $r(C)$ of the curve C along a divisor $r(D)$, where D is cut out in C by a line. Since all such divisors D are linear equivalent, we see that for any line ℓ the divisor $r(\ell\cap C)$ is cut out by a hyperplane in $|V|$. Choose ℓ such that it intersects C at d distinct points $x_1,\ldots,x_d$. Choose bases in V and W. The image of $\phi(\ell)$ in $|V^\vee\otimes W| = \mathbb{P}(\mathrm{Mat}_d)$ is a pencil of matrices $\lambda A+\mu B$. We know that there are d distinct values of (λ,μ) such that the corresponding matrix is of corank 1. Without loss of generality, we may assume that A and B are nonsingular matrices. So we have d distinct λ_i such that the matrix $A+\lambda_iB$ is singular. Let $v_1,\ldots,v_d$ be the generators of Ker$(A+\lambda_iB)$. The corresponding points in $|V|$ are equal to the points $r(x_i)$. We claim that the vectors $v_1,\ldots,v_d$ are linearly independent vectors in V. The proof is by induction on d. Assume $a_1v_1+\cdots+a_dv_d = 0$. Then $Av_i+\lambda_iBv_i = 0$ for each $i = 1,\ldots,d$ gives

$$0 = A\big(\sum_{i=1}^d a_iv_i\big) = \sum_{i=1}^d a_iAv_i = -\sum_{i=1}^d a_i\lambda_iBv_i.$$

We also have

$$0 = B\big(\sum_{i=1}^d a_iv_i\big) = \sum_{i=1}^d a_iBv_i.$$

Multiplying the second equality by λ_d and adding it to the first one, we obtain

$$\sum_{i=1}^{d-1} a_i(\lambda_d - \lambda_i)Bv_i = B\big(\sum_{i=1}^{d-1} a_i(\lambda_d - \lambda_i)v_i\big) = 0.$$

Since B is invertible, this gives

$$\sum_{i=1}^{d-1} a_i(\lambda_i - \lambda_d)v_i = 0.$$

By induction, the vectors $v_1, \ldots, v_{d-1}$ are linearly independent. Since $\lambda_i \neq \lambda_d$, we obtain $a_1 = \ldots = a_{d-1} = 0$. Since $v_d \neq 0$, we also get $a_d = 0$.

Since $v_1, \ldots, v_d$ are linearly independent, the points $r(x_i)$ span $\mathbb{P}(W)$. Hence no hyperplane contains these points. This proves $H^0(C, \mathcal{L}(-1)) = 0$. Similarly, we prove that $H^0(C, \mathcal{M}(-1)) = 0$. Applying Lemma 4.1.1 we get

$$\mathcal{L}(-1) \otimes \mathcal{M}(-1) \cong \mathcal{O}_C(d-3) = \omega_C, \tag{4.6}$$

where ω_C is the canonical sheaf on C. By duality,

$$H^i(C, \mathcal{M}(-1)) \cong H^{1-i}(C, \mathcal{L}(-1)),\ i = 0, 1.$$

This proves (iii). Let us prove (i) and (ii). Let H be a section of $\mathcal{O}_C(1)$. The exact sequence

$$0 \to \mathcal{L}(-1) \to \mathcal{L} \to \mathcal{L} \otimes \mathcal{O}_H \to 0$$

gives, by passing to cohomology and applying (iii),

$$H^1(C, \mathcal{L}) = 0.$$

Replacing $\mathcal{L}$ with $\mathcal{M}$ and repeating the argument, we obtain that $H^1(C, \mathcal{M}) = 0$. We know that $\dim H^0(C, \mathcal{L}) \geq \dim V^\vee = d$. Applying Riemann-Roch, we obtain

$$\deg(\mathcal{L}) = \dim H^0(C, \mathcal{L}) + g - 1 \geq d + g - 1.$$

Similarly, we get

$$\deg(\mathcal{M}) \geq d + g - 1.$$

Adding up, and applying Lemma 4.1.1, we obtain

$$d(d-1) = \deg \mathcal{O}_C(d-1) = \deg(\mathcal{L}) + \deg(\mathcal{M}) \geq 2d + 2g - 2 = d(d-1).$$

Thus all the inequalities in above are the equalities, and we get assertions (i) and (ii).

□

Now we would like to prove the converse. Let $\mathcal{L}$ and $\mathcal{M}$ be invertible sheaves on C satisfying (4.5) and properties from the previous Lemma hold.

Let $r : C \to |V|, l : C \to \mathbb{P}(W^\vee)$ be the maps given by the complete linear systems $|\mathcal{L}|$ and $|\mathcal{M}|$. We define $\psi : C \to |V \otimes W^\vee|$ to be the composition of $r \times l$ and the Segre map s_2. It follows from property (4.5) that the map ψ is the restriction of the map $\Psi : |E| \to |V \otimes W^\vee|$ given by a linear system of plane curves of degree $d-1$. We can view this map as a tensor in $S^{d-1}(E^\vee) \otimes V \otimes W^\vee$. In coordinates, it is a $d \times d$ matrix $A(t)$ with entries from the space of homogeneous polynomials of degree $d-1$. Since $\Psi|_C = \psi$, for any point $x \in C$, we have $\mathrm{rank} A(x) = 1$. Let M be a 2×2 submatrix of $A(t)$. Since $\det M(x) = 0$ for $x \in C$, we have $f \mid \det M$. Consider a 3×3 submatrix N of $A(t)$. We have $\det \mathrm{adj}(N) = \det(N)^2$. Since the entries of $\mathrm{adj}(N)$ are determinants of 2×2 submatrices, we see that $f^3 \mid \det(N)^2$. Since C is irreducible, this immediately implies that $f^2 \mid \det(N)$. Continuing in this way we obtain that f^{d-2} divides all cofactors of the matrix A. Thus $B = f^{2-d}\mathrm{adj}(A)$ is a matrix with entries in $E^\vee$. Since $\mathrm{rank}\, B = \mathrm{rank}\, \mathrm{adj}(A)$, and $\mathrm{rank}\, A(x) = 1$, we get that $\mathrm{rank}\, B(x) = d-1$ for any $x \in C$. So, if $\det B$ is not identically zero, we obtain that $V(\det(B))$ is a hypersurface of degree d vanishing on C, hence $\det(B) = \lambda f$ for some $\lambda \in K^*$. This shows that $C = V(\det(B))$. To see that $\det(B) \neq 0$, we have to use property (iii) of Lemma 4.1.2. Reversing the proof of this property, we see that for a general line ℓ in $|E|$ the images of the points $x_i \in \ell \cap C$ in $|V| \times \mathbb{P}(W^\vee)$ are the points (a_i, b_i) such that the a_i's span $|V|$ and the b_i's span $\mathbb{P}(W^\vee)$. The images of the x_i's in $|V \otimes W^\vee|$ under the map Ψ span a subspace L of dimension $d-1$. If we choose coordinates so that the points a_i and b_i are defined by the unit vectors $(0, \ldots, 1, \ldots, 0)$, then L corresponds to the space of diagonal matrices. The image of the line ℓ under Ψ is a Veronese curve of degree $d-1$ in L. A general point $\Psi(x), x \in \ell$, on this curve does not belong to any hyperplane in L spanned by $d-1$ points x_i's, thus it can be written as a linear combination of the points $\Psi(x_i)$ with nonzero coefficients. This represents a matrix of rank d. This shows that $\det A(x) \neq 0$ and hence $\det(B(x)) \neq 0$.

To sum up, we have proved the following theorem.

Theorem 4.1.3. *Let $C \subset \mathbb{P}^2$ be a nonsingular plane curve of degree d. Let $\mathrm{Pic}(C)^{g-1}$ be the Picard variety of isomorphism classes of invertible sheaves on C of degree $g-1$ (or divisor classes of degree $g-1$). Let $W_{g-1} \subset \mathrm{Pic}^{g-1}(C)$ be the subset parameterizing invertible sheaves $\mathcal{F}$ with $H^0(C, \mathcal{F}) \neq \{0\}$ (or effective divisors of degree $g-1$). Let $\mathcal{L}_0 \in \mathrm{Pic}^{g-1}(C) \setminus W_{g-1}$, and $\mathcal{M}_0 = \omega_C \otimes \mathcal{L}_0^{-1}$. Then $V \cong H^0(C, \mathcal{L}_0(1))^*$ and $W \cong H^0(C, \mathcal{M}_0(1))$ have dimension d and there is a unique regular map $\phi : \mathbb{P}^2 \to |V^\vee \otimes W|$ such that C is equal to the preimage of the determinantal hypersurface $\mathcal{D}_d$ and the maps $r : C \to |V|$ and $l : C \to \mathbb{P}(W^\vee)$ given by the complete linear systems $|\mathcal{L}_0(1)|$ and $|\mathcal{M}_0(1)|$ coincide with the maps $x \mapsto \mathrm{Ker}(\phi(x))$ and $x \mapsto \mathrm{Ker}({}^t\phi(x))$, respectively. Conversely, given a map $\phi : \mathbb{P}^2 \to |V^\vee \otimes W|$ such that $C = \phi^{-1}(\mathcal{D}_d)$ there exists a unique $\mathcal{L}_0 \in \mathrm{Pic}^{g-1}(C)$ such that $V \cong H^0(C, \mathcal{L}_0(1))^*$, $W \cong H^0(C, \omega_C(1) \otimes \mathcal{L}_0^{-1})$ and the map ϕ is defined by $\mathcal{L}$ as above.*

Remark 4.1.2. Let X be the set of $d \times d$ matrices $A(t)$ with entries in $E^\vee$ such that $f = \det A(t)$. The group $G = \mathrm{GL}(d) \times \mathrm{GL}(d)$ acts on the set by

$$(\sigma_1, \sigma_2) \cdot A = \sigma_1 \cdot A \cdot \sigma_2^{-1}.$$

It follows from the Theorem that the orbit space X/G is equal to $\mathrm{Pic}^{g-1}(C) \setminus W_{g-1}$.

We map $\mathcal{L}_0 \mapsto \mathcal{M}_0 = \omega_C \otimes \mathcal{L}_0^{-1}$ is an involution on $\mathrm{Pic}^{g-1} \setminus W_{g-1}$. It corresponds to the involution on X defined by taking the transpose of the matrix.

4.1.3 The symmetric case

Let us assume that the determinant representation of a plane irreducible curve C of degree d is given by a pair of equal invertible sheaves $\mathcal{L} = \mathcal{M}$. It follows from Lemmas 4.1.1 and 4.1.2 that

- $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_C(d-1)$;
- $\deg(\mathcal{L}) = \frac{1}{2}d(d-1)$;
- $H^0(C, \mathcal{L}(-1)) = \{0\}$.

Recall that the canonical sheaf ω_C is isomorphic to $\mathcal{O}_C(d-3)$. Thus

$$(\mathcal{L}(-1))^{\otimes 2} \cong \omega_C. \tag{4.7}$$

Definition 4.1. *Let X be a curve with a canonical invertible sheaf ω_X (e.g. a nonsingular curve, or a curve on a nonsingular surface). An invertible sheaf $\mathcal{N}$ whose tensor square is isomorphic to ω_X is called a* theta characteristic. *A theta characteristic is called* even *(resp.* odd*) if* $\dim\ H^0(X, \mathcal{N})$ *is even (resp. odd).*

Using this definition we can express (4.7) by saying that

$$\mathcal{L} \cong \mathcal{N}(1),$$

where $\mathcal{N}$ is an even theta characteristic (because $H^0(C, \mathcal{N}) = \{0\}$). Of course, the latter condition is stronger. An even theta characteristic with no nonzero global sections (resp. with nonzero global sections) is called a *non-effective theta characteristic* (resp. *effective theta characteristic*).

Rewriting the previous subsection in the special case $\mathcal{L} = \mathcal{M}$ we obtain that $V^\vee = H^0(C, \mathcal{L}) = H^0(C, \mathcal{M}) = W$. The maps $l = r$ given by the linear systems $\mid \mathcal{L} \mid$ and $\mid \mathcal{M} \mid$ and define a map $r \times r : C \to |V| \times |V|$. Its composition with the Segre map $|V| \times |V| \to |V \otimes V|$ and the projection to $|S^2V|$ defines a map

$$\phi : C \to |S^2V|.$$

In coordinates, it is given by

$$\psi(x) = \tilde{r}(x) \cdot {}^t\tilde{r}(x),$$

where $\tilde{r}(x)$ is the column of projective coordinates of the point $r(x)$. It is clear that the image of the map ψ is contained in the variety of rank 1 quadrics in the dual space $|V^\vee| = \mathbb{P}(V)$. It follows from the proof of Theorem 4.1.3 that there exists a linear map $\phi : \mathbb{P}^2 \to |S^2V^\vee|$ such that its composition with the rational map defined by taking the adjugate matrix equals, after restriction to C, the map ψ. The image of ϕ is a net N of quadrics in $|V|$. The image of C is the locus of singular quadrics in N. For each point $x \in C$, we denote the corresponding quadric by Q_x. The rational map l (regular if C is nonsingular) is defined by assigning to a point $x \in C$ the singular point of the quadric Q_x. The image X of C in $|V|$ is a curve of degree equal to $\deg \mathcal{L} = \frac{1}{2}d(d-1)$.

Proposition 4.1.4. *The restriction map*

$$r : H^0(|V|, \mathcal{O}_{|V|}(2)) \to H^0(X, \mathcal{O}_X(2))$$

is surjective. Under the isomorphism

$$H^0(X, \mathcal{O}_X(2)) \cong H^0(C, \mathcal{L}^{\otimes 2}) \cong H^0(C, \mathcal{O}_C(d-1)),$$

the space of quadrics in $|V|$ is identified with the space of plane curves of degree $d-1$. The net of quadrics N is identified with the linear system of first polars of the curve C.

Proof. Reversing the proof of property (iii) from Lemma 4.1.2 shows that the image of C under the map $\psi : C \to |V \otimes W^\vee|$ spans the space. In our case, this implies that the image of C under the map $C \to |S^2V \vee|$ spans the space of quadrics in the dual space. If the image of C in $|V|$ were contained in a quadric Q, then Q would be apolar to all quadrics in the dual space, a contradiction. Thus the restriction map r is injective. Since the spaces have the same dimension, it must be surjective.

The composition of the map $i : \mathbb{P}^2 \to |\mathcal{O}_{|V|}(2)|, x \mapsto Q_x$, and the isomorphism $|\mathcal{O}_{|V|}(2)| \cong |\mathcal{O}_{\mathbb{P}^2}(d-1)|$ is a map $s : \mathbb{P}^2 \to |\mathcal{O}_{\mathbb{P}^2}(d-1)|$. A similar map s' is given by the first polars $x \mapsto P_x(C)$. We have to show that the two maps coincide. Recall that $P_x(C) \cap C = \{c \in C : x \in \mathbb{T}_c(C)\}$. In the next Lemma we will show that the quadrics $Q_x, x \in \mathbb{T}_c(C)$, form the line in N of quadrics passing through the singular point of Q_c equal to $r(c)$. This shows that the quadric $Q_{r(x)}$ cuts out in $r(C)$ the divisor $r(P_x(C) \cap C)$. Thus the curves $s(x)$ and $s'(x)$ of degree $d-1$ cut out the same divisor on C, hence they coincide. □

Lemma 4.1.5. *Let $W \subset S^dV^\vee$ be a linear subspace, and $D \subset \mathbb{P}(W)$ be the locus of singular hypersurfaces. Assume $x \in D$ is a nonsingular point. Then the corresponding hypersurface has a unique ordinary double point y and the embedded tangent space $\mathbb{T}_x(D)$ is equal to the hyperplane of hypersurfaces containing y.*

Proof. Assume $W = S^dV^\vee$. Then D coincides with the discriminant hypersurface $\mathcal{D}$ of all singular degree d hypersurfaces in $|V|$. In this case the assertion follows from 1.2.1, where we described explicitly the tangent space of $\mathcal{D}$ at any point. Since $D = \mathbb{P}(W) \cap \mathcal{D}$ and $x \in D$ is a nonsingular point, the intersection is transversal and $\mathbb{T}_x(D) = \mathbb{T}_x(\mathcal{D}) \cap \mathbb{P}(W)$. This is our assertion.

□

We see that a pair $(C, \mathcal{N})$, where C is a plane irreducible curve and $\mathcal{N}$ is a non-effective even theta characteristic on C defines a net N of quadrics in $|V|$, where $V = H^0(C, \mathcal{N}(1))^\vee$. Conversely, given a net N of quadrics in $\mathbb{P}^{d-1} = |V|$. It is known that the singular locus of the discriminant hypersurface $\mathcal{D}_2(d-1)$ of quadrics in $\mathbb{P}^{d-1}$ is of codimension 2. Thus a general net N intersects $\mathcal{D}_2(d-1)$ transversally along a nonsingular curve C of degree d. This gives a representation of C as a symmetric determinant and hence defines an invertible sheaf $\mathcal{L}$ and a non-effective even theta characteristic $\mathcal{N}$. This gives a dominant rational map of varieties of dimension $(d^2+3d-16)/2$

$$G(3,S^2V^\vee)/\mathrm{PGL}(V) - \to |\mathcal{O}_{\mathbb{P}^2}(d)|/\mathrm{PGL}(3). \tag{4.8}$$

The degree of this map is equal to the number of non-effective even theta characteristics on a general curve of degree d. We will see in the next chapter that the number of even theta characteristics is equal to $2^{g-1}(2^g+1)$, where $g=(d-1)(d-2)/2$ is the genus of the curve. A curve C of odd degree $d=2k+3$ has a unique vanishing even theta characteristic equal to $\mathcal{N}=\mathcal{O}_C(k)$ with $h^0(\mathcal{N})=(k+1)(k+2)/2$. A general curve of even degree does not have vanishing even theta characteristics.

Observe that under the isomorphism from Proposition 4.1.4, the variety of quadrics of rank 1 (i.e. double hyperplanes) is mapped isomorphically to the variety of plane curves of degree $d-1$ which are everywhere tangent to the curve C. We call these curves *contact curves* of order $d-1$. Thus any symmetric determinantal representation of C determines an algebraic system of dimension $d-1$ of contact curves of degree $d-1$.

Proposition 4.1.6. *Let $C=V(f)$, where C is equal to the determinant of a $d\times d$ symmetric matrix (l_{ij}) of linear forms. Then the corresponding algebraic system of contact curves of degree $d-1$, considered as a hypersurface in $|E|\times|V|$ of bidegree $(d-1,2)$, is given by the equation*

$$\det\begin{pmatrix} l_{11} & \dots & l_{1d} & u_0\\ l_{21} & \dots & l_{2d} & u_1\\ \vdots & \vdots & \vdots & \vdots\\ l_{d1} & \dots & l_{dd} & u_{d-1}\\ u_0 & \dots & u_{d-1} & 0\end{pmatrix}=0. \tag{4.9}$$

Proof. Obviously, the *bordered determinant* (4.9) can be written in the form

$$\sum_{i,j=0}^{d-1} A_{ij}u_iu_j,$$

where A_{ij} is the (ij)-cofactor of the matrix A. For any $x\in C$, the rank of the cofactor matrix adj$A(x)$ is equal to 1. Thus the quadratic form on the dual space of $|V|$ with coordinates $u_0,\dots,u_{d-1}$ defined by the above equation is of rank 1. Hence it is equal to the double hyperplane $H^2=(\sum a_iu_i)^2$, where $[a_0,\dots,a_{d-1}]\in\mathbb{P}(W)$ belongs to the null-space of the matrix $A(x)$. Under the identification of $\mathbb{P}^{d-1}=|V|$ with $|\mathcal{O}_{\mathbb{P}^2}(\mathcal{L})|^*$, the hyperplane H in the dual space corresponds to the point $l(x)$, where $l: C\to\mathbb{P}^{d-1}$ is the map defined by the symmetric determinantal representation of C. This checks the assertion. ☐

Consider the multiplication map

$$H^0(C,\mathcal{N}(1))\otimes H^0(C,\mathcal{N}(1))\to H^0(C,K_C(2)). \tag{4.10}$$

Let us use the notation $V = H^0(C, \mathcal{N}(1)) \cong \mathbb{P}^{d-1}$ as above. Passing to the projective spaces, we get a regular map

$$|V| \times |V| \to \mathbb{P}(K_C(2)) = |\mathcal{O}_{\mathbb{P}^2}(d-1)|, \ (D_1, D_2) \mapsto D_1 + D_2. \tag{4.11}$$

It defines a hypersurface $F \subset |V| \times |V| \times |E| = \mathbb{P}^{d-1} \times \mathbb{P}^{d-1} \times \mathbb{P}^2$

$$F = \{(D_1, D_2, x) \in |V| \times |V| \times |E| : x \in D_1 + D_2\}. \tag{4.12}$$

It is equal to the divisor of zeros of a tri-homogeneous form of degree $(1, 1, d-1)$, symmetric in the variables of degree 1. In coordinates, it is equal to

$$F : \sum_{0 \le i,j \le d-1} a_{ij}(t_0, t_1, t_2) u_i v_j = 0,$$

where $a_{ij} = a_{ji}$ are homogeneous forms of degree $d-1$. The projection $F \to \mathbb{P}^{d-1} \times \mathbb{P}^{d-1}$ is a family of curves of degree $d-1$ parametrized by $\mathbb{P}^{d-1} \times \mathbb{P}^{d-1}$. The projection to $\mathbb{P}^2$ is a family of divisors of type $(1, 1)$ on $\mathbb{P}^{d-1} \times \mathbb{P}^{d-1}$. For any $x \in C$, the set of divisors (D_1, D_2) containing x is the union of two divisors $H_x \times \mathbb{P}^{d-1}$ and $\mathbb{P}^{d-1} \times H_x$, where H_x is the hypersurface of divisors in $|\mathcal{N}(1)|$ containing the point x. Since this divisor is singular, the curve C is contained in the locus Σ of points parameterizing singular fibres of $F \to \mathbb{P}^2$. Note that Σ is given by the determinant of the matrix (a_{ij}) from above, and its degree is equal to $d(d-1)$. We can find the equation of Σ using the following beautiful determinant identity due to O. Hesse [213].

Lemma 4.1.7. *Let $A = (a_{ij})$ be a square matrix of size k. For any $\mathbf{x} = (x_1, \ldots, x_k)$ and $\mathbf{y} = (y_1, \ldots, y_k)$*

$$\begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1k} & x_1 \\ a_{21} & a_{22} & \ldots & a_{2k} & x_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{k1} & a_{k2} & \ldots & a_{kk} & x_k \\ x_1 & x_2 & \ldots & x_k & 0 \end{vmatrix} \times \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1k} & y_1 \\ a_{21} & a_{22} & \ldots & a_{2k} & y_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{k1} & a_{k2} & \ldots & a_{kk} & y_k \\ y_1 & y_2 & \ldots & y_k & 0 \end{vmatrix}$$

$$- \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1k} & x_1 \\ a_{21} & a_{22} & \ldots & a_{2k} & x_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{k1} & a_{k2} & \ldots & a_{kk} & x_k \\ y_1 & y_2 & \ldots & y_k & 0 \end{vmatrix}^2 = \begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1k} \\ a_{21} & a_{22} & \ldots & a_{2k} \\ \vdots & \vdots & \vdots & \vdots \\ a_{k1} & a_{k2} & \ldots & a_{kk} \end{vmatrix} \times U,$$

where $U = U(a_{11}, \ldots, a_{kk}; x_1, \ldots, x_k; y_1, \ldots, y_k)$ is a polynomial of degree $k-2$ in variables a_{ij} and of degree 2 in variables x_i and y_j.

Replacing $\mathbf{x}, \mathbf{y}$ by $\mathbf{x}' = \alpha\mathbf{x} + \beta\mathbf{y}, \mathbf{y}' = \gamma\mathbf{x} + \delta\mathbf{y}$, the left-hand side changes by a constant multiple equal to the square of the determinant of matrix $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$. This shows that the polynomial U depends only on the Plücker coordinates p_{ij} of the line $\overline{\mathbf{x}, \mathbf{y}}$.

In the case when $C = V(|A(t)|)$, we can interpret the determinantal equality as follows. We consider $\mathbf{x}$ and $\mathbf{y}$ as projective coordinates in $\mathbb{P}^{d-1}$. By Proposition 4.1.6, the left-hand side is equal to $fg - h^2$, where $V(f) \cap C = 2D(\mathbf{x}), V(g) \cap C = 2D(\mathbf{y})$ and $V(h) \cap C = 2(D_1 + D_2)$. This shows that $fg - h^2$ vanishes on the curve $C = V(|A|)$. The residual curve is of degree $2(d-1) - d = d - 2$. Thus varying $\mathbf{x}, \mathbf{y}$, we get a family of curves of degree $d - 2$ parametrized by the Plücker coordinates of the lines spanned by the points $[x_1, \ldots, x_d], [y_1, \ldots, y_d]$. We can view this as a family of quadric hypersurfaces in the Grassmannian of lines in $\mathbb{P}^{d-1}$ parametrized by the plane $\mathbb{P}^2$.

4.1.4 Examples

Take $d = 2$. Then $\mathrm{Pic}^{g-1}(C)$ is one point represented by the divisor class of degree -1. It is obviously non-effective. Thus there is unique (up to the equivalence relation defined in Remark 4.1.2) representation of a conic as a determinant. For example,

$$t_0t_1 - t_2^2 = \det \begin{pmatrix} t_0 & t_2 \\ t_2 & t_1 \end{pmatrix}.$$

Take $d = 3$. Then $\mathrm{Pic}^{g-1}(C) = \mathrm{Pic}^0(C)$. If we fix a point $x_0 \in C$, then $x \mapsto [x - x_0]$ defines an isomorphism from $\mathrm{Pic}^0(C)$ to the curve C. The divisor $x - x_0$ is effective if and only if $x = x_0$. Thus we obtain that

$$\mathrm{Pic}^0(C) \setminus W_{g-1} = C \setminus \{x_0\}.$$

Let $\mathcal{L}_0 = \mathcal{O}_C(D)$, where D is a divisor of degree 0. Then $\mathcal{L} = \mathcal{L}_0(1) = \mathcal{O}_C(H + D)$, where H is a divisor of 3 collinear points. Similarly, $\mathcal{M}_0 = \mathcal{O}_C(-D)$ and $\mathcal{M} = \mathcal{M}_0(1) = \mathcal{O}_C(H - D)$. Note that any positive divisor of degree 3 is linearly equivalent to $H + D$ for some degree 0 divisor D. Thus any line bundle $\mathcal{L} = \mathcal{L}_0(1)$, where $\mathcal{L}_0 \in \mathrm{Pic}^0(C) \setminus W_{g-1}$ corresponds to a positive divisor of degree 3 not cut out by a line. The linear system $|\mathcal{L}|$ gives a reembedding $C \to C' \subset \mathbb{P}^2$ which is not projectively equivalent to the original embedding.

The map $r \times l$ maps C isomorphically onto a curve $X \subset \mathbb{P}^2 \times \mathbb{P}^2$. Consider the restriction homomorphism

$$\alpha : V^\vee \otimes W \cong H^0(|V|, \mathcal{O}_{|V|}(1)) \otimes H^0(|W^\vee|, \mathcal{O}_{|W^\vee|}(1))$$
$$\cong H^0(|V| \times |W^\vee|, \mathcal{O}_{|V|}(1) \boxtimes \mathcal{O}_{|W^\vee|}(1)) \xrightarrow{\alpha} H^0(X, \mathcal{O}_{|V|}(1) \boxtimes \mathcal{O}_{|W^\vee|}(1) \otimes \mathcal{O}_X)$$
$$\cong H^0(C, \mathcal{L} \otimes \mathcal{M}) \cong H^0(C, \mathcal{O}_C(2)).$$

Lemma 4.1.8. *The kernel of the restriction map α is of dimension 3. Let*

$$\sum_{i,j=0}^{2} a_{ij}^{(k)} x_i y_j = 0, \quad k = 1, 2, 3, \tag{4.13}$$

be the sections of bidegree $(1,1)$ which span the kernel. Let $X \subset \mathbb{P}^2 \times \mathbb{P}^2$ be the variety defined by these equations. Then

$$X = (r \times l)(C).$$

Proof. The target space of α is of dimension $6 = \dim H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$. The domain of α is of dimension 9. In coordinates, an element of the kernel is a matrix A such that $xAy = 0$ for any $(x, y) \in C$. Since the image of C under the Segre map is equal to the image of an elliptic curve under a map defined by the complete linear system of degree 6, it must span $\mathbb{P}^5$. Thus we have 6 linearly independent conditions on A. This shows that the kernel is of dimension 3. The projection of X to the first factor is equal to the locus of points $[t_0, t_1, t_2]$ such that the system

$$\sum_{i,j=0}^{2} a_{ij}^{(k)} t_i y_j = \sum_{j=0}^{2} (\sum_{i=0}^{2} a_{ij}^{(k)} t_i) y_j = 0, \quad k = 1, 2, 3$$

has a nontrivial solution. The condition for this is

$$\det \begin{pmatrix} \sum_{i=0}^{2} a_{i0}^{(1)} t_i & \sum_{i=0}^{2} a_{i1}^{(1)} t_i & \sum_{i=0}^{2} a_{i2}^{(1)} t_i \\ \sum_{i=0}^{2} a_{i0}^{(2)} t_i & \sum_{i=0}^{2} a_{i1}^{(2)} t_i & \sum_{i=0}^{2} a_{i2}^{(2)} t_i \\ \sum_{i=0}^{2} a_{i0}^{(3)} t_i & \sum_{i=0}^{2} a_{i1}^{(3)} t_i & \sum_{i=0}^{2} a_{i2}^{(3)} t_i \end{pmatrix} = 0. \tag{4.14}$$

Thus, replacing $[t_0, t_1, t_2]$ with unknowns t_0, t_1, t_2, we obtain that the projection is either a cubic curve C' or the whole plane. Assume that the second case occurs. Since the determinant of a matrix does not change after taking the transpose of the matrix, we see that the projection of X to the second factor is also the whole plane. This easily implies that X is a graph of a projective automorphism $\mathbb{P}^2 \to \mathbb{P}^2$. In appropriate coordinates X becomes the diagonal, and hence C embeds in $\mathbb{P}^2 \times \mathbb{P}^2$ by means of the diagonal map $\mathbb{P}^2 \to \mathbb{P}^2 \times \mathbb{P}^2$. But this means that $\mathcal{L} \cong \mathcal{M} \cong \mathcal{O}_C(1)$. This contradicts our choice of $\mathcal{L}$. Thus the projection of X and of C to the first factor is the cubic curve C' equal to C reembedded by $|\mathcal{L}|$. Similarly, the projection of X and of C to the second factor is the cubic curve C'' which is C reembedded by $|\mathcal{M}|$. This implies that $X = (r \times l)(C)$. ☐

Since any matrix $A(t)$ can be written in the form (4.14), we see that a determinantal equation of a plane cubic defines a model of the cubic as a complete intersection of three bilinear hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^2$.

4.1.5 Quadratic Cremona transformations

Note that (4.14) gives a determinantal equation for the reembedded curve $C' = r(C)$. Let us see that different plane models of the same elliptic curve differ by a birational transformation of the plane.

Let $\varphi : X \to \mathbb{P}^n$ be a regular map from a variety X to a projective space. Recall that it is defined by an invertible sheaf $\mathcal{F} = f^*\mathcal{O}_{\mathbb{P}^n}(1)$ and a set of $n+1$ sections $(s_0, \ldots, s_n)$. Two different maps differ by a projective automorphism of $\mathbb{P}^n$ if and only if they are defined by isomorphic sheaves and isomorphic sets of sections. Suppose we have an automorphism $\sigma : X \to X$. Then the composition $\varphi \circ \sigma : X \to \mathbb{P}^n$ is defined by the invertible sheaf $\sigma^*\mathcal{L}$ and sections $\sigma^*(s_0), \ldots, \sigma^*(s_n)$. Of course, the images of both maps φ and $\varphi \circ \sigma$ are the same, but there is no projective automorphism of $\mathbb{P}^n$ which induces the automorphism σ. However, in some cases one can find a birational automorphism T of $\mathbb{P}^n$ which does this job. Recall that, although T may be not defined on a closed subset $Z \subset \mathbb{P}^n$, it could be defined on the whole X. This happens, for example, when X is a nonsingular curve and $Z \cap X$ is a set of points. In fact, we know that any rational map of nonsingular projective curve to a projective variety extends to a regular map. Assume T is given by a linear system $|V|$ of hypersurfaces of degree m such that none of them vanish identically on the curve X. Let $x_1, \ldots, x_k$ be the points on $X \cap Z$. All polynomials $f \in V$ intersect X with some multiplicity. Let m_i be the minimal multiplicity (it is enough to compute it for a basis of V). Then it is easy to see that the restriction of T to X is given by a linear system defined by the line bundle $\mathcal{F} = \mathcal{O}_X(m) \otimes \mathcal{O}_X(-m_1x_1 - \ldots - m_kx_k)$. This is the invertible sheaf which defines the regular map $T \circ \varphi : X \to \mathbb{P}^n$. Sometimes this map defines a new embedding of X.

Let us apply this to our situation. Fix a group law on an elliptic curve X with the zero point x_0. Let τ_x be the translation automorphism defined by a point x. Recall that

$$\tau_x(y) = x \oplus y \sim x + y - x_0.$$

For any divisor $D = \sum n_i x_i$, we have

$$\tau_x^*(D) = \sum n_i \tau_x^{-1}(x_i) = \sum n_i(x_i \ominus x) \sim \sum n_i(x_i + x_0 - x)$$

$$= \sum n_i x_i + \deg(D)(x_0 - x).$$

In particular, we see that τ_x acts identically on divisors of degree 0 and hence on divisors of functions. This allows one to define the action of τ_x on the divisor classes.

Suppose we have two divisors D_1, D_2 of the same degree $m \neq 0$. Then $D_1 - D_2$ is of degree 0. Thus we can find a degree 0 divisor G such that $mG \sim D_1 - D_2$ (we use that the endomorphism of algebraic groups $[m] : X \to X, x \mapsto m \cdot x$ is surjective). Let $G \sim x_G - x_0$ for a unique point x_G. Then

$$\tau_{x_G}^*(D_1) = D_1 + m(x_0 - x_G) = D_1 - mG \sim D_2. \tag{4.15}$$

This shows that translations act transitively on divisor classes of the same positive degree.

Now suppose we have two embeddings of an elliptic curve $\phi_i : X \to \mathbb{P}^n, i = 1, 2,$ which are given by a complete linear systems defined by the corresponding invertible sheaves $\mathcal{L}_1, \mathcal{L}_2$. By the above we can find a point $x \in X$ such that $\tau_x^*\mathcal{L}_1 = \mathcal{L}_2$ (recall that for any divisor D and any regular map $\varphi : X \to Y$ we have $\varphi^*\mathcal{O}_Y(D) = \mathcal{O}_X(\varphi^*(D))$). This shows that the embeddings $\phi_2 : X \to \mathbb{P}^n$ and $\phi_1 \circ \tau_x : X \to \mathbb{P}^n$ are defined by the same invertible sheaf, and hence their images are projectively equivalent. But the image of $\phi_1 \circ \tau_x$ is obviously equal to the image of ϕ_1. Thus there exists a projective transformation σ which sends $\phi_1(X)$ to $\phi_2(X)$ such that, for any $y \in X$,

$$\sigma(\phi_1(\tau_x(y))) = \phi_2(y).$$

Thus if we change ϕ_1 by $\sigma\circ\phi_1$ (by choosing different basis of the linear system defining ϕ_1), we find that one can always choose bases in linear systems $|\mathcal{L}_1|$ and $|\mathcal{L}_2|$ such that the corresponding maps have the same image. In particular, any plane nonsingular cubic can be obtained as the image of an elliptic curve under a map defined by any complete linear system of degree 3.

Now let us see how this implies that a translation automorphism of a nonsingular plane cubic can be realized by a certain Cremona transformation of the plane.

Let

$$T : \mathbb{P}^2 - \to \mathbb{P}^2, \quad [t_0, t_1, t_2] \mapsto [f_0(t_0, t_1, t_2), f_1(t_0, t_1, t_2), f_2(t_0, t_1, t_2)]$$

be a rational map of $\mathbb{P}^2$ to itself given by polynomials of degree 2. The preimage of a line $V(a_0t_0 + a_1t_1 + a_2t_2)$ is the conic $V(a_0f_0 + a_1f_1 + a_2f_2)$. The preimage of a general point is equal to the intersection of the preimages of two general lines, thus the intersection of two conics from the net L of conics spanned by f_0, f_1, f_2. If we want T to define a birational map we need the intersection of two general conics to be equal to 1. This can be achieved if all conics pass through the same set of three points p_1, p_2, p_3 (*base points*). These points must be non-collinear, otherwise all polynomials have a common factor, after dividing, we get a projective transformation. Birational automorphisms of $\mathbb{P}^2$ (*Cremona transformations*) which are obtained by nets of conics through three non-collinear points are called *quadratic transformations*. If we choose a basis in $\mathbb{P}^2$ such that $p_1 = [1, 0, 0], p_2 = [0, 1, 0], p_3 = [0, 0, 1]$ and a basis in L given by the conics $V(t_1t_2), V(t_0t_2), V(t_0t_1)$, then the transformation is given by the formula

$$T : [t_0, t_1, t_2] \mapsto [t_1t_2, t_0t_2, t_0t_1]. \tag{4.16}$$

This is called the *standard Cremona transformation*. In affine coordinates, it is given by

$$T : (x, y) \mapsto (x^{-1}, y^{-1}).$$

Let C be a nonsingular cubic curve containing the base points p_1, p_2, p_3 of a quadratic transformation T. Then the restriction of T to C is given by the complete linear system $|2H - p_1 - p_2 - p_3|$, where H is a line section of C. It is of degree 3, and hence defines an embedding $\iota : C \hookrightarrow \mathbb{P}^2$ such that $\iota^*\mathcal{O}_{\mathbb{P}^2}(1) \cong \mathcal{O}_C(2H - p_1 - p_2 - p_3)$. Since $H = (2H - p_1 - p_2 - p_3) - (H - p_1 - p_2 - p_3)$, it follows from (4.15) that

$$\tau_x^*\mathcal{O}_C(1) \cong \mathcal{O}_C(2H - p_1 - p_2 - p_3),$$

where $3(x - x_0) \sim p_1 + p_2 + p_3 - H$. As we have explained earlier, this implies that there exists a projective automorphism σ such that $T' = \sigma \cdot T$ induces the translation automorphism τ_x on C.

It follows from this that the group of translations acts transitively on the set of determinantal equations of C. One can change one discriminant equation to any other one by applying a quadratic transformation of $\mathbb{P}^2$ which leaves the curve invariant and induces a translation automorphism of the curve.

Example 4.1.1. Let

$$f = t_0^2t_1 + t_1^2t_2 + t_2^2t_0 = \det\begin{pmatrix} t_0 & t_2 & t_2 \\ -t_1 & t_0 & 0 \\ -t_2 & 0 & t_1 \end{pmatrix}. \tag{4.17}$$

Apply the Cremona transformation

$$T : [t_0, t_1, t_2] \mapsto [t_0t_1, t_0t_2, t_1t_2]. \tag{4.18}$$

We have

$$(t_0t_1)^2t_0t_2 + (t_0t_2)^2t_1t_2 + (t_1t_2)^2t_0t_1 = t_0t_1t_2(t_0^2t_1 + t_1^2t_2 + t_2^2t_0).$$

Thus T transforms the curve to itself. Substituting (4.18) in the entries of the matrix $A(t)$ from (4.17), we get

$$\det\begin{pmatrix} t_0t_1 & t_1t_2 & t_1t_2 \\ -t_0t_2 & t_0t_1 & 0 \\ -t_1t_2 & 0 & t_0t_2 \end{pmatrix} = t_0t_1t_2 \det\begin{pmatrix} t_0 & t_2 & t_2 \\ -t_2 & t_1 & 0 \\ -t_1 & 0 & t_0 \end{pmatrix}.$$

Thus the new determinantal equation is

$$f = \det\begin{pmatrix} t_0 & t_2 & t_2 \\ -t_2 & t_1 & 0 \\ -t_1 & 0 & t_0 \end{pmatrix}. \tag{4.19}$$

However, it is projectively equivalent to the old one.

$$\begin{pmatrix} t_0 & t_2 & t_2 \\ -t_2 & t_1 & 0 \\ -t_1 & 0 & t_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} t_0 & t_2 & t_2 \\ -t_1 & t_0 & 0 \\ -t_2 & 0 & t_1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Let r be the right kernel map defined by the second matrix. We have

$$r([0,1,0]) = [0,0,1],\ r([0,0,1]) = [0,1,-1],\ r([1,0,0]) = [0,1,0].$$

Since the points $[0,1,-1],\ [0,1,0],\ [0,0,1]$ are on a line, we get

$$\mathcal{L} = r^*\mathcal{O}_C(1) = \mathcal{O}_C(x_1 + x_2 + x_3),$$

where $x_1 = [0,1,0], x_2 = [0,0,1], x_3 = [1,0,0]$. Thus the second determinantal equation corresponds to $\mathcal{L}_0 = \mathcal{L}(-1) = \mathcal{O}_C(x_1 + x_2 + x_3 - H)$, where H is a line section. Doing the same for the first matrix we find the same invertible sheaf $\mathcal{L}_0$. Note that $3H \sim 3(x_1 + x_2 + x_3)$ since $V(t_0t_1t_2)$ cuts out the divisor $3x_1 + 3x_2 + 3x_3$. This shows that the Cremona transformation induces an automorphism of the curve C equal to translation τ_x, where x is a 3-torsion point. But we know from Lectrure 4 that such automorphism is induced by a projective transformation. This explains why we are not getting an essentially new determinantal equation.

4.1.6 A moduli space

Let us consider the moduli space of pairs $(C, A(t))$, where C is a nonsingular plane curve of degree d, $A(t)$ is a matrix of linear forms such that $C = V(\det(A(t)))$. We say that two pairs $(C, A(t))$ and $(C, B(t))$ are isomorphic if there exists invertible matrices C and D such that $B(t) = CA(t)D$. Equivalently, we consider the space $|\mathrm{Hom}(E, \mathrm{Hom}(V, W))| = |E^\vee \otimes V^\vee \otimes W|$ modulo the natural action of the group $G = \mathrm{GL}(V) \times \mathrm{GL}(W)$ on the space $\mathrm{Hom}(V, W)$. The determinant map $A(t) \to \det(A(t))$ is obvioulsy invariant and defines a map

$$\det : |E^\vee \otimes V^\vee \otimes W|/G \to |\mathcal{O}_{|E|}(d)|.$$

We consider this map as a map of sets since there is an issue here whether the orbit space exists as an algebraic variety. But let us restrict this map over the subset $|\mathcal{O}_{|E|}(d)|^{\mathrm{ns}}$ of nonsingular plane curves of degree d. When we know that the fibre of the map $\det$ over the curve C is bijective to $\mathrm{Pic}^{g-1}(C) \setminus W_{g-1}$. There is an algebraic variety $\mathcal{P}ic_d^{g-1}$ (the *relative Picard scheme*) and a divisor $\mathcal{W}_d \subset \mathcal{P}ic_d$ which admits a morphism p

$$p : \mathcal{P}ic_d^{g-1} \setminus \mathcal{W}_d \to |\mathcal{O}_{|E|}(d)|^{ns}$$

with fibres isomorphic to $\mathrm{Pic}^{g-1}(C) \setminus W_{g-1}$. One can show that there exists a Zariski open subset $|E^\vee \otimes V^\vee \otimes W|^{\mathrm{ns}}$ of $|E^\vee \otimes V^\vee \otimes W|$ such that its quotient by G is isomorphic to $\mathcal{P}ic_d^{g-1}$ and the determinant map agrees with the projection p.

Since $\mathcal{P}ic_d^{g-1}$ contains an open subset which is covered by an open subset of a projective space, the variety $\mathcal{P}ic_d^{g-1}$ is unirational. It is a very difficult question to decide whether the variety $\mathcal{P}ic_d^{g-1}$ is rational. It is known only for $d = 3$ and $d = 4$ [167]. Let us sketch a beautiful proof of the rationality in the case $d = 3$ due to M. Van den Bergh [418].

Theorem 4.1.9. *Assume $d = 3$. Then $\mathcal{P}ic_3^0$ is a rational variety.*

Proof. A point of $\mathcal{P}ic^0$ is a pair $(C, \mathcal{L})$, where C is a nonsingular plane cubic and $\mathcal{L}$ is the isomorphism class of an invertible sheaf of degree 0. Let D be a divisor of degree 0 such that $\mathcal{O}_C(D) \cong \mathcal{L}$. Choose a line ℓ and let $H = \ell \cap C = p_1 + p_2 + p_3$. Let $p_i + D \sim q_i, i = 1, 2, 3$, where q_i is a point. Since $p_i - q_i \sim p_j - q_j$, we have $p_i + q_j \sim p_j + q_i$. This shows that the lines $\langle p_i, q_j\rangle$ and $\langle p_j, q_i\rangle$ intersect at the same point r_{ij} on C. Thus we have 9 points: $p_1, p_2, p_3, q_1, q_2, q_3, r_{12}, r_{23}, r_{13}$. We have

$$p_1 + p_2 + p_3 + q_1 + q_2 + q_3 + r_{12} + r_{23} + r_{13} \sim$$

$$\sim (p_1+p_2+p_3)+(q_1+q_2+q_3)+(H-p_1-q_2)+(H-p_1-q_3)+(H-p_2-q_3) \sim 3H$$

This easily implies that there is a cubic curve which intersects C at the nine points. Together with C we get a pencil of cubics with the nine points as the set of its base points. Let $U = \ell^3 \times (\mathbb{P}^2)^3/\mathfrak{S}_3$, where $\mathfrak{S}_3$ acts by

$$\sigma : \big((p_1, p_2, p_3), (q_1, q_2, q_3)\big) = \big((p_{\sigma(1)}, p_{\sigma(2)}, p_{\sigma(3)}), (q_{\sigma(1)}, q_{\sigma(2)}, q_{\sigma(3)})\big).$$

The variety U is easily seen to be rational. The projection to $\ell^3/\mathfrak{S}_3 \cong \mathbb{P}^3$ defines a birational isomorphism between the product of $\mathbb{P}^3$ and $(\mathbb{P}^2)^3$. For each $u = (\mathcal{P}, \mathcal{Q}) \in$

U, let $c(u)$ be the pencil of cubics through the points $p_1, p_2, p_3, q_1, q_2, q_3$ and the points $r_{ij} = \langle p_i, q \rangle$, where $(ij) = (12, (23), (13)$. Consider the set U' of pairs $(u, C), C \in c(u)$. The projection $(u, C) \mapsto u$ has fibres isomorphic to $\mathbb{P}^1$. Thus the field of rational functions on U' is isomorphic to the field of rational functions on a conic over the field $K(U)$. But this conic has a rational point. It is defined by fixing a point in $\mathbb{P}^2$ and choosing a member of the pencil passing though this point. Thus the conic is isomorphic to $\mathbb{P}^1$ and $K(U')$ is a purely transendental extension of $K(U)$. Now we define a birational map from $\mathcal{P}ic_3^0$ to U'. Each $(C, \mathcal{L})$ defines a point of U' by ordering the set $\ell \cap C$, then defining q_1, q_2, q_3 as above. The member of the corresponding pencil through p_i's, q_i's and r_{ij}'s is the curve C. Conversely, a point $(u, C) \in U'$ defines a point $(C, \mathcal{L})$ in $\mathcal{P}ic_3^0$. We define $\mathcal{L}$ to be the invertible sheaf corresponding to the divisor $q_1 + q_2 + q_3$. it is easy that these map are inverse to each other. $\square$

Remark 4.1.3. If we choose a basis in each space E, V, W, then a map $\phi : E \to \mathrm{Hom}(W, V)$ is determined by the matrices $A_i = \phi(e_i)$, where $e_1 = [1, 0, 0], e_2 = [0, 1, 0], e_3 = [0, 0, 1]$. Our moduli space is the space of triples (A_1, A_2, A_3) of $d \times d$ matrices up to the action of the group $G = \mathrm{GL}(d) \times \mathrm{GL}(d)$ simultaneously by left and right multiplication

$$(\sigma_1, \sigma_2) \cdot (A_1, A_2, A_3) = (\sigma_1^{-1} A_1 \sigma_2, \sigma_1^{-1} A_2 \sigma_2, \sigma_1^{-1} A_3 \sigma_2).$$

Consider an open subset of maps ϕ such that A_1 is an invertible matrix. Taking $(\sigma_1, \sigma_2) = (1, A_1^{-1})$, we may assume that $A_1 = I_d$ is the identity matrix. The stabilizer subgroup of (I_d, A_2, A_3) is the subgroup of (σ_1, σ_2) such that $\sigma_1 \sigma_2 = 1$. Thus our orbit space is equal to the orbit space of pairs of matrices (A, B) up to simultaneous conjugation. The determinantal curve has the affine equation

$$\det(I_d + XA + YB) = 0.$$

Compare this space with the space of matrices up to conjugation. As above this is reduced to the problem of description of the maps $E \to \mathrm{Hom}(V, W)$, where $\dim E = 2$ instead of 3. The determinantal curve is replaced with a determinantal hypersurface in $\mathbb{P}^1$ given by the equation

$$\det(I_d + XA) = 0.$$

Its roots are $(-\lambda^{-1})$, where λ are eigenvalues of the matrix A. If all roots are distinct (this corresponds to the case of a nonsingular curve!), a matrix is determined uniquely up to conjugacy by its eigenvalues, or equivalently by its characteristic polynomial. In the case of pairs of matrices, we need additional information expressed in terms of a point in $\mathrm{Pic}^{g-1} \setminus W_{g-1}$.

4.2 Determinantal equations for hypersurfaces

4.2.1 Cohen-Macauley sheaves

Recall that a finitely generated module M over a local Noetherian commutative ring A is called *Cohen-Macaulay module* if there exists a sequence $a_1, \ldots, a_n$ of elements in the maximal ideal of A such that n is equal to the dimension of the ring $A/\mathrm{Ann}(M)$ and $a_i \notin \mathrm{Ann}(M/(a_i, \ldots, a_{i-1})M), i = 2, \ldots, n$.

If A is a Noetherian commutative ring, not necessary local, a finitely generated A-module is called Cohen-Macaulay if for any prime ideal $\mathfrak{p}$ the localization $M_\mathfrak{p}$ is a Cohen-Macaulay module over $A_\mathfrak{p}$. A Noetherian commutative ring is called a *Cohen-Macaulay ring* if, considered as a module over itself, it is a Cohen-Macaulay module.

These definitions are globalized and give the notions of a Cohen-Macaulay scheme and a Cohen-Macaulay coherent sheaf.

A coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ is called *arithmetically Cohen-Macaulay* (an *ACM-sheaf*) if the corresponding module

$$\Gamma_*(\mathcal{F}) = \bigoplus_{i\in\mathbb{Z}} H^0(\mathbb{P}^n, \mathcal{F}(j))$$

is a graded Cohen-Macaulay module over the ring of polynomials $S = \Gamma_*(\mathcal{O}_{\mathbb{P}^n})$. Using a local cohomology characterization of Cohen-Macaulay modules one shows that $\mathcal{F}$ is a ACM-sheaf if and only if the following conditions are satisfied:

(i) $\mathcal{F}_x$ is a Cohen-Macaulay module over $\mathcal{O}_{\mathbb{P}^n,x}$ for each $x \in \mathbb{P}^n$;

(ii) $H^k(\mathbb{P}^n, \mathcal{F}(j)) = 0$, for $1 \leq k \leq \dim \mathrm{Supp}(\mathcal{F}) - 1$ and all $j \in \mathbb{Z}$, where $\mathrm{Supp}(\mathcal{F})$ denotes the support of $\mathcal{F}$.

It is known that for any Cohen-Macaulay module over a regular ring A

$$\mathrm{depth}(M) + \mathrm{proj}(M) = \dim A,$$

where proj denotes the projective dimension, the minimal length of a free resolution of M. A global analog of this equality for ACM-sheaves is

$$\dim \mathrm{Supp}(\mathcal{F}) + \mathrm{proj}(\mathcal{F}) = n,$$

where $\mathrm{proj}(\mathcal{F})$ denotes the projective dimension of $\mathcal{F}$, the minimal length of a projective graded resolution for the module $\Gamma_*(\mathcal{F})$.

Theorem 4.2.1. *Let $\mathcal{F}$ be an ACM-sheaf over $\mathbb{P}^n$ such that $\dim \mathrm{Supp}(\mathcal{F}) = 1$. Then there exists an exact sequence*

$$0 \to \bigoplus_{i=0}^{r} \mathcal{O}_{\mathbb{P}^n}(f_i) \xrightarrow{A} \bigoplus_{i=0}^{r} \mathcal{O}_{\mathbb{P}^n}(e_i) \to \mathcal{F} \to 0. \tag{4.20}$$

Proof. Since $\mathrm{proj}(\mathcal{F}) = 1$, we get a resolution of graded S-modules

$$0 \to \bigoplus_{i=0}^{r} S(f_i) \to \bigoplus_{i=0}^{r} S(e_i) \to \Gamma_*(\mathcal{F}) \to 0.$$

Passing to the corresponding sheaves in $\mathbb{P}^n$ we obtain the exact sequence from the assertion. $\square$

The map A is given by a $r \times r$ matrix whose ij-th entry is a homogeneous polynomial of degree $e_i - f_j$. We may assume that the resolution is minimal. To achieve this we must have $a_{ij} = 0$ whenever $e_i = f_j$.

Clearly, the support $\mathcal{F}$ is given by the determinant of the matrix A. It is a hypersurface of some degree d. We must have

$$d = (e_1 + \cdots + e_r) - (f_1 + \cdots + f_r). \tag{4.21}$$

Conversely, if $X = V(f)$ is given as a determinant of a matrix A whose entries a_{ij} are homogeneous polynomials of degree $e_i - f_j$ such that equality (4.21) holds, then we get a resolution (4.20) defined by the matrix. The cokernel $\mathcal{F}$ will be an ACM-sheaf.

Example 4.2.1. Take $\mathcal{F} = i_*\mathcal{O}_V(k)$. Then, the minimal resolution is of course

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-d+k) \to \mathcal{O}_{\mathbb{P}^n}(k) \to \mathcal{F} \to 0. \tag{4.22}$$

Here $r = 1, f_1 = -d+k, e_1 = k$. The equation is the tautological one $X = \det((f))$, where (f) is the 1×1 matrix with entry C. Note that according to the Lefschetz Theorem on Hyperplane Sections, Pic$(V) = \mathbb{Z}\mathcal{O}_V(1)$ if $n > 3$. Thus (4.20) reduces to (4.22) and we cannot get any nontrivial determinantal equations for nonsingular hypersurfaces of dimension ≥ 3.

4.2.2 Determinants with linear entries

Let X be a hypersurface of degree d in $\mathbb{P}^n$. Let $\mathcal{M}$ be an invertible sheaf on X. We will take $\mathcal{F} = \iota_*(\mathcal{M})$, where $\iota : X \hookrightarrow \mathbb{P}^n$ denotes the natural closed embedding. Then the condition (i) for a ACM-sheaf will be always satisfied (since $\mathcal{F}_x$ is isomorphic to $\mathcal{O}_{\mathbb{P}^n,x}/(\tau_x)$, where $\tau_x = 0$ is a local equation of V). Condition (ii) reads as

$$H^k(X, \mathcal{M}(j)) = 0, \quad 1 \leq k \leq n-2,\ j \in \mathbb{Z}. \tag{4.23}$$

Assume that the following additional conditions are satisfied:

$$H^0(X, \mathcal{M}(-1)) = H^{n-1}(X, \mathcal{M}(1-n)) = 0. \tag{4.24}$$

Consider the resolution (4.20), twist it by -1 and apply the exact sequence of cohomology. We must get

$$0 \to \bigoplus_{i=0}^{r} H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-f_i - 1)) \to \bigoplus_{i=0}^{r} H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-e_i - 1)) \to H^0(\mathbb{P}^n, \mathcal{F}(-1)) \to 0,$$

$$0 \to H^{n-1}(\mathbb{P}^n, \mathcal{F}(1-n)) \to \bigoplus_{i=0}^{r} H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-f_i - 2)) \to \bigoplus_{i=0}^{r} H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-e_i - 2)) \to 0.$$

Here we used the standard facts (see [206]) that

$$H^k(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(j)) = 0,\ k \neq 0, n,\ j \in \mathbb{Z},$$

$$H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(j)) \cong H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-n-1-j)).$$

Since $f_i < e_i$, (4.24) gives $e_i - 1 < 0$ and $-f_i - 2 < 0$, hence $e_i \leq 0, f_i \geq -1$. This implies $e_i = 0, f_i = -1$ for all $i = 1, \ldots, r$. Applying (4.21), we get $r = d$. So, we obtain a resolution

$$0 \to \bigoplus_{i=0}^{d} \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{A} \bigoplus_{i=0}^{d} \mathcal{O}_{\mathbb{P}^n} \to \mathcal{F} \to 0. \tag{4.25}$$

This gives a determinantal expression of X as a $d \times d$ determinant with linear forms as its entries.

It is convenient to rewrite the exact sequence in the form

$$0 \to W_1 \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{T} W_2 \otimes \mathcal{O}_{\mathbb{P}^n} \to \mathcal{F} \to 0, \tag{4.26}$$

where W_1, W_2 are some linear spaces of dimension d, and T is a linear map

$$T : V \to \mathrm{Hom}(W_1, W_2),$$

where $\mathbb{P}^n = |V|$. The determinantal hypersurface X is the preimage in $|V|$ of the variety of linear operators $W_1 \to W_2$ of rank less than d.

Applying the cohomology, we obtain a natural isomorphism

$$H^0(\mathbb{P}^n, \mathcal{F}) \cong W_2. \tag{4.27}$$

Twisting (4.25) by $\mathcal{O}_{\mathbb{P}^n}(-n)$ and applying the cohomology, we find a natural isomorphism

$$H^{n-1}(\mathbb{P}^n, \mathcal{F}(-n)) \cong W_1 \otimes H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-n-1)) \cong W_1. \tag{4.28}$$

It follows from (4.26) and (4.27) that the invertible sheaf $\mathcal{M}$ is generated by global sections and defines a morphism

$$l_T : V \to \mathbb{P}(W_2) = |W_2^\vee|.$$

For any $x \in X$ the point $l_T(x)$ is the projectivization of the cokernel of the matrix $T(v)$, where $x = [v]$ for some $v \in V$.

Twisting (4.26) by $\mathcal{O}_{\mathbb{P}^n}(1)$ and applying the functor $\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n}}(-, \mathcal{O}_{\mathbb{P}^n})$ to the exact sequence (4.25) we obtain an exact sequence

$$0 \to W_2^\vee \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{{}^tT} W_1^\vee \otimes \mathcal{O}_{\mathbb{P}^n} \to \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{F}(1), \mathcal{O}_{\mathbb{P}^n}) \to 0. \tag{4.29}$$

Now we apply Grothendieck's Duality Theorem (see [89]) to obtain a natural isomorphism of sheaves

$$\mathcal{F}' = \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{F}(1), \mathcal{O}_{\mathbb{P}^n}) \cong \iota_* \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}(1), \mathcal{O}_{\mathbb{P}^n}(d)) \cong \iota_* \mathcal{M}^\vee(d-1). \tag{4.30}$$

Let

$$\mathcal{L} = \mathcal{M}^\vee(d-1). \tag{4.31}$$

We can rewrite (4.29) in the form

$$0 \to W_2^\vee \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{{}^tT} W_1^\vee \otimes \mathcal{O}_{\mathbb{P}^n} \to \iota_*(\mathcal{L}) \to 0. \tag{4.32}$$

Applying cohomology we see that the sheaf $\mathcal{L}$ satisfies the same condition (4.24) as $\mathcal{M}$.

It follows from (4.26) and (4.27) that the invertible sheaf $\mathcal{L}$ is generated by global sections and defines a morphism

$$r_T : V \to \mathbb{P}(W_1) = |\mathcal{L}|^\vee.$$

For any $x \in V$ the point $r_T(x)$ is the projectivization of the kernel of the matrix $T(v)$, where $x = [v]$ for some $v \in V$.

4.2.3 The case of curves

Assume $n = 2$, i.e. X is a plane curve C of degree d. Then the condition (ii) for a ACM-sheaf is vacuous. The condition (4.24) becomes

$$H^0(C, \mathcal{M}(-1)) = H^1(C, \mathcal{M}(-1)) = 0.$$

We will assume that C is an irreducible and reduced curve and $\mathcal{M}$ is an invertible sheaf on C satisfying the previous conditions.

Let ω_C be the canonical sheaf of C and

$$p_a(C) = \dim H^1(C, \mathcal{O}_C) = \dim H^0(C, \omega_C)$$

be the *arithmetic genus* of C. By Riemann-Roch (the reader unfamiliar with the Riemann-Roch on a singular curve may may consult [293] or assume that C is nonsingular),

$$\deg \mathcal{M}(-1)) = h^0(\mathcal{M}(-1)) - h^1(\mathcal{M}(-1)) + p_a(C) - 1 = p_a(C) - 1.$$

Also we obtain

$$\mathcal{L} \otimes \mathcal{M} \cong \mathcal{O}_C(d-1).$$

If C is a nonsingular curve, everything agrees with the theory from the previous section.

Example 4.2.2. Let C be a plane irreducible cubic curve. Then $\mathcal{M}(-1)$ must be an invertible sheaf of degree 0 with no nonzero sections. It is known that $\mathrm{Pic}^0(C) \cong C \setminus \mathrm{Sing}(C)$ and has a structure of an algebraic group isomorphic to the multiplicative group $\mathbb{G}_m$ if C is a nodal cubic and isomorphic to the additive group $\mathbb{G}_a$ if C is a cuspidal cubic. Any nonzero element of this group defines a determinantal representation of C. For any nonzero $a \in \mathbb{C}$, we have

$$t_0t_2^2 + 2t_1^3 = \det \begin{pmatrix} \frac{1}{a^2}t_0 & t_1 & t_1 \\ t_1 & -\frac{1}{a^2}t_0 & t_1 - at_2 \\ t_1 & t_1 + at_2 & 0 \end{pmatrix}. \tag{4.33}$$

Note that, for any $t = [t_0, t_1, t_2] \in C$ the rank of the matrix is equal to 2, as it should be because the sheaf $\mathcal{L}$ is invertible. We cannot get a symmetric determinannt representation in this way because $\mathcal{L} \cong \mathcal{M}$ would imply that $\mathcal{L}$ is a non-trivial 2-torsion point of $\mathrm{Pic}(C)$. However, the additive group does not have non-trivial torsion elements. On the other hand, we have

$$t_0t_2^2 + t_1^3 = \det\begin{pmatrix} -t_1 & 0 & -t_2 \\ 0 & -t_0 & -t_1 \\ -t_2 & -t_1 & 0 \end{pmatrix}. \tag{4.34}$$

The matrices have rank 1 at the singular point $[1, 0, 0]$ of the curve. This shows that C admits symmetric determinantal representations not defined by an invertible sheaf on C. We refer to [25] for the theory of symmetric determinantal representaions of singular plane curves with certain type of singularities. One can show that any determinantal representation of a cuspidal cubic is equivalent either to one given in (4.33) or to one given in (4.34). The latter one corresponds to a non-invertible ACM sheaf $\mathcal{E}$ on C satisfying

$$\mathcal{E} \cong \mathcal{H}om_{\mathcal{O}_C}(\mathcal{E}, \omega_C).$$

This sheaf "compactifies" the Picard scheme of C.

4.2.4 The case of surfaces

Let X be a normal surface of degree d in $\mathbb{P}^3$. We are looking for an invertible sheaf $\mathcal{M}$ on X such that $\mathcal{F} = \iota_*(\mathcal{M})$ is a ACM-sheaf on $\mathbb{P}^3$ satisfying an additional assumption (4.24). It will give us a resolution (4.25). It follows from this resolution that $\mathcal{M}$ is generated by global sections. By Bertini's theorem, a general section of $\mathcal{M}$ is a nonsingular curve C. Thus we can write $\mathcal{M} = \mathcal{O}_X(C)$ for some nonsingular curve C.

Since X is a hypersurface in $\mathbb{P}^3$, its local ring is a Cohen-Macaulay module over the corresponding local ring of $\mathbb{P}^3$. Thus the first condition for an ACM sheaf is satisfied. Let us interpret the second condition $H^1(X, \mathcal{M}(j)) = 0, j \in \mathbb{Z}$. Recall that a subvariety $X \subset \mathbb{P}^n$ is called *projectively normal* if the restriction map $r : H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(j)) \to H^0(X, \mathcal{O}_X(j))$ is surjective for all j. If X is nonsingular in codimension 1, one can show that it is equivalent to requiring that the projective coordinate ring of X is normal. Suppose $Y \subset X$ for some hypersurface X. Then the restriction homomorphism r is the composition of the homomorphisms $r_1 : H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(j)) \to H^0(X, \mathcal{O}_X(j))$ and $r_2 : H^0(X, \mathcal{O}_X(j)) \to H^0(Y, \mathcal{O}_Y(j))$. It is easy to see that X is projectively normal (the cokernel of r_1 is $H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-d)) = 0$). Thus Y is projectively normal if r_2 is surjective. The exact sequence

$$0 \to \mathcal{J}_Y(j) \to \mathcal{O}_X(j) \to \mathcal{O}_Y(j) \to 0,$$

where $\mathcal{J}_Y$ is the sheaf of ideals of Y in X, shows that r_2 is surjective if and only if $H^1(Y, \mathcal{J}_Y(j)) = 0$ for all $j \in \mathbb{Z}$. Applying this to our case, where $Y = C \subset X \subset \mathbb{P}^3$, we get that C is projectively normal if and only if

$$H^1(X, \mathcal{O}_X(-C)(j)) = H^1(X, \omega_X(-j) \otimes \mathcal{O}_X(C))$$
$$= H^1(X, \mathcal{O}_X(C)(d-4-j)) = 0, \quad j \in \mathbb{Z}.$$

Here we used the adjunction formula for the canonical sheaf and the Serre Duality Theorem. Thus we see that the ACM-condition is the condition for the projective normality of C.

To get a resolution (4.25) we need the additional conditions

$$H^0(X, \mathcal{O}_X(C)(-1)) = H^2(X, \mathcal{O}_X(C)(-2)) = 0. \tag{4.35}$$

Together with the ACM condition this is equivalent to

$$\chi(\mathcal{O}_X(C)(-1)) = \chi(\mathcal{O}_X(C)(-2)) = 0. \tag{4.36}$$

Let $\mathcal{O}_X(1) = \mathcal{O}_X(H)$. Consider the exact sequence

$$0 \to \mathcal{O}_X(C)(-2) \to \mathcal{O}_X(C)(-1) \to \mathcal{O}_H(C-H) \to 0.$$

It gives

$$\chi(\mathcal{O}_H(C-H)) = \chi(\mathcal{O}_X(C)(-1)) - \chi(\mathcal{O}_X(C)(-2)).$$

By Bertini's Theorem we may assume that H is a nonsingular plane curve of degree d. By Riemann-Roch on H, we get

$$\deg(C) - d = \deg(\mathcal{O}_H(C-H)) = d(d-3)/2.$$

This gives

$$\chi(\mathcal{O}_X(C)(-1)) - \chi(\mathcal{O}_X(C)(-2)) \Longleftrightarrow \deg(C) = \tfrac{1}{2}d(d-1). \tag{4.37}$$

The exact sequence

$$0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X(C)(-1) \to \mathcal{O}_C(C-H) \to 0$$

gives

$$\begin{aligned}\chi(\mathcal{O}_C(C-H)) &= \chi(\mathcal{O}_X(C)(-1)) - \chi(\mathcal{O}_X(-1)) \\ &= \chi(\mathcal{O}_X(C)(-1)) - \chi(\mathcal{O}_{\mathbb{P}^3}(-1)) + \chi(\mathcal{O}_{\mathbb{P}^3}(-d-1)) \\ &= \chi(\mathcal{O}_X(C)(-1)) - \binom{d}{3}.\end{aligned}$$

Applying Riemann-Roch on the curve C, we get

$$\begin{aligned}\chi(\mathcal{O}_C(C-H)) &= \deg \mathcal{O}_C(C-H) + \chi(\mathcal{O}_C) = \deg \mathcal{O}_C(C + K_X - (d-3)H) + \chi(\mathcal{O}_C) \\ &= \deg K_C - (d-3)\deg(C) + \chi(\mathcal{O}_C) = -\tfrac{1}{2}(d-3)d(d-1) + g(C) - 1.\end{aligned}$$

Thus we see that

$$\chi(\mathcal{O}_X(C)(-1)) = 0 \Longleftrightarrow g(C) = \frac{1}{6}(d-2)(d-3)(2d+1).$$

Together with (4.37) we see that condition (4.35) is equivalent to the conditions

(i) C is a projectively normal curve;

(ii) $\deg(C) = \frac{1}{2}d(d-1)$;

(iii) $g(C) = \frac{1}{6}(d-2)(d-3)(2d+1)$.

Example 4.2.3. Take $d = 3$. We get $\deg(C) = 3, g(C) = 0$. Since $\chi(\mathcal{O}_X(C)(-1)) = -1$, we easily get $\chi(\mathcal{O}_X(C)) = \dim H^0(X, \mathcal{O}_X(C)) = 3$. The linear system $|C|$ maps X to $\mathbb{P}^2$. This is a birational morphism whose inverse is the blow-up of 6 points in $\mathbb{P}^2$. We will see later when we will be discussing cubic surfaces, that there are 72 such linear systems. Thus a cubic surface can be written in 72 essentially different ways as a 3×3 determinant.

Example 4.2.4. Take $d = 4$. We get $\deg(C) = 6, g(C) = 3$. The projective normality is equivalent to the condition that C is not hyperelliptic (Exercise 4.10). We also have $h^0(\mathcal{O}_X(C)) = 4$. According to Noether's Theorem, the Picard group of a general surface of degree ≥ 4 is generated by a plane section. Since a plane section of a quartic surface is of degree 4, we see that a general quartic surface does not admit a determinantal equation. The condition that X contains a curve C as above imposes one algebraic condition on the coefficients of a quartic surface.

Remark 4.2.1. Let $X = V(\det A(t))$ be a determinantal equation of a nonsingular surface of degree d in $\mathbb{P}^3$. Let $C \subset H$ be a nonsingular plane section of X. Then we obtain a determinantal equation of C. The left kernel sheaf for C is the restriction of the sheaf $\mathcal{M}$ to C, where $\mathcal{M} = \mathcal{O}_X(C)$ is defined by the resolution (4.25). Since we know that $\deg(\mathcal{M} \otimes \mathcal{O}_C) = d(d-1)/2$, we obtain another proof that $\deg(C) = d(d-1)/2$.

Remark 4.2.2. For nonsingular hypersurfaces X in $\mathbb{P}^3$ the condition on $\mathcal{M}$ defining a symmetric determinant is that $\mathcal{M}$ is isomorphic to the sheaf $\mathcal{L}$ defined as the cokernel of the transpose of the matrix twisted by -1. Applying the functor $\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^3}}(-, \mathcal{O}_X)$ to the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1)^d \to \mathcal{O}_{\mathbb{P}^3}^d \to \iota_*\mathcal{M} \to 0,$$

we obtain

$$0 \to \mathcal{O}_{\mathbb{P}^3}^d \to \mathcal{O}_{\mathbb{P}^3}(1)^d \to \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^3}}(\iota_*\mathcal{M}, \mathcal{O}_{\mathbb{P}^3}) \to 0.$$

Twisting by -1, we get

$$\iota_*\mathcal{L} = \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^3}}(\iota_*\mathcal{M}, \mathcal{O}_{\mathbb{P}^3})(-1) \cong \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^3}}(\iota_*\mathcal{M}, \mathcal{O}_{\mathbb{P}^3}(-1)).$$

By the duality, we have

$$\omega_X \cong \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^3}}(\iota_*\mathcal{O}_X, \omega_{\mathbb{P}^3}) \cong \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^3}}(\iota_*\mathcal{O}_V, \mathcal{O}_{\mathbb{P}^3}(-4)).$$

By standard properties of the sheaves $\mathcal{E}xt^i$ we have

$$\mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^3}}(\iota_*\mathcal{M}, \mathcal{O}_{\mathbb{P}^3})(-1) \cong \iota_*(\mathcal{M}^\vee \otimes \mathcal{E}xt^1_{\mathcal{O}_{\mathbb{P}^3}}(\iota_*\mathcal{O}_X, \mathcal{O}_{\mathbb{P}^3}(-4))) \otimes \mathcal{O}_{\mathbb{P}^3}(3).$$

This gives

$$\mathcal{L} = \mathcal{M}^\vee \otimes \omega_X(3).$$

Thus, if $\mathcal{M} \cong \mathcal{L}$, we must have

$$\mathcal{M}^{\otimes 2} \cong \omega_V(3) = \mathcal{O}_V(d-1).$$

We also must have $h^0(\mathcal{M}(-1)) = 0$. Note that

$$\mathrm{Pic}(X)[2] = \{0\}$$

for a nonsingular surface in $\mathbb{P}^3$, where, for any abelian group A we denote by $A[k]$ the subgroup of k-torsion elements. Thus there is at most one square root of $\mathcal{O}_X(d-1)$. When $d = 2k+1$ is odd, the square root is isomorphic to $\mathcal{M} = \mathcal{O}_V(k)$ but does not satisfy the condition $h^0(\mathcal{M}(-1)) = 0$. So, there are no symmetric determinantal equations. When $d = 2k$ we have no contradiction. However, in both cases the nonexistence of symmetric determinantal equations follows from the general fact that the determinantal variety of symmetric $d \times d$ matrices is singular in codimension 2. Thus any linear projective space of dimension 3 intersects it the singular locus and cuts out a singular surface. So, only singular surfaces admit a symmetric determinantal equation. We will return to this later.

Exercises

4.1 Show that any irreducible cubic curve admits a determinantal equation.

4.2 Let $(t_0(t_0-t_1), (t_0-t_2)(t_0-t_1), t_0(t_0-t_2))$ define a rational map from $\mathbb{P}^2$ to $\mathbb{P}^2$. Show that it is a birational map and find its inverse.

4.3 Let $C = V(f)$ be a nonsingular plane cubic, p_1, p_2, p_3 be three non-collinear points. Let (A_0, A_1, A_2) define a quadratic Cremona transformation with fundamental points p_1, p_2, p_3. Let q_1, q_2, q_3 be another set of three points such that the six points $p_1, p_2, p_3, q_1, q_2, q_3$ are cut out by a conic. Let (B_0, B_1, B_2) define a quadratic Cremona transformation with fundamental points q_1, q_2, q_3. Show that

$$F^{-3} \det \mathrm{adj} \begin{pmatrix} A_0B_0 & A_0B_1 & A_0B_2 \\ A_1B_0 & A_1B_1 & A_1B_2 \\ A_2B_0 & A_2B_1 & A_2B_2 \end{pmatrix}$$

is a determinantal equation of C.

4.4 Find a determinantal equation of the cubic curve from Example 4.1.1 which is not equivalent to the equation from the example.

4.5 Find a determinantal equation for the *Klein quartic* $V(t_0^3t_1 + t_1^3t_2 + t_2^3t_0)$.

4.6 Find determinantal equations for a nonsingular quadric surface in $\mathbb{P}^3$.

4.7 Let $V \subset \mathrm{Mat}_d$ be a linear subspace of dimension 3 of the space of $d \times d$ matrices. Show that the locus of points $x \in \mathbb{P}^{d-1}$ such that there exists $A \in V$ for which $x \in \mathrm{Ker}(A)$ is defined by $\binom{d}{3}$ equations of degree d. In particular, for any determinantal equation of a curve C, the images of C under the maps $r : \mathbb{P}^2 \to \mathbb{P}^{d-1}$ and $l : \mathbb{P}^2 \to \mathbb{P}^{d-1}$ are defined by such a system of equations.

4.8 Let $X = V(\det(A(t)))$ be a 4×4-determinantal equation of a nonsingular quartic surface X and $\mathcal{O}_X(C)$ be the corresponding invertible sheaf represented by a non-hyperelliptic curve C of genus 3 and degree 6. Show that $\mathcal{L} = \mathcal{O}_X(-C)(3)$ is isomorphic to $\mathcal{O}_X(C')$ for some other curve of genus 3 and degree 6. Find the interpretation of the sheaf $\mathcal{L}$ in terms of the determinantal equation.

4.9 Let C be a non-hyperelliptic curve of genus 3 and degree 6 in $\mathbb{P}^3$.

(i) Show that the homogeneous ideal of C in $\mathbb{P}^3$ is generated by four cubic polynomials f_0, f_1, f_2, f_3.

(ii) Show that the equation of any quartic surface containing C can be written in the form $\sum l_i f_i = 0$, where l_i are linear forms.

(iii) Show that (f_0, f_1, f_2, f_3) define a birational map f from $\mathbb{P}^3$ to $\mathbb{P}^3$. The image of any quartic containing C is another quartic surface.

(iv) Show that the map f is the right kernel map for the determinantal representation of the quartic defined by the curve C.

4.10 Show that a curve of degree 6 and genus 3 in $\mathbb{P}^3$ is projectively normal if and only if it is not hyperelliptic.

4.11 Let C be a nonsingular plane curve of degree d and $\mathcal{L}_0 \in \mathrm{Pic}^{g-1}(C)$. Assume that $h^0(\mathcal{L}_0) \neq 0$. Show that the image of C under the map given by the linear system $|\mathcal{L}_0(1)|$ is a singular curve.

Historical Notes

The fact that a general plane curve of degree d can be defined by the determinant of a symmetric $d \times d$ matrix with entries homogeneous linear forms was first proved by A. Dixon [125]. However, for curves of degree 4 this was proved almost 50 years earlier by O. Hesse [214]. He also showed that it can be done in 36 ways. For cubic curves the representation follows from the fact that any cubic curve can be written in three ways as the Hessian curve. This fact was also proven by Hesse [209], p. 89. The first modern treatment of Dixon's result was given in [22] and [409].

It was proved by L. Dickson [124] that any plane curve can be written as the determinant of not necessarily symmetric matrix with linear homogeneous forms as its entries. The relationship between linear determinantal representations of an irreducible plane curve of degree d and line bundles of degree $d(d-1)/2$ was first established in [91]. This was later elaborated by V. Vinnikov [421].

The theory of linear determinant representation for cubic surfaces was developed by L. Cremona [106]. Dickson proves in [124] that a general homogeneous form of degree $d > 2$ in r variables cannot be represented as a linear determinant unless $r = 3$ or $r = 4, d \leq 3$. We refer to [25] for a survey of modern development of determinantal representations of hypersurfaces.

Chapter 5

Theta characteristics

5.1 Odd and even theta characteristics

5.1.1 First definitions and examples

We have already defined a theta characteristic, odd and even, on a nonsingular curve C (see section 4.1.3). In this chapter we will study them in more details .

It follows from the definition that two theta characteristics, considered as divisor classes of degree $g-1$, differ by a 2-torsion divisor class. Since the 2-torsion subgroup $\mathrm{Jac}(C)[2]$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{2g}$, there are 2^{2g} theta characteristics. However, in general, there is no canonical identification between the set $\mathrm{TChar}(C)$ of theta characteristics on C and the set $\mathrm{Jac}(C)[2]$. One can say only that $\mathrm{TChar}(C)$ is an affine space over the vector space of $\mathrm{Jac}(C)[2] \cong \mathbb{F}_2^{2g}$.

There is one more structure on $\mathrm{TChar}(C)$. Recall that the subgroup of 2-torsion points $\mathrm{Jac}(C)[2]$ is equipped with a natural symmetric bilinear form over $\mathbb{F}_2$, called the *Weil pairing*. It is defined as follows (see [9], Appendix B). Let ϵ, ϵ' be two 2-torsion divisor classes. Choose their representatives D, D' with disjoint supports. Write $\mathrm{div}(f) = 2D, \mathrm{div}(f') = 2D'$. Then $f(D')/f'(D) = \pm 1$. Here $f(\sum_i x_i) = \prod_i f(x_i)$. Now we set

$$\langle \epsilon, \epsilon' \rangle = \begin{cases} 1 & \text{if} f(D')/f'(D) = -1 \\ 0 & \text{otherwise.} \end{cases}$$

Note that the Weil pairing is a symplectic form, i.e. satisfies $\langle \epsilon, \epsilon \rangle = 0$. One can show that it is a nondegenerate symplectic form.

For any $\vartheta \in \mathrm{TChar}(C)$ define the function

$$q_\vartheta : \mathrm{Jac}(C)[2] \to \mathbb{F}_2, \ \epsilon \mapsto h^0(\vartheta + \epsilon) + h^0(\vartheta).$$

Proposition 5.1.1. *The function q_ϑ is a quadratic form on* $\mathrm{Jac}(C)[2]$ *whose associated symmetric bilinear form is equal to the Weil pairing.*

Later we shall see that there are two types of quadratic forms associated to a fixed nondegenerate symplectic form: even and odd. They agree with our definition of an even and odd theta characteristic. The number of even (odd) theta characteristics is equal to $2^{g-1}(2^g+1)$ $(2^{g-1}(2^g-1))$.

An odd theta characteristic ϑ is obviously effective, i.e. $h^0(\vartheta) > 0$. If C is a canonical curve, then divisor $D \in |\vartheta|$ satisfies the property that $2D$ is cut out by a hyperplane H in the space $|K_C|^\vee$, where C is embedded. Such a hyperplane is called a *bitangent hyperplane*. It follows from above that a canonical curve either has $2^{g-1}(2^g-1)$ bitangent hyperplanes or infinitely many. The latter case happens if and only if there exists a theta characteristic ϑ with $h^0(\vartheta) > 1$. Such a theta characteristic is called *vanishing theta characteristic*. An example of a vanishing odd theta characteristic is the divisor class of a line section of a plane quintic curve. An example of a vanishing even theta characteristic is the unique g_3^1 on a canonical curve of genus 4 lying on a singular quadric.

The geometric interpretation of an even theta characteristic is more subtle. If C is a plane curve we explained in the previous chapter how a non-vanishing (equivalently, non-effective) even theta characteristic determines a symmetric linear determinantal representation of C. The only known geometrical construction related to canonical curves is the Scorza construction of a quartic hypersurface associated to a canonical curve and a non-effective theta characteristic. We discuss this construction in section 5.5.

5.1.2 Quadratic forms over a field of characteristic 2

Recall that a quadratic form on a vector space V over a field F is a map $q : V \to F$ such that $q(av) = a^2q(v)$ for any $a \in F$ and any $v \in V$ and the map

$$b_q : V \times V \to F, \quad (v,w) \mapsto q(v+w) - q(v) - q(w)$$

is bilinear (it is called the *polar bilinear form*). We have $b_q(v,v) = 2q(v)$ for any $v \in V$. In particular, q can be reconstructed from b_q if $\mathrm{char}(F) \neq 2$. In the case when $\mathrm{char}(F) = 2$, we get $b_q(v,v) \equiv 0$, hence b_q is a symplectic bilinear form. Two quadratic forms q, q' have the same polar bilinear form if and only if $q - q' = l$, where $l(v+w) = l(v) + l(w), l(av) = a^2l(v)$ for any $v, w \in V, a \in F$. If F is a finite field of characteristic 2, $\sqrt{l}$ is a linear form on V, and we obtain

$$b_q = b_{q'} \Longleftrightarrow q = q' + \ell^2 \tag{5.1}$$

for a unique linear form $\ell : V \to F$.

Let $e_1, \ldots, e_n$ be a basis in V and $A = (a_{ij}) = (b_q(e_i, e_j))$ be the matrix of the bilinear form b_q. It is a symmetric matrix with zeros on the diagonal if $\mathrm{char}(F) = 2$. It follows from the definition that

$$q(\sum_{i=1}^n x_ie_i) = \sum_{i=1}^n x_i^2 q(e_i) + \sum_{1 \le i < j \le n} x_ix_ja_{ij}.$$

The *rank* of a quadratic form is the rank of the matrix A of the polar bilinear form. A quadratic form is called *nondegenerate* if the rank is equal to $\dim V$. In coordinate-free way this is the rank of the linear map $V \to V^\vee$ defined by b_q. The kernel of this map is called the *radical* of b_q. The restriction of q to the radical is identically zero. The quadratic form q arises from a nondegenerate quadratic form on the quotient space. In the following we assume that q is nondegenerate.

A subspace E of V is called *singular* if $q|E \equiv 0$. Each singular subspace is an *isotropic subspace* with respect to b_q, i.e., $b_q(v, w) = 0$ for any $v, w \in E$. The converse is true only if $\mathrm{char}(F) \neq 2$.

Assume $\mathrm{char}(F) = 2$. Since b_q is a nondegenerate symplectic form, $n = 2k$, and there exists a basis $e_1, \cdots, e_n$ such that the matrix of b_q is equal to

$$J_k = \begin{pmatrix} 0_k & I_k \\ I_k & 0_k \end{pmatrix}. \tag{5.2}$$

Thus

$$q(\sum_{i=1}^n x_i e_i) = \sum_{i=1}^n x_i^2 q(e_i) + \sum_{i=1}^k x_i x_{i+k}.$$

Assume additionally that $F^* = F^{*2}$, i.e., each element in F is a square (i.e. F is a finite or algebraically closed field). Then, we can further reduce q to the form

$$q(\sum_{i=1}^{2k} x_i e_i) = (\sum_{i=1}^n \alpha_i x_i)^2 + \sum_{i=1}^k x_i x_{i+k}, \tag{5.3}$$

where $q(e_i) = \alpha_i^2,\ i = 1, \ldots, n$. This makes (5.1) more explicit. Fix a nondegenerate symplectic form $\langle , \rangle : V \times V \to F$. Each linear function on V is given by $\ell(v) = \langle v, \eta \rangle$ for a unique $\eta \in V$. By (5.1), two quadratic forms q, q' with polar bilinear form equal to $\langle , \rangle$ satisfy

$$q(v) = q'(v) + \langle v, \eta \rangle^2$$

for a unique $\eta \in V$. Choose a standard symplectic basis (i.e. the matrix of the bilinear form with respect to this basis is equal to (5.2)). The quadratic form defined by

$$q_0(\sum_{I=1}^{2k} x_i e_i) = \sum_{i=1}^k x_i x_{i+k}$$

has the polar bilinear form equal to the standard symplectic form. Any other form with the same polar bilinear form is defined by

$$q(v) = q_0(v) + \langle v, \eta_q \rangle^2,$$

where

$$\eta_q = \sum_{i=1}^{2k} \sqrt{q(e_i)} e_i.$$

From now on we assume that $F = \mathbb{F}_2$, the field of two elements. In this case $a^2 = a$ for any $a \in \mathbb{F}_2$. The formula (5.1) shows that the set $Q(V)$ of quadratic forms associated to the standard symplectic form is an affine space over V with addition $q+\eta, q \in Q(V), \eta \in V$, defined by

$$(q+\eta)(v) = q(v) + \langle v, \eta \rangle = q(v+\eta) + q(\eta). \tag{5.4}$$

The number

$$\mathrm{Arf}(q) = \sum_{i=1}^{k} q(e_i)q(e_{i+k}) \tag{5.5}$$

is called the *Arf invariant* of q. One can show that it is independent of the choice of a standard symplectic basis. A quadratic form $q \in Q(V)$ is called *even* (resp. *odd*) if $\mathrm{Arf}(q) = 0$ (resp. $\mathrm{Arf}(q) = 1$).

If we choose a standard symplectic basis for b_q and write q in the form $q_0 + \eta_q$, then we obtain

$$\mathrm{Arf}(q) = \sum_{i=1}^{k} \alpha_i \alpha_{i+k} = q_0(\eta_q) = q(\eta_q). \tag{5.6}$$

In particular, if $q' = q + v = q_0 + \eta_q + v$,

$$\mathrm{Arf}(q+v) + \mathrm{Arf}(q) = q_0(\eta_q + v) + q_0(\eta_q) = q_0(v) + \langle v, \eta_q \rangle = q(v). \tag{5.7}$$

It follows from (5.6) that the number of even (resp. odd) quadratic forms is equal to the cardinality of the set $q_0^{-1}(0)$ (resp. $q_0^{-1}(1)$). We have

$$|q_0^{-1}(0)| = 2^{k-1}(2^k+1), \quad |q_0^{-1}(1)| = 2^{k-1}(2^k-1). \tag{5.8}$$

This is easy to prove by using induction on k.

Let $\mathrm{Sp}(V)$ be the group of linear automorphisms of the symplectic space V . If we choose a standard symplectic basis then

$$\mathrm{Sp}(V) \cong \mathrm{Sp}(2k, \mathbb{F}_2) = \{X \in \mathrm{GL}(2k)(\mathbb{F}_2) : {}^tX \cdot J_k \cdot X = J_k\}.$$

It is easy to see by induction on k that

$$|\mathrm{Sp}(2k, \mathbb{F}_2)| = 2^{k^2}(2^{2k}-1)(2^{2k-2}-1)\cdots(2^2-1). \tag{5.9}$$

The group $\mathrm{Sp}(V)$ has 2 orbits in $Q(V)$, the set of even and the set of odd quadratic forms. An even quadratic form is equivalent to the form q_0 and an odd quadratic form is equivalent to the form

$$q_1 = q_0 + e_k + e_{2k},$$

where $(e_1, \ldots, e_{2k})$ is the standard symplectic basis. Explicitly,

$$q_1(\sum_{i=1}^{2k} x_i e_i) = \sum_{i=1}^{k} x_i x_{i+k} + x_k^2 + x_{2k}^2.$$

The stabilizer subgroup $\mathrm{Sp}(V)^+$ (resp. $\mathrm{Sp}(V)^-$) of an even quadratic form (resp. an odd quadratic form) is a subgroup of $\mathrm{Sp}(V)$ of index $2^{k-1}(2^k+1)$ (resp. $2^{k-1}(2^k-1)$). If $V = \mathbb{F}_2^{2k}$ with the symplectic form defined by the matrix J_k, then $\mathrm{Sp}(V)^+$ (resp. $\mathrm{Sp}(V)^-$) is denoted by $\mathrm{O}(2k, \mathbb{F}_2)^+$ (resp. $\mathrm{O}(2k, \mathbb{F}_2)^-$).

5.2 Hyperelliptic curves

5.2.1 Equations of hyperelliptic curves

Let us first describe explicitly theta characteristics on hyperelliptic curves. Recall that a hyperelliptic curve of genus g is a nonsingular projective curve X admitting a degree 2 map $\varphi : C \to \mathbb{P}^1$. By Hurwitz's formula, there are $2g+2$ branch points $p_1, \ldots, p_{2g+2}$ in $\mathbb{P}^1$. Let $f_{2g+2}(t_0, t_1)$ be a binary form of degree $2g+2$ whose zeros are the branch points. The equation of C in $\mathbb{P}(1,1,g+1)$ is

$$t_2^2 + f_{2g+2}(t_0, t_1) = 0. \tag{5.10}$$

Recall that a weighted projective space $\mathbb{P}(\mathbf{q}) = \mathbb{P}(q_0, \ldots, q_n)$ is defined as the quotient of $\mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}^\vee$, where $\mathbb{C}^\vee$ acts by

$$t : [z_0, \ldots, z_n] \mapsto [t^{q_0} z_0, \ldots, t^{q_n} z_n].$$

A more general definition of $\mathbb{P}(\mathbf{q})$ which works over $\mathbb{Z}$ is $\mathbb{P}(\mathbf{q}) = \text{Proj } \mathbb{Z}[T_0, \ldots, T_n]$, where the grading is defined by setting $\deg T_i = q_i$. Here $\mathbf{q} = (q_0, \ldots, q_n)$ are integers ≥ 1. We refer to [129] or [230] for the theory of weighted projective spaces and their subvarieties. Note that a hypersurface in $\mathbb{P}(\mathbf{q})$ is defined by a homogeneous polynomial where the unknowns are homogeneous of degree q_i. Thus equation (5.10) defines a hypersurface of degree $2g+2$. Although, in general, $\mathbb{P}(\mathbf{q})$ is a singular variety, it admits a canonical sheaf

$$\omega_{\mathbb{P}(\mathbf{q})} = \mathcal{O}_{\mathbb{P}(\mathbf{q})}(-|\mathbf{q}|),$$

where $|\mathbf{q}| = q_0 + \cdots + q_n$. Here the Serre sheaves are understood in the sense of theory of projective spectrums of graded algebras. There is also the adjunction formula for a hypersurface $X \subset \mathbb{P}(\mathbf{q})$ of degree d

$$\omega_X = \mathcal{O}_X(d - |\mathbf{q}|). \tag{5.11}$$

In the case of a hyperelliptic curve, we have

$$\omega_C = \mathcal{O}_C(g-1).$$

The morphism $\varphi : C \to \mathbb{P}^1$ corresponds to the projection $[t_0, t_1, t_2] \to [t_0, t_1]$ and we obtain that

$$\omega_C = \varphi^* \mathcal{O}_{\mathbb{P}^1}(g-1).$$

The weighted projective space $\mathbb{P}(1,1,g+1)$ is isomorphic to the projective cone in $\mathbb{P}^{g+2}$ over the Veronese curve $v_{g+1}(\mathbb{P}^1) \subset \mathbb{P}^{g+1}$. The hyperelliptic curve is isomorphic to the intersection of this cone and a quadric hypersurface in $\mathbb{P}^{g+1}$ not passing through the vertex of the cone. The projection from the vertex to the Veronese curve is the double cover $\varphi : C \to \mathbb{P}^1$. The canonical linear system $|K_C|$ maps C to $\mathbb{P}^g$ with the image equal to the Veronese curve $v_{g-1}(\mathbb{P}^1)$.

5.2.2 2-torsion points on a hyperelliptic curve

Let $c_1, \ldots, c_{2g+2}$ be the ramification points of the map φ. We assume that $\varphi(c_i) = p_i$. Obviously, $2c_i - 2c_j \sim 0$, hence the divisor class of $c_i - c_j$ is of order 2 in $\mathrm{Pic}(C)$. Also, for any subset I of the set $B_g = \{1, \ldots, 2g+2\}$, we have

$$\alpha_I = \sum_{i \in I} c_i - \#I c_{2g+2} = \sum_{i \in I}(c_i - c_{2g+2}) \in \mathrm{Pic}(C)[2].$$

Now observe that

$$\alpha_{B_g} = \sum_{i \in B_g} c_i - (2g+2)c_{2g+2} = \mathrm{div}(\phi) \sim 0, \tag{5.12}$$

where $\phi = t_2/(bt_0 - at_1)^{g+1}$ and $p_{2g+2} = (a,b)$ (we consider the fraction modulo the equation (5.10) defining C). Thus

$$c_i - c_j \sim 2c_i + \sum_{k \in B_g \setminus \{j\}} c_k - (2g+2)c_{2g+2} \sim \alpha_{B_g \setminus \{i,j\}}.$$

Adding to α_I the zero divisor $c_{2g+2} - c_{2g+2}$ we can always assume that $\#S$ is even. Also adding the principal divisor α_{B_g}, we obtain that $\alpha_I = \alpha_{\bar{I}}$, where $\bar{I}$ denotes $B_g \setminus I$.

Let $\mathbb{F}_2^{B_g} \cong \mathbb{F}_2^{2g+2}$ be the $\mathbb{F}_2$-vector space of functions $B_g \to \mathbb{F}_2$, or, equivalently, subsets of B_g. The sum is defined by the symmetric sum of subsets

$$I + J = I \cup J \setminus (I \cap J).$$

The subsets of even cardinality form a hyperplane. It contains the subsets $\emptyset$ and B_g as a subspace of dimension 1. Let E_g denote the factor space. Elements of E_g are represented by subsets of even cardinality up to the complementary set (*bifid maps* in terminology of A. Cayley). We have

$$E_g \cong \mathbb{F}_2^{2g},$$

hence the correspondence $I \mapsto \alpha_I$ defines an isomorphism

$$E_g \cong \mathrm{Pic}(C)[2]. \tag{5.13}$$

Note that E_g carries a natural symmetric bilinear form

$$e : E_g \times E_g \to \mathbb{F}_2, \quad e(I,J) = \#I \cap J \mod 2. \tag{5.14}$$

This form is symplectic (i.e. $e(I,I) = 0$ for any I) and nondegenerate. If we choose a basis represented by the subsets

$$A_i = \{2i-1, 2i\}, \quad B_i = \{2i, 2i+1\}, \quad i = 1, \ldots, g, \tag{5.15}$$

then the matrix of the bilinear form e will be equal to J_g from (5.2)

Under isomorphism (5.13), this bilinear form corresponds to the Weil pairing on 2-torsion points of the Jacobian variety of C.

Remark 5.2.1. The symmetric group $\mathfrak{S}_{2g+2}$ acts on E_g via its action on B_g and preserves the symplectic form e. This defines a homomorphism

$$s_g : \mathfrak{S}_{2g+2} \to \mathrm{Sp}(2g, \mathbb{F}_2).$$

If $g = 1$, $\mathrm{Sp}(2, \mathbb{F}_2) \cong \mathfrak{S}_3$, and the homomorphism s_1 has the kernel isomorphic to the Klein group $(\mathbb{Z}/2\mathbb{Z})^2$. If $g = 2$, the homomorphism s_2 is an isomorphism. If $g > 2$, the homomorphism s_g is injective but not surjective.

5.2.3 Theta characteristics on a hyperelliptic curve

For any subset T of B_g set

$$\vartheta_T = \sum_{i \in T} c_i + (g - 1 - \#T c_{2g+2}) = \alpha_T + (g-1)c_{2g+2}.$$

We have

$$2\vartheta_T \sim 2\alpha_T + (2g-2)c_{2g+2} \sim (2g-2)c_{2g+2}.$$

It follows from the proof of the Hurwitz formula that

$$K_C = \varphi^*(K_{\mathbb{P}^1}) + \sum_{i \in B_g} c_i.$$

Choose a representative of $K_{\mathbb{P}^1}$ equal to $-2p_{2g+2}$ and use (5.12) to obtain

$$K_C \sim (2g-2)c_{2g+2}.$$

Thus we obtain that ϑ_T is a theta characteristic. Again adding and subtracting c_{2g+2} we may assume that $\#T \equiv g+1 \mod 2$. Since T and $\bar{T}$ define the same theta characteristic, we will consider the subsets up to taking the complementary set. We obtain a set Q_g which has a natural structure of an affine space over E_g, the addition is defined by

$$\vartheta_T + \alpha_I = \vartheta_{T+I}.$$

Thus all theta characteristics are uniquely represented by the divisor classes ϑ_T, where $T \in Q_g$.

An example of an affine space over $V = \mathbb{F}_2^{2g}$ is the space of quadratic forms $q : \mathbb{F}_2^{2g} \to \mathbb{F}_2$ whose associated symmetric bilinear form b_q coincides with the standard symplectic form defined by the matrix (5.2). We identify V with its dual $V^\vee$ by means of b_0 and set $q + l = q + l^2$ for any $l \in V^\vee$.

For any $T \in Q_g$ we define the quadratic form q_T on E_g by

$$q_T(I) = \tfrac{1}{2}(\#(T+I) - \#T) = \#T \cap I + \tfrac{1}{2}\#I = \tfrac{1}{2}\#I + e(I,T) \mod 2.$$

We have (all equalities are modulo 2)

$$q_T(I+J) + q_T(I) + q_T(J) = \tfrac{1}{2}(\#(I+J) + \#| + \#J) + e(I+J,T) + e(I,T) + e(J,T)$$
$$= \tfrac{1}{2}(2\#I + 2\#J - 2\#I \cap J) = \#I \cap J.$$

Thus each theta characteristic can be identified with an element of the space $Q_g = Q(E_g)$ of quadratic forms on E_g with polar form e.

Also notice that

$$(q_T + \alpha_I)(J) = q_T(J) + e(I,J) = \tfrac{1}{2}\#J + e(T,J) + e(I,J)$$
$$= \tfrac{1}{2}\#J + e(T+I,J) = q_{T+I}(J).$$

Lemma 5.2.1. *Let ϑ_T be a theta characteristic on a hyperelliptic curve C of genus g identified with a quadratic form on E_g. Then the following properties are equivalent:*

(i) $\#T \equiv g+1 \mod 4$*;*

(ii) $h^0(\vartheta_T) \equiv 0 \mod 2$*;*

(iii) q_T *is even.*

Proof. Without loss of generality, we may assume that p_{2g+2} is the point $(0,1)$ at infinity in $\mathbb{P}^1$. Then the field of rational functions on C is generated by the functions $y = t_2/t_0$ and $x = t_1/t_0$. We have

$$\vartheta_T = \sum_{i\in T} c_i + (g-1-\#T)c_{2g+2} \sim (g-1+\#T)c_{2g+2} - \sum_{i\in T} c_i.$$

Any function ϕ from the space $L(\vartheta_T) = \{\phi : \mathrm{div}(\phi) + \vartheta_T \geq 0\}$ has a unique pole at c_{2g+2} of order $< 2g+1$. Since the function y has a pole of order $2g+1$ at c_{2g+2}, we see that $\phi = \varphi^*(p(x))$, where $p(x)$ is a polynomial of degree $\leq \frac{1}{2}(g-1+\#T)$ in x. Thus $L(\vartheta_T)$ is isomorphic to the linear space of polynomials $p(x)$ of degree $\leq \frac{1}{2}(g-1+\#T)$ with zeros at $p_i, i \in T$. The dimension of this space is equal to $\frac{1}{2}(g+1-\#T)$. This proves the equivalence of (i) and (ii).

Let

$$U = \{1,3,\ldots,2g+1\} \subset B_g \tag{5.16}$$

be the subset of odd numbers in B_g. If we take the standard symplectic basis in E_g defined in (5.15), then we obtain that $q_U = q_0$ is the standard quadratic form associated to the standard symplectic basis. It follows from (5.6) that q_T is an even quadratic form if and only if $T = U + I$, where $q_U(I) = 0$. Let I consists of k even numbers and s odd numbers. Then $q_U(I) = \#U\cap I + \frac{1}{2}\#I = m + \frac{1}{2}(k+m) = 0 \mod 2$. Thus $\#T = \#(U+S) = \#U + \#I - 2\#U\cap S = (g+1)+(k+m)-2m = g+1+k-m$. Then $m + \frac{1}{2}(k+m)$ is even, hence $3m+k \equiv 0 \mod 4$. This implies that $k-m \equiv 0 \mod 4$ and $\#T \equiv g+1 \mod 4$. Conversely, if $\#T \equiv g+1 \mod 4$, then $k-m \equiv 0 \mod 4$ and $q_U(I) = 0$. This proves the lemma. $\square$

5.2.4 Families of curves with odd or even theta characteristic

Let $\mathcal{X} \to S$ be a smooth projective morphism whose fibre X_s over a point $s \in S$ is a curve of genus $g > 0$ over the residue field $\kappa(s)$ of s. Let $\mathbf{Pic}^n_{\mathcal{X}/\mathcal{S}} \to S$ be the *relative Picard scheme* of $\mathcal{X}/S$. It represents the functor on the category of S-schemes defined by assigning to a S-scheme T the set of isomorphism classes of invertible sheaves on

$X \times_S T$ of relative degree n over T modulo tensor product with invertible sheaves coming from T. The S-scheme $\mathbf{Pic}^n_{\mathcal{X}/S} \to S$ is a smooth projective scheme over S. Its fibre over a point $s \in S$ is isomorphic to the Picard variety $\mathbf{Pic}^n_{\mathcal{X}_s/\kappa(s)}$ over the field $\kappa(s)$. The relative Picard scheme comes with a universal invertible sheaf $\mathcal{U}$ on $\mathcal{X} \times_S \mathbf{Pic}^n_{\mathcal{X}/S}$ (locally in étale topology). For any point $y \in \mathbf{Pic}^n_{\mathcal{X}/S}$ over a point $s \in S$, the restriction of $\mathcal{U}$ to the fibre of the second projection over y is an invertible sheaf $\mathcal{U}_y$ on $X_s \otimes_{\kappa(s)} \kappa(y)$ representing a point in $\mathrm{Pic}^n(\mathcal{X}_s \otimes \kappa(y))$ defined by y.

For any integer m, raising a relative invertible sheaf into m-th power defines a morphism

$$[m] : \mathbf{Pic}^n_{\mathcal{X}/S} \to \mathbf{Pic}^{mn}_{\mathcal{X}/S}.$$

Taking $n = 2g-2$ and $m = 2$, the preimage of the section defined by the relative canonical class $\omega_{\mathcal{X}/S}$ is a closed subscheme of $\mathbf{Pic}^{g-1}_{\mathcal{X}/S}$. It defines a finite cover

$$\mathcal{TC}_{\mathcal{X}/S} \to S$$

of degree 2^{2g}. The pull-back of $\mathcal{U}$ to $\mathcal{TC}_{\mathcal{X}/S}$ defines an invertible sheaf $\mathcal{T}$ over $\mathcal{P} = \mathcal{X} \times_S \mathcal{TC}_{\mathcal{X}/S}$ satisfying $\mathcal{T}^{\otimes 2} \cong \omega_{\mathcal{P}/\mathcal{TC}_{\mathcal{X}/S}}$. By a theorem of Mumford [295], the parity of a theta characteristic is preserved in an algebraic family, thus the function $\mathcal{TC}_{\mathcal{X}/S} \to \mathbb{Z}/2\mathbb{Z}$ defined by $y \mapsto \dim H^0(U_y, \mathcal{T}_y) \mod 2$ is constant on each connected component of $\mathcal{TC}_{\mathcal{X}/S}$. Let $\mathcal{TC}^{\mathrm{ev}}_{\mathcal{X}/S}$ (resp. $\mathcal{TC}^{\mathrm{odd}}_{\mathcal{X}/S}$) be the closed subset of $\mathcal{TC}_{\mathcal{X}/S}$, where this function takes the value 0 (resp. 1). The projection $\mathcal{TC}^{\mathrm{ev}}_{\mathcal{X}/S} \to S$ (resp. $\mathcal{TC}^{\mathrm{odd}}_{\mathcal{X}/S} \to S$) is a finite cover of degree $2^{g-1}(2^g+1)$ (resp. $2^{g-1}(2^g-1)$).

It follows from above that $\mathcal{TC}_{\mathcal{X}/S}$ has at least two connected components.

Now take $S = |\mathcal{O}_{\mathbb{P}^2}(d)|^{\mathrm{ns}}$ to be the space of nonsingular plane curves C of degree d and $\mathcal{X} \to |\mathcal{O}_{\mathbb{P}^2}(d)|^{\mathrm{ns}}$ be the universal curve defined by $\{(x, C) : x \in C\}$. We set

$$\mathcal{TC}_d = \mathcal{TC}_{\mathcal{X}/S},\ \mathcal{TC}^{\mathrm{ev/odd}}_d = \mathcal{TC}^{\mathrm{ev/odd}}_{\mathcal{X}/S}.$$

The proof of the following proposition can be found in [23].

Proposition 5.2.2. *If d is even or $d = 3$, $\mathcal{TC}_d$ consists of two irreducible components $\mathcal{TC}^{ev}_d$ and $\mathcal{TC}^{odd}_d$. If $d \equiv 1 \mod 4$, then $\mathcal{TC}^{ev}_d$ is irreducible but $\mathcal{TC}^{odd}_d$ has two irreducible components, one of which is the section of $\mathcal{TC}_d \to |\mathcal{O}_{\mathbb{P}^2}(d)|^{\mathrm{ns}}$ defined by $\mathcal{O}_{\mathbb{P}^2}((d-3)/2)$. If $d \equiv 3 \mod 4$, then $\mathcal{TC}^{odd}_d$ is irreducible but $\mathcal{TC}^{ev}_d$ has two irreducible components, one of which is the section of $\mathcal{TC}_d \to |\mathcal{O}_{\mathbb{P}^2}(d)|^{\mathrm{ns}}$ defined by $\mathcal{O}_{\mathbb{P}^2}((d-3)/2)$.*

Let $\mathcal{TC}^0_d$ be the open subset of $\mathcal{TC}^{\mathrm{ev}}_d$ corresponding to the pairs (C, ϑ) with $h^0(\vartheta) = 0$. It follows from the theory of symmetric determinantal representations of plane curves that $\mathcal{TC}^0_d/\mathrm{PGL}(3)$ is an irreducible variety covered by an open subset of a Grassmannian. Since the algebraic group $\mathrm{PGL}(3)$ is connected and acts freely on a Zariski open subset of $\mathcal{TC}^0_d$, we obtain that $\mathcal{TC}^0_d$ is irreducible. It follows from the previous proposition that

$$\mathcal{TC}^0_d = \mathcal{TC}^{\mathrm{ev}}_d \quad \text{if } d \not\equiv 3 \mod 4. \tag{5.17}$$

Note that there exist coarse moduli space $\mathcal{M}^{ev}_g$ and $\mathcal{M}^{\mathrm{odd}}_g$ of curves of genus together with an even (odd) theta characteristic. We refer to [95] for the proof of irreducibility of these varieties and for construction of certain compactifications of these spaces.

5.3 Theta functions

5.3.1 Jacobian variety

Recall the classical definition of the Jacobian variety of a nonsingular projective curve C of genus g over $\mathbb{C}$. We consider C as a compact oriented 2-dimensional manifold of genus g. We view the linear space $H^0(C, K_C)$ as the space of holomorphic 1-forms on C. By integration over 1-dimensional cycles we get a homomorphism of $\mathbb{Z}$-modules

$$\iota : H_1(C, \mathbb{Z}) \to H^0(C, K_C)^*, \iota(\gamma)(\omega) = \int_\gamma \omega.$$

The image of this map is a lattice Λ of rank $2g$ in the complex space $H^0(C, K_C)^*$. The quotient by this lattice

$$\mathrm{Jac}(C) = H^0(C, K_C)^*/\Lambda$$

is a complex g-dimensional torus. It is called the *Jacobian variety* of C.

Recall that the cap product

$$\cap : H_1(C, \mathbb{Z}) \times H_1(C, \mathbb{Z}) \to H_2(C, \mathbb{Z}) \cong \mathbb{Z}$$

defines a nondegenerate symplectic form on group $H_1(C, \mathbb{Z}) \cong \mathbb{Z}^{2g}$ with a nondegenerate symplectic form. Let $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ be a standard symplectic basis, i.e.,

$$(\alpha_i, \alpha_j) = (\beta_i, \beta_j) = 0, \quad (\alpha_i, \beta_j) = \delta_{ij}.$$

We choose a basis $\omega_1, \ldots, \omega_g$ of holomorphic 1-differentials on C such that

$$\int_{\alpha_i} \omega_j = \delta_{ij}. \tag{5.18}$$

Let

$$\tau_{ij} = \int_{\beta_i} \omega_i.$$

The complex matrix $\tau = (\tau_{ij})$ is called the *period matrix*. The basis $\omega_1, \ldots, \omega_g$ identifies $H^0(C, K_C)^*$ with $\mathbb{C}^g$ and the period matrix identifies the lattice Λ with the lattice $\Lambda_\tau = [\tau\ I_g]\mathbb{Z}^{2g}$, where $[\tau\ I_g]$ denotes the block-matrix of size $g \times 2g$. The period matrix $\tau = \Re(\tau) + \sqrt{-1}\Im(\tau)$ satisfies

$$^t\tau = \tau, \quad \Im(\tau) > 0.$$

As is well-known (see [197]) this implies that $\mathrm{Jac}(C)$ is a projective algebraic group, i.e. an abelian variety. It is isomorphic to the Picard scheme $\mathbf{Pic}^0_{C/\mathbb{C}}$.

We consider any divisor $D = \sum n_x x$ on C as a 0-cycle on C. The divisors of degree 0 are boundaries, i.e. $D = \partial\gamma$ for some 1-chain β. By integrating over β we get a linear function on $H^0(C, K_C)$ whose coset modulo $\Lambda = \iota(H_1(C, \mathbb{Z}))$ does not depend on the choice of β. This defines a homomorphism of groups $p : \mathrm{Div}^0(C) \to$

Jac(C). The *Abel-Jacobi Theorem* asserts that p is zero on principal divisors (Abel's part), and surjective (Jacobi's part). This defines an isomorphism of abelian groups

$$\mathrm{aj} : \mathrm{Pic}^0(C) \to \mathrm{Jac}(C) \tag{5.19}$$

which is called the *Abel-Jacobi map*. For any positive integer d let $\mathrm{Pic}^d(C)$ denote the set of divisor classes of degree d. The group $\mathrm{Pic}^0(C)$ acts simply transitively on $\mathrm{Pic}^d(C)$ via addition of divisors. There is a canonical map

$$u_d : C^{(d)} \to \mathrm{Pic}^d(C),\ D \mapsto [D],$$

where we identify the symmetric product with the set of effective divisors of degree d. One can show that $\mathrm{Pic}^d(C)$ can be equipped with a structure of a projective algebraic variety (isomorphic to the Picard scheme $\mathbf{Pic}^d_{C/\mathbb{C}}$) such that the map u_d is a morphism of algebraic varieties. Its fibres are projective spaces, the complete linear systems corresponding to the divisor classes of degree d. The action of $\mathrm{Pic}^0(C) = \mathrm{Jac}(C)$ on $\mathrm{Pic}^d(C)$ is an algebraic action equipping $\mathrm{Pic}^d(C)$ with a structure of a torsor over the Jacobian variety.

Let

$$W^r_{g-1} = \{[D] \in \mathrm{Pic}^{g-1}(C) : h^0(D) \geq r+1\}.$$

In particular, W^0_{g-1} was denoted by W_{g-1} in Theorem 4.1.3, where we showed that the invertible sheaves $\mathcal{L}_0 \in \mathrm{Pic}^{g-1}(C)$ defining a determinantal equation of a plane curve of genus g belong to the set $\mathrm{Pic}^{g-1}(C) \setminus W^0_{g-1}$. The fundamental property of the loci W^r_{g-1} is given by the following *Riemann-Kempf Theorem*.

Theorem 5.3.1.

$$W^r_{g-1} = \{x \in W_{g-1} : \mathrm{mult}_x W_{g-1} \geq r+1\}.$$

In particular, we get

$$W^1_{g-1} = \mathrm{Sing}(W_{g-1}).$$

From now on we will identify $\mathrm{Pic}^0(C)$ with the points on the Jacobian variety Jac(C) by means of the Abel-Jacobi map. For any theta characteristic ϑ the subset

$$\Theta = W_{g-1} - \vartheta \subset \mathrm{Jac}(C)$$

is a hypersurface in Jac(C). It has the property that

$$h^0(\Theta) = 1, \quad [-1]^*(\Theta) = \Theta, \tag{5.20}$$

where $[m]$ is the multiplication by an integer m in the group variety Jac(C). Conversely, any divisor on Jac(C) satisfying these properties is equal to W_{g-1} translated by a theta characteristic. This follows from the fact that a divisor D on an abelian variety A satisfying $h^0(D) = 1$ defines a bijective map $A \to \mathrm{Pic}^0(A)$ by sending a point $x \in A$ to the divisor $t_x^*D - D$, where t_x is the translation map $a \mapsto a + x$ in the group variety, and $\mathrm{Pic}^0(A)$ is the group of divisor classes algebraically equivalent to zero. This fact implies that any two divisors satisfying properties (5.20) differ by translation by a 2-torsion point.

We call a divisor satisfying (5.20) a *symmetric theta divisor*. An abelian variety that contains such a divisor is called a *principally polarized abelian variety*.

Let $\Theta = W_{g-1} - \theta$ be a symmetric theta divisor on $\mathrm{Jac}(C)$. Applying Theorem 5.3.1 we obtain that, for any 2-torsion point $\epsilon \in \mathrm{Jac}(C)$, we have

$$\mathrm{mult}_\epsilon \Theta = h^0(\vartheta + \epsilon). \tag{5.21}$$

In particular, $\epsilon \in \Theta$ if and only if $\theta + \epsilon$ is an effective theta characteristic. According to ϑ, the symmetric theta divisors are divided into two groups: even and odd theta divisors.

5.3.2 Theta functions

The preimage of Θ under the quotient map $\mathrm{Jac}(C) = H^0(C, K_C)^*/\Lambda$ is a hypersurface in the complex linear space $V = H^0(C, K_C)^*$ equal to the zero set of some holomorphic function $\phi : V \to \mathbb{C}$. This function ϕ is not invariant with respect to translations by Λ (only constants are because the quotient is compact). However, it has the property that, for any $v \in V$ and any $\lambda \in \Lambda$,

$$\phi(v + \lambda) = e_\lambda(v)(\gamma)\phi(v),$$

where e_λ is an invertible holomorphic function on V. Such a function is called a *theta function*. The set of zeros of ϕ does not change if we replace ϕ with $\phi\alpha$, where α is an invertible holomorphic function on V. The functions $e_\lambda(v)$ will change into functions $e_{\lambda'}(v) = e_\lambda(v)\phi(v + \lambda)\phi^{-1}(v)$. One can show that, after choosing an appropriate α one may assume that

$$e_\lambda(v) = \exp(2\pi i(a_\gamma(v) + b_\gamma)),$$

where a_γ is a linear function and b_γ is constant. We will assume that such a choice has been made.

It turns out that the theta function corresponding to a symmetric theta divisor Θ can be given in coordinates defined by a choice of a normalized basis (5.18) by the following expression

$$\theta\left[\begin{smallmatrix}\boldsymbol{\epsilon}\\ \boldsymbol{\eta}\end{smallmatrix}\right](\mathbf{z};\tau) = \sum_{r\in\mathbb{Z}^g} \exp \pi i\big[(\mathbf{r} + \tfrac{1}{2}\boldsymbol{\epsilon}) \cdot \tau \cdot (\mathbf{r} + \tfrac{1}{2}\boldsymbol{\epsilon}) + 2(\mathbf{z} + \tfrac{1}{2}\boldsymbol{\eta}) \cdot (\mathbf{r} + \tfrac{1}{2}\boldsymbol{\epsilon})\big], \tag{5.22}$$

where $\boldsymbol{\epsilon}, \boldsymbol{\eta} \in \{0,1\}^g$ considered as a column or a raw vector from $\mathbb{F}_2^g$. The function defined by this expression is called a *theta function with characteristic*. The invertible function $e_\lambda(z_1, \ldots, z_g)$ for such a function is given by the expression

$$e_\lambda(\mathbf{z}) = \exp -\pi i(\mathbf{m} \cdot \tau \cdot \mathbf{m} - 2\mathbf{z} \cdot \mathbf{m} - \boldsymbol{\epsilon} \cdot \mathbf{n} + \boldsymbol{\eta} \cdot \mathbf{m}),$$

where we write $\lambda = \tau \cdot \mathbf{m} + \mathbf{n}$ for some $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^g$. One can check that

$$\theta\left[\begin{smallmatrix}\boldsymbol{\epsilon}\\ \boldsymbol{\eta}\end{smallmatrix}\right](-\mathbf{z};\tau) = \exp(\pi i \boldsymbol{\epsilon} \cdot \boldsymbol{\eta})\theta\left[\begin{smallmatrix}\boldsymbol{\epsilon}\\ \boldsymbol{\eta}\end{smallmatrix}\right](\mathbf{z};\tau). \tag{5.23}$$

This shows that $\theta\left[\begin{smallmatrix}\boldsymbol{\epsilon}\\ \boldsymbol{\eta}\end{smallmatrix}\right](-\mathbf{z};\tau)$ is an odd (resp. even) function if and only if $\boldsymbol{\epsilon}\cdot\boldsymbol{\eta} = 1$ (resp. 0). In particular, $\theta\left[\begin{smallmatrix}\boldsymbol{\epsilon}\\ \boldsymbol{\eta}\end{smallmatrix}\right](0;\tau) = 0$ if the function is odd. It follows from (5.21) that $\theta\left[\begin{smallmatrix}\boldsymbol{\epsilon}\\ \boldsymbol{\eta}\end{smallmatrix}\right](0;\tau) = 0$ if θ is an odd theta characteristic or an effective even theta characteristic.

Taking $\boldsymbol{\epsilon},\boldsymbol{\eta} = 0$, we obtain the *Riemann theta function*

$$\theta(\mathbf{z};\tau) = \sum_{\mathbf{r}\in\mathbb{Z}^g} \exp \pi i(\mathbf{r}\cdot\tau\cdot\mathbf{r} + 2\mathbf{z}\cdot\mathbf{r}).$$

All other theta functions with characteristic are obtained from $\theta(\mathbf{z};\tau)$ by translate

$$\theta\left[\begin{smallmatrix}\boldsymbol{\epsilon}\\ \boldsymbol{\eta}\end{smallmatrix}\right](\mathbf{z};\tau) = \exp \pi i(\boldsymbol{\epsilon}\cdot\boldsymbol{\eta} + \boldsymbol{\epsilon}\cdot\tau\cdot\boldsymbol{\epsilon})\theta(\mathbf{z} + \tfrac{1}{2}\tau\cdot\boldsymbol{\eta} + \tfrac{1}{2}\boldsymbol{\epsilon};\tau).$$

In this way points on $\mathbb{C}^g$ of the form $\frac{1}{2}\tau\cdot\boldsymbol{\epsilon} + \frac{1}{2}\boldsymbol{\eta}$ are identified with elements of the 2-torsion group $\frac{1}{2}\Lambda/\Lambda$ of $\mathrm{Jac}(C)$. The theta divisor corresponding to the Riemann theta function is equal to W_{g-1} translated by a certain theta characteristic κ called the *Riemann constant*. Of course, there is no any distinguished theta characteristic, the definition of κ depends on the choice of a symplectic basis in $H_1(C,\mathbb{Z})$.

The multiplicity m of a point on a theta divisor $\Theta = W_{g-1} - \vartheta$ is equal to the multiplicity of the corresponding theta function defined by vanishing partial derivatives up to order $m-1$. Thus the quadratic form defined by θ can be redefined in terms of the corresponding theta function as

$$q_\vartheta(\tfrac{1}{2}\tau\cdot\boldsymbol{\epsilon}' + \tfrac{1}{2}\boldsymbol{\eta}') = \mathrm{mult}_0\theta\left[\begin{smallmatrix}\boldsymbol{\epsilon}+\boldsymbol{\epsilon}'\\ \boldsymbol{\eta}+\boldsymbol{\eta}'\end{smallmatrix}\right](\mathbf{z},\tau) + \mathrm{mult}_0\theta\left[\begin{smallmatrix}\boldsymbol{\epsilon}\\ \boldsymbol{\eta}\end{smallmatrix}\right](\mathbf{z},\tau).$$

It follows from (5.23) that this number is equal to

$$\boldsymbol{\epsilon}\cdot\boldsymbol{\eta}' + \boldsymbol{\eta}\cdot\boldsymbol{\eta}' + \boldsymbol{\eta}'\cdot\boldsymbol{\eta}'. \tag{5.24}$$

A choice of a symplectic basis in $H_1(C,\mathbb{Z})$ defines a standard symplectic basis in $H_1(C,\mathbb{F}_2) \cong \frac{1}{2}\Lambda/\Lambda = \mathrm{Jac}(C)[2]$. Thus we can identify 2-torsion points $\frac{1}{2}\tau\cdot\boldsymbol{\epsilon}' + \frac{1}{2}\boldsymbol{\eta}'$ with vectors $(\boldsymbol{\epsilon}',\boldsymbol{\eta}') \in \mathbb{F}_2^{2g}$. The quadratic form corresponding to the Riemann theta function is the standard one

$$q_0((\boldsymbol{\epsilon}',\boldsymbol{\eta}')) = \boldsymbol{\epsilon}'\cdot\boldsymbol{\eta}'.$$

The quadratic form corresponding to $\theta\left[\begin{smallmatrix}\boldsymbol{\epsilon}\\ \boldsymbol{\eta}\end{smallmatrix}\right](\mathbf{z};\tau)$ is given by (5.24). The Arf invariant of this quadratic form is equal to

$$\mathrm{Arf}(q_\vartheta) = \boldsymbol{\epsilon}\cdot\boldsymbol{\eta} = q_0((\boldsymbol{\epsilon},\boldsymbol{\eta})).$$

5.3.3 Hyperelliptic curves again

In this case we can compute the Riemann constant explicitly. Recall that we identify 2-torsion points with subsets of even cardinality of the set $B_g = \{1,\ldots,2g+2\}$ which we can identify with the set of ramification or branch points. Let us define a standard symplectic basis in C by choosing the 1-cycle α_i to be the path which goes from c_{2i-1} to c_{2i} along one sheet of the Riemann surface C and returns to c_{2i-1} along the other sheet. Similarly, we define the 1-cycle β_i by choosing the points c_{2i} and c_{2i+1}. Choose g holomorphic forms ω_j normalized by the condition (5.18). Let τ be the corresponding period matrix. Notice that each holomorphic 1-form changes sign when we switch the sheets. This gives

$$\tfrac{1}{2}\delta_{ij} = \tfrac{1}{2}\int_{\alpha_i}\omega_j = \int_{c_{2i-1}}^{c_{2i}}\omega_j = \int_{c_{2i-1}}^{c_{2g+2}}\omega_j - \int_{c_{2i}}^{c_{2g+2}}\omega_j$$
$$= \int_{c_{2i-1}}^{c_{2g+2}}\omega_j + \int_{c_{2i}}^{c_{2g+2}}\omega_j - 2\int_{c_{2i}}^{c_{2g+2}}\omega_j.$$

Since
$$2\Big(\int_{c_{2i}}^{c_{2g+2}}\omega_1,\dots,\int_{c_{2i}}^{c_{2g+2}}\omega_g\Big) = \mathrm{aj}(2c_{2i}-2c_{2g+2}) = 0,$$

we obtain
$$\iota(c_{2i-1}+c_{2i}-2c_{2g+2}) = \tfrac{1}{2}\mathbf{e}_i \mod \Lambda_\tau,$$

where, as usual, $\mathbf{e}_i$ denotes the i-th unit vector. Let A_i, B_i be defined as in (5.15). We obtain that
$$\mathrm{aj}(\alpha_{A_i}) = \tfrac{1}{2}\mathbf{e}_i \mod \Lambda_\tau.$$

Similarly, we find that

$$\mathrm{aj}_{c_{2g+2}}(\alpha_{B_i}) = \tfrac{1}{2}\tau\cdot\mathbf{e}_i \mod \Lambda_\tau.$$

Now we can match the set Q_g with the set of theta functions with characteristics. Recall that the set $U = \{1,3,\dots,2g+1\}$ plays the role of the standard quadratic form. We have
$$q_U(A_i) = q_U(B_i) = 0, \quad i = 1,\dots,g.$$

Comparing it with (5.24), we see that the theta function $\theta\left[\begin{smallmatrix}\boldsymbol{\epsilon}\\ \boldsymbol{\eta}\end{smallmatrix}\right](\mathbf{z};\tau)$ corresponding to ϑ_U must coincide with the function $\theta(\mathbf{z};\tau)$. This shows that

$$\iota_{c_{2g+2}}^{g-1}(\vartheta_U) = \iota_{c_{2g+2}}(\vartheta_U - k_{c_{2g+2}}) = 0.$$

Thus the Riemann constant κ corresponds to the theta characteristic ϑ_U. This allows one to match theta characteristics with theta functions with theta characteristics.

Write any subset I of E_g in the form

$$I = \sum_{i=1}^{g}\epsilon_i A_i + \sum_{i=1}^{g}\eta_i B_i,$$

where $\boldsymbol{\epsilon} = (\epsilon_1,\dots,\epsilon_g),\ \boldsymbol{\eta} = (\eta_1,\dots,\eta_g)$ are binary vectors. Then

$$\vartheta_{U+I} \longleftrightarrow \theta\left[\begin{smallmatrix}\boldsymbol{\epsilon}\\ \boldsymbol{\eta}\end{smallmatrix}\right](\mathbf{z};\tau).$$

In particular,
$$\vartheta_{U+I} \in \mathrm{TChar}(C)^{\mathrm{ev}} \Longleftrightarrow \epsilon\cdot\eta = 0 \mod 2.$$

Example 5.3.1. We give the list of theta characteristics for small genus. We also list 2-torsion points at which the corresponding theta function vanishes.

$g = 1$

3 even "thetas":

$$\vartheta_{12} = \theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (\alpha_{12}),$$
$$\vartheta_{13} = \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (\alpha_{13}),$$
$$\vartheta_{14} = \theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (\alpha_{14}).$$

1 odd theta

$$\vartheta_{\emptyset} = \theta \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (\alpha_{\emptyset}).$$

$g = 2$

10 even thetas:

$$\vartheta_{123} = \theta \begin{bmatrix} 01 \\ 10 \end{bmatrix} \quad (\alpha_{12}, \alpha_{23}, \alpha_{13}, \alpha_{45}, \alpha_{46}, \alpha_{56}),$$
$$\vartheta_{124} = \theta \begin{bmatrix} 00 \\ 10 \end{bmatrix} \quad (\alpha_{12}, \alpha_{24}, \alpha_{14}, \alpha_{35}, \alpha_{36}, \alpha_{56}),$$
$$\vartheta_{125} = \theta \begin{bmatrix} 00 \\ 11 \end{bmatrix} \quad (\alpha_{12}, \alpha_{25}, \alpha_{15}, \alpha_{34}, \alpha_{36}, \alpha_{46}),$$
$$\vartheta_{126} = \theta \begin{bmatrix} 11 \\ 11 \end{bmatrix} \quad (\alpha_{12}, \alpha_{16}, \alpha_{26}, \alpha_{34}, \alpha_{35}, \alpha_{45}),$$
$$\vartheta_{234} = \theta \begin{bmatrix} 10 \\ 01 \end{bmatrix} \quad (\alpha_{23}, \alpha_{34}, \alpha_{24}, \alpha_{15}, \alpha_{56}, \alpha_{16}),$$
$$\vartheta_{235} = \theta \begin{bmatrix} 10 \\ 00 \end{bmatrix} \quad (\alpha_{23}, \alpha_{25}, \alpha_{35}, \alpha_{14}, \alpha_{16}, \alpha_{46}),$$
$$\vartheta_{236} = \theta \begin{bmatrix} 01 \\ 00 \end{bmatrix} \quad (\alpha_{23}, \alpha_{26}, \alpha_{36}, \alpha_{14}, \alpha_{45}, \alpha_{15}),$$
$$\vartheta_{245} = \theta \begin{bmatrix} 11 \\ 00 \end{bmatrix} \quad (\alpha_{24}, \alpha_{25}, \alpha_{13}, \alpha_{45}, \alpha_{16}, \alpha_{36}),$$
$$\vartheta_{246} = \theta \begin{bmatrix} 00 \\ 00 \end{bmatrix} \quad (\alpha_{26}, \alpha_{24}, \alpha_{13}, \alpha_{35}, \alpha_{46}, \alpha_{15}),$$
$$\vartheta_{256} = \theta \begin{bmatrix} 00 \\ 01 \end{bmatrix} \quad (\alpha_{26}, \alpha_{25}, \alpha_{13}, \alpha_{14}, \alpha_{34}, \alpha_{56}).$$

6 odd thetas

$$\vartheta_{1} = \theta \begin{bmatrix} 01 \\ 01 \end{bmatrix} \quad (\alpha_{\emptyset}, \alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{15}, \alpha_{16}),$$
$$\vartheta_{2} = \theta \begin{bmatrix} 11 \\ 01 \end{bmatrix} \quad (\alpha_{\emptyset}, \alpha_{12}, \alpha_{23}, \alpha_{24}, \alpha_{25}, \alpha_{26}),$$
$$\vartheta_{3} = \theta \begin{bmatrix} 11 \\ 01 \end{bmatrix} \quad (\alpha_{\emptyset}, \alpha_{13}, \alpha_{23}, \alpha_{34}, \alpha_{35}, \alpha_{36}),$$
$$\vartheta_{4} = \theta \begin{bmatrix} 10 \\ 10 \end{bmatrix} \quad (\alpha_{\emptyset}, \alpha_{14}, \alpha_{24}, \alpha_{34}, \alpha_{45}, \alpha_{46}),$$
$$\vartheta_{5} = \theta \begin{bmatrix} 10 \\ 11 \end{bmatrix} \quad (\alpha_{\emptyset}, \alpha_{15}, \alpha_{35}, \alpha_{45}, \alpha_{25}, \alpha_{56}),$$
$$\vartheta_{6} = \theta \begin{bmatrix} 01 \\ 11 \end{bmatrix} \quad (\alpha_{\emptyset}, \alpha_{16}, \alpha_{26}, \alpha_{36}, \alpha_{46}, \alpha_{56}).$$

$g = 3$

36 even thetas $\vartheta_{\emptyset}, \vartheta_{ijkl}$,

28 odd thetas ϑ_{ij}.

$g = 4$

136 even thetas $\vartheta_{i}, \vartheta_{ijklm}$

120 odd thetas ϑ_{ijk}.

5.4 Odd theta characteristics

5.4.1 Syzygetic triads

We have already remarked that effective theta characteristics on a canonical curve $C \subset \mathbb{P}^{g-1}$ correspond to hyperplanes everywhere tangent to C. We call them *bitangent hyperplanes* (not to be confused with hyperplanes tangent at ≥ 2 points).

An odd theta characteristic is effective and determines a bitangent hyperplane, a unique one if it is non-vanishing. In this section we will study the configuration of bitangent hyperplanes to a canonical curve. Let us note here that a general canonical curve is determined uniquely by the configuration of its bitangent hyperplanes [42].

From now on we fix a nondegenerate symplectic space (V, ω) of dimension $2g$ over $\mathbb{F}_2$. Let $Q(V)$ be the affine space of quadratic forms with associated symmetric bilinear form equal to ω. The Arf invariant divides $Q(V)$ into the union of two sets $Q(V)_+$ and $Q(V)_-$, of even or odd quadratic forms. Recall that $Q(V)_-$ is interpreted as the set of odd theta characteristics when $V = \text{Pic}(C)$ and ω is the Weil pairing. For any $q \in Q(V)$ and $v \in V$, we have

$$q(v) = \text{Arf}(q + v) + \text{Arf}(q).$$

Thus the function Arf is the symplectic analog of the function $h^0(\vartheta) \mod 2$ for theta characteristics.

The set $\tilde{V} = V \coprod Q(V)$ is equipped with a structure of a $\mathbb{Z}/2\mathbb{Z}$-graded vector space over $\mathbb{F}_2$. It complements the addition on V (the 0-th graded piece) and the structure of an affine space on $Q(V)$ (the 1-th graded piece) by setting $q + q' := v$, where $q' = q + v$. One can also extend the symplectic form on V to $\tilde{V}$ by setting

$$\omega(q, q') = q(q + q'), \quad \omega(q, v) = \omega(v, q) = q(v).$$

Definition 5.1. *A set of three elements q_1, q_2, q_3 in $Q(V)$ is called a* syzygetic triad *(resp.* azygetic triad*) if*

$$\text{Arf}(q_1) + \text{Arf}(q_2) + \text{Arf}(q_3) + \text{Arf}(q_1 + q_2 + q_3) = 0 \ (\textit{resp.} \ = 1).$$

A subset of $k \geq 3$ elements in $Q(V)$ is called an azygetic set *if any subset of three elements is azygetic.*

Note that a syzygetic triad defines a set of four quadrics in $Q(V)$ that add up to zero. Such a set is called a *syzygetic tetrad*. Obviously, any subset of three elements in a syzygetic tetrad is a syzygetic triad.

Another observation is that three elements in $Q(V)_-$ form an azygetic triad if their sum is an element in $Q(V)_+$.

For any odd theta characteristic ϑ any divisor $D_\eta \in |\vartheta|$ is of degree $g - 1$. The condition is that four odd theta characteristics ϑ_i form a syzygetic tetrad means that the sum of divisors D_{ϑ_i} are cut out by a quadric in $\mathbb{P}^{g-1}$. The converse is true if C does not have vanishing even theta characteristic.

Let us now compute the number of syzygetic tetrads.

Lemma 5.4.1. *Let q_1, q_2, q_3 be a set of three elemenst on $Q(V)$. The following properties are equivalent:*

(i) q_1, q_2, q_3 *is a syzygetic triad;*

(ii) $q_1(q_2 + q_3) = \mathrm{Arf}(q_2) + \mathrm{Arf}(q_3)$*;*

(iii) $\omega(q_1 + q_2, q_1 + q_3) = 0.$

Proof. The equivalence of (i) and (ii) follows immediately from the identity

$$q_1(q_2 + q_3) = \mathrm{Arf}(q_1) + \mathrm{Arf}(q_1 + q_2 + q_3).$$

We have

$$\omega(q_1 + q_2, q_1 + q_3) = q_1(q_1 + q_3) + q_2(q_1 + q_3)$$
$$= \mathrm{Arf}(q_1) + \mathrm{Arf}(q_3) + \mathrm{Arf}(q_2) + \mathrm{Arf}(q_1 + q_2 + q_3).$$

This shows the equivalence of (ii) and (iii). □

Proposition 5.4.2. *Let $q_1, q_2 \in Q(V)_-$. The number of ways in which the pair can be extended to a syzygetic triad of odd theta characteristics is equal to $2(2^{g-1}+1)(2^{g-2}-1)$.*

Proof. Assume that q_1, q_2, q_3 is a syzygetic triad in $Q(V)_-$. By the previous lemma, $q_1(q_2 + q_3) = 0$. Also, we have $q_2(q_2 + q_3) = \mathrm{Arf}(q_3) + \mathrm{Arf}(q_2) = 0$. Thus q_1 and q_2 vanish at $v_0 = q_2 + q_3$. Conversely, assume $v \in V$ satisfies $q_1(v) = q_2(v) = 0$ and $v \neq q_1 + q_2$ so that $q_3 = q_2 + v \neq q_1, q_2$. We have $\mathrm{Arf}(q_3) = \mathrm{Arf}(q_2) + q_2(v) = 1$, hence $q_3 \in Q(V)_-$. Since $q_1(v) = q_1(q_2 + q_3) = 0$, by the previous Lemma q_1, q_2, q_3 is a syzygetic triad.

Thus the number of the ways we can extend q_1, q_2 to a syzygetic triad q_1, q_2, q_3 is equal to the cardinality of the set

$$Z = q_1^{-1}(0) \cap q_2^{-1}(0) \setminus \{0, v_0\},$$

where $v_0 = q_1 + q_1$. It follows from (5.6) that $v \in Z$ satisfies $\omega(v, v_0) = q_2(v) + q_1(v) = 0$. Thus any $v \in Z$ is a representative of a nonzero element in $W = v_0^{\perp}/v_0 \cong \mathbb{F}_2^{2g-2}$ on which q_1 and q_2 vanish. It is clear that q_1 and q_2 induce the same quadratic form q on W. It is an odd quadratic form. Indeed, we can choose a symplectic basis in V by taking as a first vector the vector v_0. Then computing the Arf invariant of q_1 we see that it is equal to the Arf invariant of the quadratic form q. Thus we get

$$\#Z = 2(\#Q(W)_- - 1) = 2(2^{g-2}(2^{g-1} - 1) - 1) = 2(2^{g-1} + 1)(2^{g-2} - 1).$$

□

Corollary 5.4.3. *Let t_g be the the number of syzygetic tetrads of odd theta characteristics on a nonsingular curve of genus g. Then*

$$t_g = \frac{1}{3}2^{g-3}(2^{2g} - 1)(2^{2g-2} - 1)(2^{g-2} - 1).$$

Proof. Let I be the set of triples (q_1, q_2, T), where $q_1, q_2 \in Q(V)_-$ and T is a syzygetic tetrad containing q_1, q_2. We count $\#I$ in two ways by projecting I to the set $\mathcal{P}$ of unordered pairs of distinct elements $Q(V)_-$ and to the set of syzygetic tetrads. Since each tetrad contains 6 pairs from the set $\mathcal{P}$, and each pair can be extended in $(2^{g-1}+1)(2^{g-2}-1)$ ways to a syzygetic tetrad, we get

$$\#I = (2^{g-1}+1)(2^{g-2}-1)\binom{2^{g-1}(2^g-1)}{2} = 6t_g.$$

This gives

$$t_g = \frac{1}{3}2^{g-3}(2^{2g}-1)(2^{2g-2}-1)(2^{g-2}-1).$$

□

Let V be a vector space with a symplectic or symmetric bilinear form. Recall that a linear subspace L is called isotropic if the restriction of the bilinear form to L is identically zero.

Corollary 5.4.4. *Let $\{q_1, q_2, q_3, q_4\}$ be a syzygetic tetrad in $Q(V)_-$. Then $P = \{q_1 + q_i, \ldots, q_4 + q_i\}$ is an isotropic 2-dimensional subspace in (V, ω) which does not depend on the choice of q_i.*

Proof. It follows from Lemma 5.4.1 (iii) that P is an isotropic subspace. The equality $q_1 + \cdots + q_4 = 0$ gives

$$q_k + q_l = q_i + q_j, \tag{5.25}$$

where $\{i, j, k, l\} = \{1, 2, 3, 4\}$. This shows that the subspace P of V formed by the vectors $q_j + q_i, j = 1, \ldots, 4$, is independent on the choice of i. One of its bases is the set $(q_1 + q_4, q_2 + q_4)$. □

5.4.2 Steiner complexes

Let $\mathcal{P}$ be the set of unordered pairs of distinct elements in $Q(V)_-$. The addition map in $Q(V)_- \times Q(V)_- \to V$ defines a map

$$s : \mathcal{P} \to V \setminus \{0\}.$$

Definition 5.2. *The union of pairs from the same fibre $s^{-1}(v)$ of the map s is called a* Steiner compex. *It is denoted by $\Sigma(v)$.*

It follows from (5.25) that any two pairs from a syzygetic tetrad belong to the same Steiner complex. Conversely, let $\{q_1, q_1'\}, \{q_2, q_2'\}$ be two pairs from $\Sigma(v)$. We have $(q_1 + q_1') + (q_2 + q_2') = v + v = 0$, showing that the tetrad (q_1, q_1', q_2, q_2') is syzygetic.

Proposition 5.4.5. *There are $2^{2g} - 1$ Steiner complexes. Each Steiner complex consists of $2^{g-1}(2^{g-1}-1)$ elements paired by translation $q \mapsto q + v$. An odd quadratic form q belongs to a Steiner complex $\Sigma(v)$ if and only if $q(v) = 0$.*

Proof. Since $2^{2g}-1 = \#(V\setminus\{0\})$, it suffices to show that the map $s : \mathcal{P} \to V\setminus\{0\}$ is surjective. The symplectic group $\mathrm{Sp}(V,\omega)$ acts transitively on $V\setminus\{0\}$ and on $\mathcal{P}$, and the map s is obviously equivariant. Thus its image is a non-empty G-invariant subset of $V\setminus\{0\}$. It must coincide with the whole set.

By (5.7), we have $q(v) = \mathrm{Arf}(q+v)+\mathrm{Arf}(q)$. If $q \in \Sigma(v)$, then $q+v \in Q(V)_-$, hence $\mathrm{Arf}(q+v) = \mathrm{Arf}(q) = 1$ and we get $q(v) = 0$. Conversely, if $q(v) = 0$ and $q \in \Sigma(v)$, we get $q+v \in Q(V)_-$ and hence $q \in \Sigma(v)$. This proves the last assertion. □

Lemma 5.4.6. *Let $\Sigma(v), \Sigma(v')$ be two Steiner complexes. Then*

$$\#\Sigma(v)\cap\Sigma(v') = \begin{cases} 2^{g-1}(2^{g-2}-1) & \textit{if } \omega(v,v') = 0\\ 2^{g-2}(2^{g-1}-1) & \textit{if } \omega(v,v') \neq 0.\end{cases}$$

Proof. Let $q \in \Sigma(v)\cap\Sigma(v')$. Then we have $q+q' = v, q+q'' = v'$ for some $q' \in \Sigma(v), q'' \in \Sigma(v')$. This implies that

$$q(v) = q(v') = 0. \tag{5.26}$$

Conversely, if these equalities hold, then $q+v, q+v' \in Q(V)_-$, $q,q' \in \Sigma(v)$, and $q,q'' \in \Sigma(v')$. Thus we have reduced our problem to linear algebra. We want to show that the number of elements in $Q(V)_-$ which vanish at 2 nonzero vectors $v,v' \in V$ is equal to $2^{g-1}(2^{g-2}-1)$ or $2^{g-2}(2^{g-1}-1)$ depending on whether $\omega(v,v') = 0$ or 1. Let q be one such quadratic form. Suppose we have another q' with this property. Write $q' = q+v_0$ for some v_0. We have $q(v_0) = 0$ since q' is odd and

$$\omega(v_0,v) = \omega(v_0,v') = 0.$$

Let L be the plane spanned by v,v'. Assume $\omega(v,v') = 1$, then we can include v,v' in a standard symplectic basis. Computing the Arf invariant, we find that the restriction of q to $L^\perp$ is an odd quadratic form. Thus it has $2^{g-2}(2^{g-1}-1)$ zeroes. Each zero gives us a solution for v_0. Assume $\omega(v,v') = 0$. Then L is a singular plane for q since $q(v) = q(v') = q(v+v') = 0$. Consider $W = L^\perp/L \cong \mathbb{F}_2^{2g-4}$. The form q has $2^{g-3}(2^{g-2}-1)$ zeros in W. Any representative v_0 of these zeros defines the quadratic form $q+v_0$ vanishing at v,v'. Any quadratic form we are looking for is obtained in this way. The number of such representatives is equal to $2^{g-1}(2^{g-2}-1)$.

□

Definition 5.3. *Two Steiner complexes $\Sigma(v)$ and $\Sigma(v')$ are called* syzygetic *(resp.* azygetic *if $\omega(v,v') = 0$)(resp. $\omega(v,v') = 1$).*

Theorem 5.4.7. *The union of three mutually syzygetic Steiner complexes $\Sigma(v), \Sigma(v')$ and $\Sigma(v+v')$ is equal to $Q(V)_-$.*

Proof. Since
$$\omega(v+v',v) = \omega(v+v',v') = 0,$$
we obtain that the Steiner complex $\Sigma(v+v')$ is syzygetic to $\Sigma(v)$ and $\Sigma(v')$. Suppose $q \in \Sigma(v)\cap\Sigma(v')$. Then $q(v+v') = q(v)+q(v')+\omega(v,v') = 0$. This implies that

$\Sigma(v)\cap\Sigma(v')\subset\Sigma(v+v')$ and hence $\Sigma(v),\Sigma(v'),\Sigma(v+v')$ share the same set of $2^{g-1}(2^{g-2}-1)$ elements. This gives

$$\begin{aligned}\#\Sigma(v)\cup\Sigma(v')\cup\Sigma(v+v') &= 6\cdot 2^{g-2}(2^{g-1}-1)-2\cdot 2^{g-1}(2^{g-2}-1)\\ &= 2^{g-1}(2^g-1)=\#Q(V)_-.\end{aligned}$$

□

Definition 5.4. *A set of three mutually syzygetic Steiner complexes is called a* syzygetic triad *of Steiner complexes. A set of three Steiner complexes corresponding to vectors forming a non-isotropic plane is called* azygetic triad *of Steiner complexes.*

Let $\Sigma(v_i), i=1,2,3$ be a azygetic triad of Steiner complexes. Then $\#\Sigma(v_1)\cap\Sigma(v_2)=2^{g-2}(2^{g-1}-1)$. Each set $\Sigma(v_1)\setminus(\Sigma(v_1)\cap\Sigma(v_1))$ and $\Sigma(v_1)\setminus(\Sigma(v_1)\cap\Sigma(v_1))$ consists of $2^{g-2}(2^{g-1}-1)$ elements. The union of these sets forms the Steiner complex $\Sigma(v_3)$. The number of azygetic triads of Steiner complexes is equal to $\frac{1}{3}2^{2g-2}(2^{2g}-1)$ (= the number of non-isortropic planes). We leave the proofs to the reader.

Let $\mathcal{S}_4(V)$ denote the set of syzygetic tetrads. By Corollary 5.4.4, each $T\in\mathcal{S}_4(V)$ defines an isotropic plane P_T in V. Let $\mathrm{Iso}_k(V)$ denote the set of k-dimensional isotropic subspaces in V.

Proposition 5.4.8. *Let $\mathcal{S}_4(V)$ be the set of syzygetic tetrads. For each tetrad T let P_T denote the corresponding isotropic plane. The map*

$$\mathcal{S}_4(V)\to\mathrm{Iso}_2(V),\ T\mapsto P_T,$$

is surjective. The fibre over a plane T consists of $2^{g-3}(2^{g-2}-1)$ tetrads forming a partition of the intersection of the Steiner complexes $\Sigma(v)$, where $v\in P\setminus\{0\}$.

Proof. The surjectivity of this map is proved along the same lines as we proved Proposition 5.4.5. We use the fact the symplectic group $\mathrm{Sp}(V,\omega)$ acts transitively on the set of isotropic subspaces of the same dimension. Let $T=\{q_1,\ldots,q_4\}\in\mathcal{S}_4(V)$. By definition, $P_T\setminus\{0\}=\{q_1+q_2,q_1+q_3,q_1+q_4\}$. Suppose we have another tetrad $T'=\{q_1',\ldots,q_4'\}$ with $P_T=P_{T'}$. Suppose $T\cap T'\neq\emptyset$. Without loss of generality, we may assume that $q_1'=q_1$. Then, after reindexing, we get $q_1+q_i=q_1+q_i'$, hence $q_i=q_i'$ and $T=T'$. Thus the tetrads T with $P_T=P$ are disjoint. Obviously, any $q\in T$ belongs to the intersection of the Steiner complexes $\Sigma(v),\ v\in P\setminus\{0\}$. It remains to apply Lemma 5.4.6. □

A closer look at the proof of Lemma 5.4.6 shows that the fibre over P can be identified with the set $Q(P^\perp/P)_-$.

Combining Proposition 5.4.8 with the computation of the number t_g of syzygetic tetrads, we obtain the known number of isotropic planes in V:

$$\#\mathrm{Iso}_2(V)=\frac{1}{3}(2^{2g}-1)(2^{2g-2}-1). \tag{5.27}$$

Let $\mathrm{Iso}_2(v)$ be the set of isotropic planes containing a nonzero vector $v \in V$. The set $\mathrm{Iso}_2(v)$ is naturally identified with nonzero elements in the symplectic space $(v^\perp/v, \omega')$, where ω' is defined by the restriction of ω to $v^\perp$. We can transfer the symplectic form ω' to $\mathrm{Iso}_2(v)$. We obtain $\omega'(P,Q) = 0$ if and only if $P+Q$ is an isotropic 3-subspace.

Let us consider the set $\mathcal{S}_4(V,v) = \alpha^{-1}(\mathrm{Iso}_2(v))$. This set consists of syzygetic tetrads that are invariant with respect to the translation by v. In particular, each tetrad from $\mathcal{S}_4(V,v)$ is contained in $\Sigma(v)$. We can identify the set $\mathcal{S}_4(V,v)$ with the set of cardinality 2 subsets of $\Sigma(v)/\langle v\rangle$.

There is a natural pairing on $\mathcal{S}_4(V,v)$ defined by

$$\langle T, T'\rangle = \tfrac{1}{2}\#T\cap T' \mod 2. \tag{5.28}$$

Proposition 5.4.9. *For any* $T, T' \in \mathcal{S}_4(V,v)$,

$$\omega'(P_T, P_{T'}) = \langle T, T'\rangle.$$

Proof. Let $X = \{\{T,T'\} \subset \mathcal{S}_4(V) : \alpha_v(T) \neq \alpha_v(T')\}$, $Y = \{\{P,P'\} \subset \mathrm{Iso}_2(v)\}$. We have a natural map $\tilde{\alpha}_v : X \to Y$ induced by α_v. The pairing ω' defines a function $\phi : Y \to \mathbb{F}_2$. The corresponding partition of Y consists of two orbits of the stabilizer group $G = \mathrm{Sp}(V,\omega)_v$ on Y. Suppose $\{T_1,T_2\}$ and $\{T_1',T_2'\}$ are mapped to the same subset $\{P,P'\}$. Without loss of generality, we may assume that T_1, T_1' are mapped to P. Thus

$$\langle T_1+T_2', T_2+T_1'\rangle = \langle T_1,T_2\rangle + \langle T_1',T_2'\rangle + \langle T_1,T_1'\rangle + \langle T_2,T_2'\rangle = \langle T_1,T_2\rangle + \langle T_1',T_2'\rangle.$$

This shows that the function $X \to \mathbb{F}_2$ defined by the pairing (5.28) is constant on fibres of $\tilde{\alpha}_v$. Thus it defines a map $\phi' : Y \to \mathbb{F}_2$. Both functions are invariant with respect to the group G. This immediately implies that their two level sets either coincide or are switched. However, $\#\mathrm{Iso}_2(v) = 2^{2g-2}-1$ and hence the cardinality of Y is equal to $(2^{2g-2}-1)(2^{2g-3}-1)$. Since this number is odd, the two orbits are of different cardinalities. Since the map $\tilde{\alpha}_v$ is G-equinvariant, the level sets must coincide. □

5.4.3 Fundamental sets

Recall that a standard symplectic basis in (V,ω) consists of vectors $(v_1,\ldots,v_{2g})$ such that $\omega(v_i,v_j) = 0$ unless $j = g+i$. Suppose we have an ordered set S of $2g+1$ vectors $(u_1,\ldots,u_{2g+1})$ satisfying $\omega(u_i,u_j) = 1$ unless $i = j$. It defines a standard symplectic basis by setting

$$v_i = u_1+\cdots+u_{2i-2}+u_{2i-1},\ v_{i+g} = u_1+\cdots+u_{2i-2}+u_{2i},\ i = 1,\ldots,g.$$

Conversely, we can solve the u_i's from the v_i's uniquely to reconstruct the set S from a standard symplectic basis.

Definition 5.5. *A set of* $2g+1$ *vectors* $(u_1,\ldots,u_{2g+1})$ *satisfying* $\omega(u_i,u_j) = 1$ *unless* $i = j$ *is called a* normal system *in* (V,ω).

We have established a bijective correspondence between normal systems and standard symplectic bases.

Recall that a symplectic form ω defines a nondegenerate null-system in V, i.e. a bijective linear map $f: V \to V^\vee$ such that $f(v)(v) = 0$ for all $v \in V$. Fix a basis $(e_1, \ldots, e_{2g})$ in V and the dual basis $(t_1, \ldots, t_{2g})$ in $V^\vee$ and consider vectors $u_i = e_1 + \cdots + e_{2g} - e_i, i = 1, \ldots, 2g$ and $u_{2g+1} = e_1 + \cdots + e_{2g}$. Then there exists a unique null-system $V \to V^\vee$ that sends u_i to t_i and u_{2g+1} to $t_{2g+1} = t_1 + \cdots + t_{2g}$. The vectors $u_1, \ldots, u_{2g+1}$ form a normal system in the corresponding symplectic space.

Let $(u_1, \ldots, u_{2g+1})$ be a normal system. Denote the corresponding vectors by $p_{i,2g+2}$. We will identify nonzero vectors in V with points in the projective space $|V|$. For any $i, j \neq 2g+2$ consider the line spanned by $p_{i,2g+2}$ and $p_{i,2g+2}$. Let p_{ij} be the third nonzero point in this line. Now do the same with points p_{ij} and p_{kl} with the disjoint sets of indices. Denote this point by p_{ijkl}. Note that the residual point on the line spanned by p_{ij} and p_{jk} is equal to p_{ik}. Continuing in this way we will be able to index all points in $|V|$ with subsets of even cardinality (up to complementary sets) of the set $B_g = \{1, \ldots, 2g+2\}$. This notation will agree with the notation of 2-torsion points for hyperelliptic curves of genus g. For example, we have

$$\omega(p_I, p_J) = \#I \cap J \mod 2.$$

It is easy to compute the number of normal systems. It is equal to the number of standard symplectic bases in (V, ω). The group $\mathrm{Sp}(V, \omega)$ acts simply transitively on such bases, so their number is equal to

$$\#\mathrm{Sp}(2g, \mathbb{F}_2) = 2^{g^2}(2^{2g} - 1)(2^{2g-2} - 1) \cdots (2^2 - 1). \tag{5.29}$$

Now we introduce the analog of a normal system for quadratic forms in $Q(V)$.

Definition 5.6. *A* fundamental set *in $Q(V)$ is an ordered azygetic set of $2g+2$ elements in $Q(V)$.*

The number $2g + 2$ is the largest possible cardinality of a set such that any three elements are azygetic. This follows from the following immediate corollary of Lemma 5.4.1.

Lemma 5.4.10. *Let $B = (q_1, \ldots, q_k)$ be an azygetic set. Then the set $(q_1+q_2, \ldots, q_1+q_k)$ is a normal system in the symplectic subspace of dimension $k-2$ spanned by these vectors.*

The Lemma shows that any fundamental set in $Q(V)$ defines a normal system in V, and hence a standard symplectic basis. Conversely, starting from a normal system $(u_1, \ldots, u_{2g+1})$ and any $q \in Q(V)$ we can define a fundamental set $(q_1, \ldots, q_{2g+2})$ by

$$q_1 = q, q_2 = q + u_1, \ldots, q_{2g+2} = q + u_{2g+1}.$$

Since elements in a fundamental system add up to zero, we get that the elements of a fundamental set also add up to zero.

Proposition 5.4.11. *There exists a fundamental set with all or all but one quadratic forms are even or odd. The number of odd quadratic forms in such a basis is congruent to $g+1$ modulo 4.*

Proof. Let $(u_1,\ldots,u_{2g+1})$ be a normal system and $(t_1,\ldots,t_{2g+1})$ be its image under the map $V\to V^\vee$ defined by ω. Consider the quadratic form

$$q=\sum_{1\le i<j\le 2g+1} t_it_j.$$

It is immediately checked that

$$q(u_k)\equiv \binom{2g}{2}=g(2g-1)\equiv g \mod 4.$$

Passing to the associated symplectic basis we can compute the Arf invariant of q to get

$$\mathrm{Arf}(q)=\begin{cases}1 & \text{if } g\equiv 1 \mod 2\\ 0 & \text{otherwise.}\end{cases}$$

This implies that

$$\mathrm{Arf}(q+t_k^2)=\mathrm{Arf}(q)+q(u_k)=\begin{cases}0 & \text{if } g\equiv 0,3 \mod 4\\ & \text{otherwise.}\end{cases}$$

Consider the fundamental set formed by the quadrics $q, 2q+y_k^2, k=1,\ldots,2g+1$. Thus if $g\equiv 0 \mod 4$ the set consists of all even quadratic forms. If $g\equiv 1 \mod 4$, the quadratic form q is odd, all other quadratic forms are even. If $g\equiv 2 \mod 4$, all quadratic forms are odd. Finally, if $g\equiv 3 \mod 4$, then q is even, all other quadratic forms are odd. □

Definition 5.7. *A fundamental set with all or all but one quadratic forms are even or odd is called a* normal fundamental set.

One can show (see [86], p. 271) that any normal fundamental set is obtained as in the proof of the previous proposition.

Choose a normal fundamental set $(q_1,\ldots,q_{2g+2})$ such that all the first $2g+1$ quadrics are of the same type. Any quadratic form $q\in Q(V)$ can be written in the form

$$q_{2g+2}+\sum_{i\in I}t_i^2=q+\sum_{i\in I}t_i^2,$$

where I is a subset of $[1,2g+1]$. We denote such a quadratic form by q_S, where $S=I\cup\{2g+2\}$ considered as a subset of $1,2g+2]$ modulo the complementary set. We can and will always assume that

$$\#S\equiv g+1 \mod 2.$$

The quadratic form q_S can be characterized by the property that it vanishes on points p_{ij}, where $i\in S$ and $j\in\{1,\ldots,2g+2\}$.

The following properties can be checked.

- $q_S + q_T = p_{S+T}$;
- $q_S + p_I = q_{S+I}$;
- $q_S(p_T) = 0$ if and only if $\#S \cap T + \frac{1}{2}\#S \equiv 0 \mod 2$;
- $q_S \in Q(V)_+$ if and only if $\#S \equiv g + 1 \mod 4$.

Again we see that a choice of a fundamental set defines the notation of quadratic forms which agrees with the notation of theta characteristics for hyperelliptic curves.

Since fundamental sets are in a bijective correspondence with normal systems their number is given by (5.29).

5.5 Scorza correspondence

5.5.1 Correspondences on an algebraic curve

A *correspondence* of degree d between nonsingular curves C_1 and C_2 is a non-constant morphism T from C_1 to the d-th symmetric product $C_2^{(d)}$ of C_2. A correspondence can be defined by its graph $\Gamma_T \subset C_1 \times C_2$. If $Z \subset C_2^{(d)} \times C_2$ is the incidence variety (the projection $Z \to C_2^{(d)}$ is the universal family for the functor represented by $C_2^{(d)}$), then Γ_T is the inverse image of Z under the morphism $T \times \mathrm{id} : C_1 \times C_2 \to C_2^{(d)} \times C_2$. Set-theoretically,

$$\Gamma_T = \{(x, y) \in C_1 \times C_2 : y \in T(x)\}.$$

We have

$$T(x) = \Gamma_T \cap (\{x\} \times C_2), \tag{5.30}$$

where the intersection is scheme-theoretical.

One can extend the map (5.30) to any divisors on C_1 by setting $T(D) = p_1^*(D) \cap \Gamma_T$. It is clear that a principal divisor goes to a principal divisor. Taking divisors of degree 0, we obtain a homomorphism of the Jacobian varieties

$$\phi_T : \mathrm{Jac}(C_1) \to \mathrm{Jac}(C_2).$$

The projection $\Gamma_T \to C_1$ is a finite map of degree d. Since T is not constant, the projection to C_2 is a finite map of degree d'. It defines a correspondence $C_2 \to C_1^{(d')}$ which is denoted by T^{-1} and is called the *inverse correspondence*. Its graph is equal to the image of T under the switch of factors map $C_1 \times C_2 \to C_2 \times C_1$.

We will be dealing mostly with correspondences $T : C \to C^{(d)}$ and will identify T with its graph Γ_T. If d is the degree of T and d' is the degree of T^{-1} we say that T is the correspondence of type (d, d'). A correspondence is *symmetric* if $T = T^{-1}$ We assume that T does not contain the diagonal Δ of $C \times C$. A *united point* of a correspondence is a common point with the diagonal. It comes with the multiplicity.

A correspondence $T : C \to C^{(d)}$ has *valence* ν if the divisor class of $T(x) + \nu x$ does not depend on x.

Proposition 5.5.1. *The following properties are equivalent:*

(i) *the cohomology class* $[T]$ *in* $H^2(C \times C, \mathbb{Z})$ *is equal to*

$$[T] = (d' + \nu)[\{x\} \times C] + (d + \nu)[C \times \{x\}] - \nu[\Delta],$$

where x *is any point on* C*;*

(ii) *the divisor class of* $T(x) + \nu x$ *does not depend on* x*;*

(iii) *the homomorphism* ϕ_T *is equal to homomorphism* $[-\nu] : \mathrm{Jac}(C) \to \mathrm{Jac}(C)$ *of the multiplication by* $-\nu$*.*

Proof. (i) $\Rightarrow$ (ii) Let $p_1, p_2 : C \times C \to C$ be the projections. We use the well-known fact that the natural homomorphism of the Picard varieties

$$p_1^*(\mathrm{Pic}^0(C)) \oplus p_2^*(\mathrm{Pic}^0(C)) \to \mathrm{Pic}^0(C \times C)$$

is an isomorphism. Fix a point $x_0 \in C$ and consider the divisor $T + \nu\Delta - (d' + \nu)(\{x_0\} \times C) - (d + \nu)(C \times \{x_0\})$. By assumption, it is algebraically equivalent to zero. Thus

$$T + \nu\Delta \sim p_1^*(D_1) + p_2^*(D_2)$$

for some divisors D_1, D_2 on C. Thus the divisor class $T(x) + \nu x$ is equal to the divisor class of the restriction of $p_2^*(D_2)$ to $\{x\} \times C$. Obviously, it is equal to the divisor class of D_2, hence is independent on x.

(ii) $\Leftrightarrow$ (iii) This follows from the definition of the homomorphism ϕ_T.

(ii) $\Rightarrow$ (i). We know that there exists a divisor D on C such that the restriction $T + \nu\Delta - p_2^*(D)$ to any fibre of p_1 is linearly equivalent to zero. By the seesaw principle ([294] Chapter 2, Corollary 6), $T + \nu\Delta - p_2^*(D) \sim p_1^*(D')$ for some divisor D' on C. This implies that $[T] = \deg D'[\{x\} \times C] + \deg D[C \times \{x\}] - \nu[\Delta]$. Taking the intersections with fibres of the projections, we find that $d' = \deg D' - \nu$ and $d = \deg D - \nu$.

□

Note that for a general curve C of genus $g > 2$

$$\mathrm{End}(\mathrm{Jac}(C)) \cong \mathbb{Z}$$

(see [256]), so any correspondence has valence. An example of a correspondence without valence is the graph of an automorphism of order > 2 of C.

Observe that the proof of the proposition shows that for a correspondence R with valence ν

$$T \sim p_1^*(D') + p_2^*(D) - \nu\Delta, \tag{5.31}$$

where D is the divisor class of $T(x) + \nu x$ and D' is the divisor class of $T^{-1}(x) + \nu x$. It follows from the proposition that the correspondence T^{-1} has valence ν.

The next corollary is known as the *Cayley-Brill formula*.

Corollary 5.5.2. *Let* T *be a correspondence of type* (a, b) *on a nonsingular projective curve* C *of genus* g*. Assume that* T *has valence equal to* ν*. Then the number of united points of* T *is equal to*

$$d + d' + 2\nu g.$$

This immediately follows from (5.31) and the formula $\Delta \cdot \Delta = 2 - 2g$.

Example 5.5.1. Let C be a nonsingular complete intersection of a nonsingular quadric Q and a cubic in $\mathbb{P}^3$. In other words, C is a canonical curve of genus 4 curve without vanishing even theta characteristic. For any point $x \in C$, the tangent plane $\mathbb{T}_x(Q)$ cuts out the divisor $2x + D_1 + D_2$, where $|x + D_1|$ and $|x + D_2|$ are the two g^1_3's on C defined by the two rulings of the quadrics. Consider the correspondence T on $C \times C$ defined by $T(x) = D_1 + D_2$. This is a symmetric correspondence of type $(4, 4)$ with valence 2. Its 24 united points correspond to the ramification points of the two g^1_3's.

For any two correspondences T_1 and T_2 on C one defines the *composition of correspondences* by considering $C \times C \times C$ with the projections $p_{ij} : C \times C \times C \to C \times C$ onto two factors and setting

$$T_1 \circ T_2 = (p_{13})_*\big(p^*_{12}(T_1) \cap p^*_{23}(T_2)\big).$$

Set-theoretically

$$T_1 \circ T_2 = \{(x, y) \in C \times C : \exists z \in C : (x, z) \in T_1, (z, y) \in T_2\}.$$

Also $T_1 \circ T_2(x) = T_1(T_2(x))$. Note that if $T_1 = T_2^{-1}$ and T_2 is of type (d, d') we have $T_1(T_2(x)) - dx > 0$. Thus the graph of $T_1 \circ T_2$ contains $d\Delta$. We modify the definition of the composition by setting $T_1 \diamondsuit T_2 = T_1 \circ T_2 - s\Delta$, where s is the largest positive multiple of the diagonal component of $T_1 \circ T_2$.

Proposition 5.5.3. *Let $T_1 \circ T_2 = T_1 \diamondsuit T_2 + s\Delta$. Suppose that T_i is of type (d_i, d'_i) and valency ν_i. Then $T_1 \diamondsuit T_2$ is of type $(d_1d_2 - s, d'_1d'_2 - s)$ and valency $-\nu_1\nu_2 + s$.*

Proof. Applying Proposition 5.5.1, we can write

$$[T_1] = (d'_1 + \nu_1)[\{x\} \times C] + (d_1 + \nu_1)[C \times \{x\}] - \nu_1[\Delta],$$

$$[T_2] = (d'_2 + \nu_2)[\{x\} \times C] + (d_2 + \nu_2)[C \times \{x\}] - \nu_2[\Delta].$$

Easy computation with intersections gives

$$[T_1 \diamondsuit T_2] = (d'_1d'_2 - \nu_1\nu_2)[\{x\} \times C] + (d_1d_2 - \nu_1\nu_2)[C \times \{x\}] + (\nu_1\nu_2 - s)[\Delta].$$

$$= (d'_1d'_2 - s + \nu)[\{x\} \times C] + (d_1d_2 - s + \nu)[C \times \{x\}] + \nu[\Delta],$$

where $\nu = -\nu_1\nu_2 + s$. This proves the assertion. □

Example 5.5.2. In [15], vol. 6, p. 11, the symmetric correspondence $T \diamondsuit T^{-1}$ is called the *direct lateral correspondence*. If (r, s) is the type of T and γ is its valency, then it is easy to see that $T \circ T = T \diamondsuit T^{-1} + s\Delta$, and we obtain that the type of $T \diamondsuit T^{-1}$ is equal to $(s(r - 1), s(r - 1))$ and valency $s - \gamma^2$. This agrees with Baker's formula.

Here is one application of a direct lateral correspondence. Consider a correspondence of valency 2 on a plane nonsingular curve C of degree d such that $T(x) = \mathbb{T}_c(C) \cap C - 2x$. In other words, $T(x)$ are the remaining $d - 2$ intersection points of the tangent at x with C. For any point $y \in C$ the inverse correspondence assigns to y the divisor $P_y(C) - 2y$, where $P_y(C)$ is the first polar. A united point of $T \diamondsuit T^{-1}$

is one of the two points of the intersection of a bitangent with the curve. We have $s = d(d-1)-2, r = d-2, \nu = 2$. Applying the Cayley-Brill formula, we find that the number b of bitangents is expressed by the following formula

$$2b = 2(d(d-1)-2)(d-3)+(d-1)(d-2)(d(d-1)-6) = d(d-2)(d^2-9). \quad (5.32)$$

As in the case of bitangents to the plane quartic, there exists a plane curve of degree $(d-2)(d^2-9)$ (a *bitangential curve* which cuts out on C the set of tangency points of bitangents (see [356], pp. 342-357).

5.5.2 Scorza correspondence

Let C be a nonsingular projective curve of genus $g > 0$ and ϑ be a non-effective theta-characteristic on C.

Let

$$d_1 : C \times C \to \mathrm{Jac}(C),\ (x,y) \mapsto [x-y] \quad (5.33)$$

be the difference map. Let $\Theta = W_{g-1} - \vartheta$ be symmetric theta divisor corresponding to ϑ. Define

$$R_\vartheta = d_1^{-1}(\Theta).$$

Set-theoretically,

$$(R_\vartheta)_{\text{red}} = \{(x,y) \in C \times C : h^0(x+\vartheta-y) > 0\}.$$

Lemma 5.5.4. *R_ϑ is a symmetric correspondence of type (g,g), with valence equal to -1 and without united points.*

Proof. Since Θ is a symmetric theta divisor, the divisor $d_1^{-1}(\Theta)$ is invariant with respect to the switch of the factors of $X \times X$. This shows that R_ϑ is symmetric.

Fix a point x_0 and consider the map $i : C \to \mathrm{Jac}(C)$ defined by $i(x) = [x - x_0]$. It is known (see [30], Chapter 11, Corollary (2.2)) that

$$\Theta \cdot \iota_*(C) = (C \times \{x_0\}) \cdot d_1^*(\Theta) = g.$$

This shows that R_ϑ is of type (g,g). Also it shows that $R_\vartheta(x_0) - x_0 + \vartheta \in W_{g-1}$. For any point $x \in C$, we have $h^0(\vartheta + x) = 1$ because ϑ is non-effective. Thus $R_\vartheta(x)$ is the unique effective divisor linearly equivalent to $x + \vartheta$. By definition, the valence of R_ϑ is equal to -1. Applying the Cayley-Brill formula we obtain that R_ϑ has no united points. □

Definition 5.8. *The correspondence R_ϑ is called the* Scorza correspondence.

Example 5.5.3. Assume $g = 1$ and fix a point on C equipping C with a structure of an elliptic curve. Then ϑ is a non-trivial 2-torsion point. The Scorza correspondence R_ϑ is the graph of the translation automorphism defined by η.

In general R_ϑ could be neither reduced nor irreducible correspondence. However, for general curve X of genus g everything is as expected.

Proposition 5.5.5. *Assume C is general in the sense that* $\mathrm{End}(\mathrm{Jac}(C)) \cong \mathbb{Z}$*. Then R_ϑ is reduced and irreducible.*

Proof. The assumption that $\mathrm{End}(\mathrm{Jac}(C)) \cong \mathbb{Z}$ implies that any correspondence on $C \times C$ has valence. This implies that the Scorza correspondence is irreducible curve and reduced. In fact, it is easy to see that the valence of the sum of two correspondences is equal to the sum of valences. Since R_ϑ has no united points, it follows from the Cayley-Brill formula that the valence of each part must be negative. Since the valence of R_ϑ is equal to -1, we get a contradiction. □

It follows from (5.31) that the divisor class of R_ϑ is equal to

$$R_\vartheta \sim p_1^*(\vartheta) + p_2^*(\vartheta) + \Delta. \tag{5.34}$$

Since $K_{C\times C} = p_1^*(K_C)+p_2^*(K_C)$, applying the adjunction formula and using that $\Delta \cap R = \emptyset$ and the fact that $p_1^*(\vartheta) = p_2^*(\vartheta)$, we easily find

$$\omega_{R_\vartheta} = 3p_1^*(\omega_C). \tag{5.35}$$

In particular, the arithmetic genus of R_ϑ is given by

$$p_a(R_\vartheta) = 3g(g-1) + 1. \tag{5.36}$$

Note that the curve R_ϑ is very special, for example, it admits a fixed-point free involution defined by the switching the factors of $X \times X$.

Proposition 5.5.6. *Assume that C is not hyperelliptic. Let R be a symmetric correspondence on $C \times C$ of type (g, g), without united points and some valence. Then there exists a unique non-effective theta characteristic ϑ on C such that $R = R_\vartheta$.*

Proof. It follows from the Cayley-Brill formula that the valence ν of R is equal to -1. Thus the divisor class of $R(x) - x$ does not depend on x. Since R has no united points, the divisor class $D = R(x) - x$ is not effective, i.e., $h^0(R(x) - x) = 0$. Consider the difference map $d_1 : C \times C \to \mathrm{Jac}(C)$. For any $(x, y) \in R$, the divisor $R(x) - y \sim D + x - y$ is effective of degree $g - 1$. Thus $d_1(R) + D \subset W_{g-1}^0$. Let $\sigma : X \times X \to X \times X$ be the switch of the factors. Then

$$\phi(R) = d_1(\sigma(R)) = [-1](d_1(R)) \subset [-1](W_{g-1}^0 - D) \subset W_{g-1}^0 + D',$$

where $D' = K_C - D$. Since $R \cap \Delta = \emptyset$ and C is not hyperelliptic the equality $d_1(x, y) = d_1(x', y')$ implies $(x, y) = (x', y')$. Thus the difference map d_1 is injective on R. This gives

$$R = d_1^{-1}(W_{g-1}^0 - D) = d_1^{-1}(W_{g-1}^0 - D').$$

Restricting to $\{x\} \times C$ we see that the divisor classes D and D' are equal. Hence D is a theta characteristic ϑ. By assumption, $h^0(R(x) - x) = h^0(\vartheta) = 0$, hence ϑ is non-effective. The uniqueness of ϑ follows from formula (5.34). □

Let $x, y \in R_\vartheta$. Then the sum of two positive divisors $(R_\vartheta(x) - y) + (R_\vartheta(y) - x)$ is linearly equivalent to $x + \vartheta - y + y + \vartheta - x = 2\vartheta = K_C$. This defines a map

$$\gamma : R_\vartheta \to |K_C|, \ (x, y) \mapsto (R_\vartheta(x) - y) + (R_\vartheta(y) - x). \tag{5.37}$$

Recall from [197], p. 360, that the theta divisor Θ defines the *Gauss map*

$$\mathcal{G} : \Theta^0 \to |K_C|,$$

where Θ^0 is the open subset of nonsingular points of Θ. It assigns to a point z the tangent space $T_z(\Theta)$ considered as a hyperplane in $T_z(\mathrm{Jac}(C)) \cong H^1(C, \mathcal{O}_C) \cong H^0(C, \mathcal{O}_C(K_C))^\vee$. More geometrically, $\mathcal{G}$ assigns to $D - \vartheta$ the linear span of the divisor D in the canonical space $|K_C|^\vee$ (see [9], p. 246). Since the hyperplane section of the canonical C by the hyperplane $\gamma(x, y)$ contains the divisors $R(x) - y$ (and $R(y) - x$), and they do not move, we see that

$$\gamma = \mathcal{G} \circ d_1.$$

Lemma 5.5.7.

$$\gamma^*(\mathcal{O}_{|K_C|}(1)) \cong \mathcal{O}_{R_\vartheta}(R_\vartheta) \cong p_1^*(K_C).$$

Proof. The Gauss map $\mathcal{G}$ is given by the normal line bundle $\mathcal{O}_\Theta(\Theta)$. Thus the map γ is given by the line bundle

$$d_1^*(\mathcal{O}_\Theta(\Theta)) = \mathcal{O}_{R_\vartheta}(d_1^*(\Theta) \cong \mathcal{O}_{R_\vartheta}(R_\vartheta).$$

It remains to apply formula (5.34). □

The Gauss map is a finite map of degree $\binom{2g-2}{g-1}$. It factors through the map $\Theta^0 \to \Theta^0/(\iota)$, where ι is the negation involution on $\mathrm{Jac}(C)$. The map γ also factors through the involution of $X \times X$. Thus the degree of the map $R_\vartheta \to \gamma(R_\vartheta)$ is equal to $2d(\vartheta)$, where $d(\vartheta)$ is some numerical invariant of the theta characteristic ϑ. We call it the *Scorza invariant*.

Let

$$\Gamma(\vartheta) := \gamma(R_\vartheta).$$

We considered it as a curve embedded in $|K_C|$. Applying Lemma 5.5.7, we obtain

Corollary 5.5.8.

$$\deg \Gamma(\vartheta) = \frac{g(g-1)}{d(\vartheta)}.$$

Remark 5.5.1. Let C be a canonical curve of genus g and R_ϑ be a Scorza correspondence on C. For any $x, y \in C$ consider the degree $2g$ divisor $D(x, y) = R_\vartheta(x) + R_\vartheta(y) \in |K_C + x + y|$. Since $|2K_C - (K_C + x + y)| = |K_C - x - y|$ we obtain that the linear system of quadrics through $D(x, y)$ is of dimension $\frac{1}{2}g(g+1) - 2g = \dim |\mathcal{O}_{\mathbb{P}^{g-1}}(2)| - 2g + 1$. This shows that the set $D(x, y)$ imposes one less condition on quadrics passing through this set. For example, when $g = 3$ we get that $D(x, y)$ is on a conic. If $g = 3$ it is the base set of a net of quadrics. We refer to [130] and [158] for projective geometry of sets imposing one less condition on quadrics (called *self-associated sets*).

5.5.3 Scorza quartic hypersurfaces

The following construction due to G. Scorza needs some generality assumption on C.

Definition 5.9. *A pair* (C,ϑ) *is called* Scorza general *if the following properties are satisfied*

(i) R_ϑ *is a connected nonsingular curve;*

(ii) $d(\vartheta) = 1$*;*

(iii) $\Gamma(\vartheta)$ *is not contained in a quadric.*

We will see in the next chapter that a general canonical curve of genus 3 is Scorza general. For higher genus this was proven in [405].

We continue to assume that C is non-hyperelliptic. Consider the canonical embedding $C \hookrightarrow |K_C|^\vee \cong \mathbb{P}^{g-1}$ and identify C with its image (the canonical model of C). For any $x \in C$, the divisor $R_\vartheta(x)$ consists of g points y_i. If all of them distinct we have g hyperplanes $\gamma(x,y_i) = \langle R_\vartheta(x) - y_i\rangle$, or, g points on the curve $\Gamma(\vartheta)$. More generally, we have a map $C \to C^{(g)}$ defined by the projection $p_1 : R_\vartheta \to C$. The composition of this map with the map $\gamma^{(g)} : C^{(g)} \to \Gamma(\vartheta)^{(g)}$ is a regular map $\phi : C \to \Gamma(\vartheta)^{(g)}$. Let $H \cap C = x_1 + \cdots + x_{2g-2}$ be a hyperplane section of C. Adding up the images of the points x_i under the map ϕ we obtain $g(2g-2)$ points on $\Gamma(\vartheta)$.

Proposition 5.5.9. *Let* $D = x_1 + \cdots + x_{2g-2}$ *be a canonical divisor on* C*. Assume* (C,ϑ) *is Scorza general. Then the divisors*

$$\phi(D) = \sum_{i=1}^{2g-2} \phi(x_i)$$

span the linear system of divisors on $\Gamma(\vartheta)$ *which are cut out by quadrics.*

Proof. First note that the degree of the divisor is equal to $2\deg\Gamma(\vartheta)$. Let $(x,y) \in R_\vartheta$ and $D_{x,y} = \gamma(x,y) = (R_\vartheta(x) - y) + (R_\vartheta(y) - x) \in |K_C|$. For any $x_i \in R_\vartheta(x) - y$, the divisor $\gamma(x,x_i)$ contains y. Similarly, for any $x_j \in R_\vartheta(y) - x$, the divisor $\gamma(y,x_j)$ contains x. This means that $\phi(D_{x,y})$ is cut out by the quadric $Q_{x,y}$ equal to the sum of two hyperplanes $\check{H}_x, \check{H}_y$ corresponding to the points $x,y \in C \subset |K_C|^\vee$ via the duality. The image of $|K_C|$ in $\Gamma(\vartheta)^{(g(2g-2))}$ spans a linear system L (since any map of a rational variety to Jac$(\Gamma(\vartheta))$ is constant). Since $\Gamma(\vartheta)$ is not contained in a quadric, it generates $|K_C|$. This shows that all divisors in L are cut out by quadrics. The quadrics $Q_{x,y}$ span the space of quadrics in $|K_C|$ since otherwise there exists a quadric in $|K_C|^\vee$ apolar to all quadrics $Q_{x,y}$. This would imply that for a fixed $x \in C$, the divisor $R_\vartheta(x)$ lies in a hyperplane, the polar hyperplane of the quadric with respect to the point x. However, because ϑ is non-effective, $\langle R_\vartheta(x)\rangle$ spans $\mathbb{P}^{g-1}$. Thus $\dim L \ge g(g+1)/2$, and, since no quadrics contains $\Gamma(\vartheta)$, L coincides with the linear system of divisors on $\Gamma(\vartheta)$ cut out by quadrics. □

Let $E = H^0(C,\omega_C)^\vee$ so that $|K_C| = \mathbb{P}(E^\vee)$ and $|K_C|^\vee = |E|$. We can identify the space of quadrics in $|E|$ with $\mathbb{P}(S^2E)$. Using the previous proposition we obtain a map $\mathbb{P}(E^\vee) \to \mathbb{P}(S^2E)$. The restriction of this map to the curve $\Gamma(\vartheta)$ is given by the linear system $|\mathcal{O}_{\Gamma(\vartheta)}(2)|$. This shows that the map is given by quadratic polynomials, so defines a linear map

$$\alpha : S^2(E^\vee) \to S^2(E).$$

The proof of the proposition implies that this map is bijective.

Theorem 5.5.10. *Assume (C, ϑ) is Scorza general. There exists a unique quartic hypersurface $V(f)$ in $|E| = \mathbb{P}^{g-1}$ such that the inverse linear map α^{-1} is equal to the polarization map $\psi \mapsto D_\psi(f)$.*

Proof. Consider $\alpha^{-1} : S^2(E) \to S^2(E^\vee)$ as a tensor $U \in S^2(E^\vee) \otimes S^2(E^\vee) \subset (E^\vee)^{\otimes 4}$ viewed as a 4-multilinear map $E^4 \to \mathbb{C}$. It is enough to show that U is totally symmetric. Then α^{-1} is defined by the apolarity map associated to a quartic hypersurface. Fix a reduced divisor $R_\vartheta(x) = x_1 + \cdots + x_g$. Let H_i be the hyperplane in $|E|$ spanned by $R_\vartheta(x) - x_i$. Choose a basis $(t_1, \ldots, t_g)$ in $E^\vee$ such that $H_i = V(t_i)$. It follows from the proof of Proposition 5.5.9 that the quadratic map $\mathbb{P}(E^\vee) \to \mathbb{P}(S^2E)$ assigns to the hyperplane H_i the quadric Q_{x,x_i} equal to the union of two hyperplanes associated to x and x_i via the duality. The corresponding linear map α satisfies

$$\alpha(t_j^2) = \xi_j(\sum_{i=1}^g b_i \xi_i),\ j = 1, \ldots, g,$$

where $(\xi_1, \ldots, \xi_g)$ is the dual basis to $(t_1, \ldots, t_g)$, and $(b_1, \ldots, b_g)$ are the coordinates of the point x. This implies that

$$U(\xi_j, \sum_{i=1}^g b_i\xi_i, \xi_k, \xi_m) = \begin{cases} 1 & \text{if } j = k = m, \\ 0 & \text{otherwise} \end{cases} = U(\xi_k, \sum_{i=1}^g b_i\xi_i, \xi_j, \xi_m).$$

This shows that U is symmetric in the first and the third arguments when the second argument belongs to the curve $\Gamma(\vartheta)$. Since the curve $\Gamma(\vartheta)$ spans $\mathbb{P}(E^\vee)$, this is always true. It remains to use that U is symmetric in the first and the second arguments, as well as in the third and the forth arguments. □

Definition 5.10. *Let (C, ϑ) be Scorza general pair consisting of a canonical curve of genus g and a non-effective theta characteristic ϑ. Then the quartic hypersurface $V(f)$ is called the* Scorza quartic hypersurface *associated to (C, ϑ).*

We will study the Scorza quartic plane curves in the case $g = 3$. Very little is known about Scorza hypersurfaces for general canonical curves of genus > 3. We do not even know whether they are nonsingular. However, it follows from the construction that it is always a nondegenerate in the sense of section 1.4.1.

5.5.4 Theta functions and bitangents

Let C be a nonsingular curve of genus $g > 0$. Fixing a point c_0 on C allows one to define an isomorphism of algebraic varieties

$$\mathrm{Pic}^d(C) \to \mathrm{Jac}(C), \quad [D] \mapsto [D - dc_0].$$

The composition of this map with the map $u_d : C^{(d)} \to \mathrm{Pic}^d(C)$ is called the *Abel-Jacobi map*

$$u_d(c_0) : C^{(d)} \to \mathrm{Jac}(C).$$

If no confusion arises, we drop c_0 from this notation. For $d = 1$, this map defines an embedding

$$u_1 : C \hookrightarrow \mathrm{Jac}(C).$$

For the simplicty of the notation we will identify C with its image. For any $c \in C$ the tangent space of C at a point c is a one-dimensional subspace of the tangent space of $\mathrm{Jac}(C)$ at c. By the unique translation automorphism we identify this space with $T_0\mathrm{Jac}(C)$ at the zero point. Under the Abel-Jacobi map the space of holomorphic 1-forms on $\mathrm{Jac}(C)$ is identified with the space of holomorphic forms on C. Thus we can identify $T_0\mathrm{Jac}(C)$ with the dual space $H^0(C, K_C)^\vee$. As a result we obtain the *canonical map* of C

$$\varphi : C \to \mathbb{P}(H^0(C, K_C)^\vee) = |K_C|^\vee \cong \mathbb{P}^{g-1}.$$

If C is not hyperelliptic the canonical map is an embedding.

We continue to identify $H^0(C, K_C)^\vee$ with $T_0\mathrm{Jac}(C)$. A symmetric odd theta divisor $\Theta = W_{g-1} - \vartheta$ contains the origin of $\mathrm{Jac}(C)$. If $h^0(\vartheta) = 1$, this point is nonsingular and hence Θ defines a hyperplane in $T_0(\mathrm{Jac}(C))$, the tangent hyperplane $T_0\Theta$. Passing to the projectivization we have a hyperplane in $|K_C|^\vee$.

Proposition 5.5.11. *The hyperplane in $|K_C|^\vee$ defined by Θ is a bitangent hyperplane to the image $\varphi(C)$ under the canonical map.*

Proof. Consider the difference map (5.33) $d_1 : C \times C \to \mathrm{Jac}(C)$. In the case when Θ is an even divisor, we proved in (5.34) that

$$d_1^*(\Theta) \sim p_1^*(\theta) + p_2^*(\theta) + \Delta. \tag{5.38}$$

Since two theta divisors are algebraically equivalent the same is true for an odd theta divisor. The only difference is that $d_1^*(\Theta)$ contains the diagonal Δ as the preimage of 0. It follows from the definition of the Abel-Jacobi map $u_1(c_0)$ that

$$u_1(c_0)(C) \cap \Theta = d_1^{-1}(\Theta) \cap p_1^{-1}(c_0) = c_0 + D_\vartheta,$$

where D_ϑ is the unique effective divisor linearly equivalent to ϑ. Let $\mathcal{G} : \Theta^{\mathrm{ns}} \to \mathbb{P}(T_0\mathrm{Jac}(C))$ be the Gauss map defined by translation of the tangent space at a nonsingular point of Θ to the origin. It follows from the proof of Torelli Theorem [9] that the Gauss map ramifies at any point where Θ meets $u_1(C)$. So, the image of the Gauss map intersects the canonical image with multiplicity ≥ 2 at each point. This proves the assertion. $\square$

More explicitly, the equation of the bitangent hyperplane corresponding to Θ is given by the linear term of the Taylor expansion of the theta function $\theta\left[\begin{smallmatrix}\boldsymbol{\epsilon}\\ \boldsymbol{\eta}\end{smallmatrix}\right]$ corresponding to Θ. Note that the linear term is a linear function on $H^0(C, K_C)^\vee$, hence can be identified with a holomorphic differential

$$h_\Theta = \sum_{i=1}^{g} \frac{\partial\theta\left[\begin{smallmatrix}\boldsymbol{\epsilon}\\ \boldsymbol{\eta}\end{smallmatrix}\right](z,\tau)}{\partial z_i}(0)\omega_i,$$

where $(z_1, \ldots, z_g)$ are coordinates in $H^0(C, K_C)^\vee$ defined by a normalized basis ω_1, $\ldots, \omega_g$ of $H^0(C, K_C)$. The sections of $H^0(\mathrm{Jac}(C), \mathcal{O}_{\mathrm{Jac}(C)}(\Theta)) \cong \mathbb{C}$ can be identified with holomorphic *half-order* differentials. To make this more precise, i.e. describe how to get a square root of a holomorphic 1-form, we use the following result (see [162], Proposition 2.2).

Proposition 5.5.12. *Let Θ be a symmetric odd theta divisor defined by the theta function $\theta\left[\begin{smallmatrix}\boldsymbol{\epsilon}\\ \boldsymbol{\eta}\end{smallmatrix}\right]$. Then for all $x, y \in C$*

$$\theta\left[\begin{smallmatrix}\boldsymbol{\epsilon}\\ \boldsymbol{\eta}\end{smallmatrix}\right](d_1(x-y))^2 = h_\Theta(\varphi(x))h_\Theta(\varphi(y))E(x,y)^2,$$

where $E(x, y)$ is a certain section of $\mathcal{O}_{C\times C}(\Delta)$ (the prime-form*).*

An attentive reader should notice that the equality is not well-defined in many ways. First, the vector $\varphi(x)$ is defined only up to proportionality and the value of a section of a line bundle is also defined only up to proportionality. To make sense of this equality we pass to the universal cover of $\mathrm{Jac}(C)$ identified with $H^0(C, K_C)^\vee$ and to the universal cover U of $C \times C$ and extend the difference map and the map φ to the map of universal covers. Then the prime-form is defined by a certain holomorphic function on U and everything makes sense. As the equality of the corresponding line bundles, the assertion trivially follows from (5.38).

Let

$$\mathfrak{r}\left[\begin{smallmatrix}\boldsymbol{\epsilon}\\ \boldsymbol{\eta}\end{smallmatrix}\right](x,y) = \frac{\theta\left[\begin{smallmatrix}\boldsymbol{\epsilon}\\ \boldsymbol{\eta}\end{smallmatrix}\right](d_1(x-y))}{E(x,y)}.$$

Since $E(x, y) = -E(y, x)$ and $\theta\left[\begin{smallmatrix}\boldsymbol{\epsilon}\\ \boldsymbol{\eta}\end{smallmatrix}\right]$ is an odd function, we have $\mathfrak{r}\left[\begin{smallmatrix}\boldsymbol{\epsilon}\\ \boldsymbol{\eta}\end{smallmatrix}\right](x, y) = \mathfrak{r}\left[\begin{smallmatrix}\boldsymbol{\epsilon}\\ \boldsymbol{\eta}\end{smallmatrix}\right](y, x)$ for any $x, y \in C \times C \setminus \Delta$. It satisfies

$$\mathfrak{r}\left[\begin{smallmatrix}\boldsymbol{\epsilon}\\ \boldsymbol{\eta}\end{smallmatrix}\right](x,y)^2 = h_\Theta(\varphi(x))h_\Theta(\varphi(y)). \tag{5.39}$$

Note that $E(x, y)$ satisfies $E(x, y) = -E(y, x)$, since $\theta\left[\begin{smallmatrix}\boldsymbol{\epsilon}\\ \boldsymbol{\eta}\end{smallmatrix}\right]$ is an odd function, we have $\mathfrak{r}\left[\begin{smallmatrix}\boldsymbol{\epsilon}\\ \boldsymbol{\eta}\end{smallmatrix}\right](x, y) = \mathfrak{r}\left[\begin{smallmatrix}\boldsymbol{\epsilon}\\ \boldsymbol{\eta}\end{smallmatrix}\right](y, x)$ for any $x, y \in C \times C \setminus \Delta$.

Now let us fix a point $y = c_0$, so we can define the *root function* on C. It is a rational function on the universal cover of C defined by $\mathfrak{r}\left[\begin{smallmatrix}\boldsymbol{\epsilon}\\ \boldsymbol{\eta}\end{smallmatrix}\right](x, c_0)$.

Thus every honest bitangent hyperplane of the canonical curve defines a root-function.

Suppose we have two odd theta functions $\theta\left[\begin{smallmatrix}\boldsymbol{\epsilon}\\ \boldsymbol{\eta}\end{smallmatrix}\right], \theta\left[\begin{smallmatrix}\boldsymbol{\epsilon}'\\ \boldsymbol{\eta}'\end{smallmatrix}\right]$. Then the ratio of the corresponding root functions is equal to $\frac{\theta\left[\begin{smallmatrix}\boldsymbol{\epsilon}\\ \boldsymbol{\eta}\end{smallmatrix}\right](d_1(x-c_0))}{\theta\left[\begin{smallmatrix}\boldsymbol{\epsilon}'\\ \boldsymbol{\eta}'\end{smallmatrix}\right](d_1(x-c_0))}$ and its square is a rational function on C, defined uniquely up to a constant factor depending on the choice of c_0. Its divisor is equal to the difference $2\vartheta - 2\vartheta'$. Thus we can view the ratio as a section of $K_X^{\frac{1}{2}}$ with divisor $\theta - \theta'$. This section is not defined on C but on the double cover of C corresponding to the 2-torsion point $\vartheta - \vartheta'$. If we have two pairs $\vartheta_1, \vartheta', \vartheta_2, \vartheta_2'$ of odd theta characteristics satisfying $\vartheta_1 - \vartheta' = \vartheta_2 - \vartheta_2' = \epsilon$, i.e. forming a syzygetic tetrad, the product of the two ratios is a rational function on C with divisor $\vartheta_1 + \vartheta_2' - \vartheta' - \vartheta_2$. Following Riemann [337] and Weber [422], we denote this function by $\sqrt{\frac{\vartheta_1\vartheta_1'}{\vartheta_2\vartheta_2'}}$. By Riemann-Roch, $h^0(\vartheta_1 + \vartheta_2') = h^0(K_C + \epsilon) = g - 1$, hence any

g pairs $(\vartheta_1, \vartheta'), \ldots, (\vartheta_g, \vartheta'_g)$ of odd theta characteristics in a Steiner complex define g linearly independent functions $\sqrt{\frac{\vartheta_1\vartheta'_1}{\vartheta_g\vartheta'_g}}, \ldots, \sqrt{\frac{\vartheta_{g-1}\vartheta'_{g-1}}{\vartheta_g\vartheta'_g}}$. After scaling, and getting rid of squares by using (5.39) we obtain a polynomial in $h_{\Theta_1}(\varphi(x)), \ldots, h_{\Theta_g}(\varphi(x))$ vanishing on the canonical image of C.

Example 5.5.4. Let $g = 3$. We take three pairs of odd theta functions and get the equation

$$\sqrt{\vartheta_1\vartheta'_1} + \sqrt{\vartheta_2\vartheta'_2} + \sqrt{\vartheta_3\vartheta'_3} = 0. \tag{5.40}$$

After getting rid of squares, we obtain the quartic equation of C

$$(lm)^2 + (pq)^2 + (rs)^2 - 2lmpq = 2lmrs - 2pqrs = (lm - pq - rs)^2 - 4lmpq = 0, \tag{5.41}$$

where l, m, p, q, rs are the linear functions in z_1, z_2, z_3 defining the linear terms of the Taylor expansion at 0 of the odd theta functions corresponding to three pairs in a Steiner complex. The number of possible ways to write the equation of a plane quartic in this form is equal to $63 \cdot 20 = 1260$.

Remark 5.5.2. For any non-zero 2-torsion point, the linear system $|K_C + \epsilon|$ maps C to $\mathbb{P}^{g-2}$, the map is called the *Prym canonical map*. We have seen that the root functions $\sqrt{\frac{\vartheta_1\vartheta'_1}{\vartheta_2\vartheta'_2}}$ belong to $H^0(C, K_C + \epsilon)$ and can be used to define the Prym canonical map. For $g = 3$, the map is a degree 4 cover of $\mathbb{P}^1$ and we expressed the quartic equation of C as a degree 4 cover of $\mathbb{P}^1$.

Exercises

5.1 Find 3 non-equivalent symmetric determinant expressions for the cubic curve given by a Weierstrass equation $t_0t_2^2 + t_1^3 + at_1t_0^2 + bt_0^3 = 0$.

5.2 Find a symmetric determinant expression for the Fermat quartic $V(t_0^4 + t_1^4 + t_2^4)$.

5.3 Let C be an irreducible plane curve of degree d with a $(d-2)$-multiple point. Show that its normalization is a hyperelliptic curve of genus $g = d-2$. Conversely, show that any hyperelliptic curve of genus g admits such a plane model.

5.4 Show that a nonsingular curve of genus 2 has a vanishing theta characteristic but a nonsingular curve of genus 3 has a vanishing theta characteristic if and only if it is a hyperelliptic curve.

5.5 Show that a nonsingular non-hyperelliptic curve of genus 4 has a vanishing theta characteristic if and only if its canonical model lies on a quadratic cone.

5.6 Show that a nonsingular plane curve of degree 5 does not have a vanishing theta characteristic.

5.7 Find the number of vanishing theta characteristics on a hyperelliptic curve of genus g.

5.8 Compute the number of syzygetic tetrads contained in a Steiner complex.

5.9 Show that the composition of two correspondences (defined as the composition of the multi-valued maps defined by the correspondences) with valences ν and ν' is a correspondence with valence $-\nu\nu'$.

5.10 Let $f : X \to \mathbb{P}^1$ be a non-constant rational function on a nonsingular projective curve X. Consider the fibred product $X \times_{\mathbb{P}^1} X$ as a correspondence on $X \times X$. Show that it has valence and compute the valence. Show that the Cayley-Brill formula is equivalent to the Hurwitz formula.

5.11 Suppose that a nonsingular projective curve X admits a non-constant map to a curve of genus > 0. Show that there is a correspondence on X without valence.

5.12 Show that any correspondence on a nonsingular plane cubic has valence unless the cubic is harmonic or equianharmonic.

5.13 Describe all symmetric correspondences of type $(4, 4)$ with valence 1 on a canonical curve of genus 4.

5.14 Let R_ϑ be the Scorza correspondence on a curve C. Prove that a point $(x, y) \in R_\vartheta$ is singular if and only if x and y are ramification points of the projections $R_\vartheta \to C$.

Historical Notes

It is a too large task to discuss the history of theta functions. We only mention that the connection between odd theta functions with characteristics and bitangents to a quartic curves goes back to Riemann [337], [422]. There are numerous expositions of the theory of theta functions and jacobian varieties (e.g. [9], [83], [295]). The theory of fundamental sets of theta characteristics goes back to A. Göpel and J. Rosenhein. Its good exposition can be found in Krazer's book [260]. As an abstract symplectic geometry over the field of two elements it is presented in Coble's book [87] which we followed. Some additional material can be found in [86] (see also a modern exposition in [346]).

The theory of correspondences on an algebraic curve originates from the *Charles' Principle of Correspondenc* [68] which is the special case of the Cayley-Brill formula in the case $g = 0$. We have already encountered with its application to Poncelet polygons in Chapter 2. This application was first found by A. Cayley [61]. He was also the first to extend Chasles's Principle to higher genus [61] although with incomplete proof. The first proof of the Cayley-Brill formula was given by A. Brill [38]. The notion of valence (die Werthigeit) was introduced by Brill. The fact that only correspondences with valence exist on a general curve was first pointed out by A. Hurwitz [227]. He also showed the existence of correspondences without valence. A good reference to many problems solved by the theory of correspondences is Baker's book [15], vol. 6. We refer to [389] for a fuller history of the theory of correspondences.

The number of bitangents to a plane curve was first computed by J. Plücker [319], [320]. The equations of bitangential curves were given by A. Cayley [53], G. Salmon [356] and O. Dersch [123].. The number of bitangents of a plane curve is due to J. Plücker [319].

The study of correspondences of type (g, g) with valence -1 was initiated by G. Scorza [370], [371]. His construction of a quartic hypersurface associated to a non-effective theta characteristic on a canonical curve of genus g was given in [372]. A modern exposition of Scorza' theory was first given in [132].

Chapter 6

Plane Quartics

6.1 Bitangents

6.1.1 28 bitangents

A nonsingular plane quartic C is a non-hyperelliptic genus 3 curve embedded by its canonical linear system $|K_C|$. It has no vanishing theta characteristics, so the only effective theta characteristics are odd ones. The number of them is $28 = 2^2(2^3 - 1)$. Thus C has exactly 28 bitangents . Each bitangent is tangent to C at two points that may coincide. In the latter case a bitangent is called a *flex bitangent*.

We can specialize the results from section 5 of the previous chapter to the case $g = 3$ taking $V = \mathrm{Pic}(C)[2]$ with the symplectic form ω defined by the Weil pairing. Elements of $Q(V)_-$ will be identified with bitangents.

The union of bitangents forming a syzygetic tetrad cuts out in C is a divisor of degree 8 equal to $C \cap Q$ for some conic Q. There are $t_3 = 315$ syzygetic tetrads. There is a bijection between the set of syzygetic tetrads and the set of isotropic planes in $\mathrm{Pic}(C)[2]$.

There are 63 Steiner complexes of bitangents. Each complex consists of 6 pairs of bitangents ℓ_i, ℓ_i' such that the divisor class of $\ell_i \cap C - \ell_i' \cap C$ is a fixed nonzero 2-torsion divisor class.

Two Steiner complexes have either four or six common bitangents, dependent on whether they are syzygetic or not. Each isotropic plane in $\mathrm{Pic}(C)[2]$ defines three Steiner complexes with common four bitangents. Two azygetic Steiner complexes have 6 common bitangents. The number of azygetic triads is equal to 336.

In the following we will often identify an odd theta characteristic with the corresponding bitangent.

Let $\ell_i = V(l_i),\ i = 1, \ldots, 4$, be four syzygetic bitangents and $Q = V(q)$ be the corresponding conic. Since $V(l_1l_2l_3l_4)$ and $V(q^2)$ cut out the same divisor on C we obtain that C can be given by an equation

$$f = l_1l_2l_3l_4 + q^2 = 0. \tag{6.1}$$

Conversely, if f can be written in the form (6.1), the linear forms l_i define four syzygetic bitangents. So we see that f can be written as in (6.1) in only finitely many ways. This is confirmed by "counting constants". We have 12 constants for the linear forms and 6 constants for quadratic forms, they are defined up to scaling by $\lambda_1, \ldots, \lambda_5$ subject to the condition $\lambda_1 \cdots \lambda_4 = \lambda_5^2$. Thus we have 14 parameters for quartic curves represented in the form (6.1). This is the same as the number of parameters for plane quartics.

Let $l = 0, m = 0, p = 0, q = 0, r = 0, s = 0$ be the equations of 6 bitangents such that (l, m, p, q) and (l, m, r, s) are two syzygetic tetrads of bitangents. In other words, the three pairs $(l, m), (p, q), (r, s)$ is a part of the set of 6 pairs in a Steiner complex of bitangents. By (6.1), we can write

$$f = lmpq - a^2 = lmrs - b^2$$

for some quadratic forms a, b. Subtracting we have

$$lm(pq - rs) = (a + b)(a - b).$$

If l divides $a + b'$ and m divides $a - b$, then the quadric $V(\frac{1}{2}[(a + b) + (a - b)])$ passes through the point $l \cap m$. But this is impossible since no two bitangents intersect at a point on the quartic. Thus we obtain that lm divides either $a + b$ or $a - b$. Without loss of generality, we get $lm = a + b, \quad pq - rs = a - b$, and hence $a = \frac{1}{2}(lm + pq - rs)$. Thus we can define the quartic by the equation

$$-4f = -4lmpq + (lm + pq - rs)^2 = (lm)^2 - 2lmpq - 2lmrs - 2pqrs + (pq)^2 + (rs)^2 = 0. \tag{6.2}$$

It is easy to see that this is equivalent to the equation

$$\sqrt{lm} + \sqrt{pq} + \sqrt{rs} = 0. \tag{6.3}$$

Thus we see that a nonsingular quartic can be written in 315 ways in the form of (6.1) and in $1260 = \binom{6}{3} \cdot 63$ ways in the form of (6.3). In the previous chapter we found this equation by using theta functions.

Remark 6.1.1. Consider the orbit space $X = (\mathbb{C}^3)^6/T$, where

$$T = \{(z_1, z_2, z_3, z_4, z_5, z_6) \in (\mathbb{C}^*)^6 : z_1 z_2 = z_3 z_4 = z_5 z_6\}$$

is a 14-dimensional algebraic torus. Any orbit $T(l, m, p, q, r, s) \in X$ defines the quartic curve $V(\sqrt{lm} + \sqrt{pq} + \sqrt{rs})$. We have shown that the map $X \to |\mathcal{O}_{\mathbb{P}^2}(4)| \cong \mathbb{P}^{14}$ is of degree 1260. The group $\mathrm{PGL}(3)$ acts naturally on both spaces. One can show that $X/PGL(3)$ is a rational variety and we get a map $X/\mathrm{PGL}(3) \to |\mathcal{O}_{\mathbb{P}^2}(4)|/\mathrm{PGL}(3) \cong \mathcal{M}_3$ of degree 1260.

The projection from the intersection point of two bitangents defines a g_4^1 with two non-reduced members. The intersection point of three bitangents gives a g_4^1 which is not expected on a general curve of genus 3. It is not known the maximal possible number of triple points of the arrangements of 28 lines formed by the bitangents. However, we can prove the following.

Proposition 6.1.1. *No three bitangents forming an azygetic triad can intersect at one point.*

Proof. Let $\vartheta_1, \vartheta_2, \vartheta_3$ be the corresponding odd theta characteristics. The 2-torsion divisor classes $\epsilon_{ij} = \vartheta_i - \vartheta_j$ form a non-isotropic plane. Let ϵ be a non-zero point in the orthogonal complement. Then $q_{\eta_i}(\epsilon) + q_{\eta_j}(\epsilon) + \langle \eta_{ij}, \epsilon \rangle = 0$ implies that q_{η_i} takes the same value at ϵ. We can always choose ϵ such that this value is equal to 0. Thus the three bitangents belong to the same Steiner complex $\Sigma(\epsilon)$. Obviously, no two differ by ϵ, hence we can form 3 pairs from them. These pairs can be used to define the equation (6.3) of C. It follows from this equation that the intersection point of the three bitangents lies on C. But this is impossible because C is nonsingular. $\square$

Remark 6.1.2. A natural question is whether the set of bitangents determines the quartic, i.e. whether two quartics with the same set of bitangents coincide. Surprizingly it has not been answered by the ancients. Only recently it was proven that the answer is yes: [41] (for general curve), [266] (for any nonsingular curve).

6.1.2 Aronhold sets

We know that in the case $g = 3$ a normal fundamental set of 8 theta characteristics contains 7 odd theta characteristics. The corresponding unordered set of 7 bitangents is called an *Aronhold set* . It follows from (5.29) that the number of Aronhold sets is equal to $\#\mathrm{Sp}(6, \mathbb{F}_2)/7! = 288$.

A choice of an ordered Aronhold set defines a unique normal fundamental set that contains it. The eighth theta characteristic is equal to the sum of the characteristics from the Aronhold set. Thus an Aronhold set can be defined as an azygetic set of seven bitangents.

A choice of an ordered Aronhold set allows one to index all 2-torsion divisor classes (resp. odd theta characteristics) by subsets of even cardinality (resp. of cardinality 2) of $\{1, \ldots, 8\}$, up to complementary set. Thus we have 63 2-torsion classes $\epsilon_{ab}, \epsilon_{abcd}$ and 28 bitangents ℓ_{ij} corresponding to 28 odd theta characteristics ϑ_{ij}. The bitangents from the Aronhold set correspond to the subsets $(18, 28, \ldots, 78)$.

We also know that $\vartheta_A - \vartheta_B = \epsilon_{A+B}$. This implies, for example, that four bitangents $\ell_A, \ell_B, \ell_C, \ell_D$ form a syzygetic tetrad if and only if $A + B + C + D = 0$.

Following Cayley we denote a pair of numbers from the set $\{1, \ldots, 8\}$ by a vertical line $|$. If two pairs have a common number we make them intersect. For example, we have

- Pairs of bitangents: 210 of type $||$ and 168 of type $\vee$.
- Triads of bitangents:
  1. (sygetic) 420 of type $\sqcup$, 840 azygetic of type $|||$,
  2. (asyzygetic) 56 of type $\triangle$, 1680 of type $\vee\,|$, and 280 of type $\vee\!\!/$;
- Tetrads of bitangents:
  1. (syzygetic) 105 azygetic of type $||||$, 210 of type $\square$,

2. (asygetic) 560 of type $|\,\triangle$, 280 of type $\vee\!\!\angle$, 1680 of type $|\vee\!\!/$, 2520 of type $\vee\vee$.

3. (non syzygetic but containing a syzygetic triad) 2520 of type $||\,\vee$, 5040 of type $|\,\sqcup$, 3360 of type $\sqcup\!\ulcorner$, 840 of type $\triangle\!\!/$, 3360 of type $\vee\!\!/\!\wedge$.

There are two types of Aronhold sets: $\ast$, $\vee\!\!\angle\ \triangle$. They are represented by the sets $(12,13,14,15,16,17,18)$ and $(12,13,23,45,46,47,48)$. The number of the former type is 8, the number of the latter type is 280. Note that the different types correspond to orbits of the subgroup of $\mathrm{Sp}(6,\mathbb{F}_2)$ isomorphic to the permutation group $\mathfrak{S}_8$. For example, we have two orbits of $\mathfrak{S}_8$ on the set of Aronhold sets consisting of 8 and 280 elements.

Lemma 6.1.2. *Three odd theta characteristics $\vartheta_1,\vartheta_2,\vartheta_3$ in a Steiner complex $\Sigma(\epsilon)$, no two of which differ by ϵ, are azygetic.*

Proof. Let $\vartheta_i' = \vartheta_i+\epsilon, i = 1,2,3$. Then $\{\vartheta_1,\vartheta_1',\vartheta_2,\vartheta_2'\}$ and $\{\vartheta_1,\vartheta_1',\vartheta_3,\vartheta_3'\}$ are syzygetic and have two common theta characteristics. By Proposition 5.4.9, the corresponding isotropic planes do not span an isotropic 3-space. Thus $\langle\vartheta_1-\vartheta_2,\vartheta_3-\vartheta_1\rangle = 1$, hence $\vartheta_1,\vartheta_2,\vartheta_3$ is an azygetic triad. □

The previous Lemma suggests a way to construct an Aronhold set from a Steiner set $\Sigma(\epsilon)$. Choose another Steiner set $\Sigma(\eta)$ azygetic to the first one. They intersect at 6 odd theta characteristics $\vartheta_1,\ldots,\vartheta_6$, no two of which differ by ϵ. Consider the set $\{\vartheta_1,\ldots,\vartheta_5,\vartheta_6+\epsilon,\vartheta_6+\eta\}$. We claim that this is an Aronhold set. By the previous Lemma all triads $\vartheta_i,\vartheta_j,\vartheta_k, i,j,k\le 5$ are azygetic. Any triad $\vartheta_i,\vartheta_6+\epsilon,\vartheta_6+\eta, i\le 5$, is azygetic too. In fact $q_{\vartheta_i}((\vartheta_6+\epsilon)-(\vartheta_6+\eta)) = q_{\vartheta_i}(\epsilon+\eta)\ne 0$ since $\vartheta_i\notin\Sigma(\epsilon+\eta)$. So the assertion follows from Lemma 5.4.1. We leave to the reader to check that remaining triads $\{\vartheta_i,\vartheta_j,\vartheta_6+\epsilon\},\{\vartheta_i,\vartheta_j,\vartheta_6+\eta\}, i\le 5$, are azygetic.

Proposition 6.1.3. *Any six lines in an Aronhold set are contained in a unique Steiner complex.*

We use that the symplectic group $\mathrm{Sp}(6,\mathbb{F}_2)$ acts transitively on the set of Aronhold sets. So it is enough to check the assertion for one Aronhold set. Let it correspond to the index set $(12,13,14,15,16,17,18)$. It is enough to check that the first six are contained in a unique Steiner complex. For this it is enough to exhibit a 2-torsion divisor class ϵ such that $q_{\vartheta_I}(\epsilon) = 0$ for the first six subsets I and show its uniqueness. Equivalently we have to show that there exists a unique subset J of $[1,8]$ of cardinality 2 or 4 such that it contains exactly one element from each I. Obviously, the only such subset is $\{1,8\}$.

Recall that a Steiner subset of theta characteristics on a genus 3 curve consists of 12 elements. A subset of 6 elements will be called a *hexad*.

Corollary 6.1.4. *Any Steiner complex contains 2^6 azygetic hexads. Half of them are contained in another Steiner complex, necessarily azygetic to the first one. Any other hexad can be extended to a unique Aronhold set.*

Proof. Let $\Sigma(\epsilon)$ be a Steiner complex consisting of 6 pairs of odd theta characteristics. Consider it as G-set, where $G = (\mathbb{Z}/2\mathbb{Z})^6$ whose elements, identified with subsets I of $[1,6]$, act by switching elements in i-th pairs, $i \in I$. It is clear that G acts simply transitively on the set of azygetic sextupes in $\Sigma(\epsilon)$. For any azygetic complex $\Sigma(\eta)$ the intersection $\Sigma(\epsilon) \cap \Sigma(\eta)$ is an azygetic hexad. Note that two syzygetic complexes have only 4 bitangents in common. The number of such hexads is equal to $2^6 - \#\epsilon^\perp = 2^6 - 2^5 = 2^5$. Thus the set of azygetic hexads contained in a unique Steiner complex is equal to $2^5 \cdot 63$. But this number is equal to the number $7 \cdot 288$ of subsets of cardinality 6 of Aronhold sets. By the previous proposition, all such sets are contained in a unique Steiner complex. $\square$

Let $(\vartheta_1, \ldots, \vartheta_7)$ be an Aronhold set. We use the corresponding normal fundamental set to index its elements by the subsets $(18, 28, \ldots, 78)$. By Proposition 6.1.3 the hexad $\vartheta_2, \ldots, \vartheta_7$ is contained in a unique Steiner complex $\Sigma(\epsilon)$. Let $\eta_2 = \vartheta_2 + \epsilon$. The only 2-torsion point at which all quadrics $q_{28}, \ldots, q_{78}$ vanish is the point p_{18} corresponding to the subset $\{1,8\}$. Thus $q_{\eta_2} = q_{28} + p_{18} = q_{12}$. This shows that the bitangent defined by η_2 corresponds to (12). Similarly, we see that the bitangents corresponding to $\eta_i + \epsilon, i = 3, \ldots, 7$ corresponds to $(1i)$.

6.1.3 Riemann's equations for bitangents

Here we show how to write equations of all bitangents knowing the equations of an Aronhold set of bitangents.

Let $\ell_0, \ldots, \ell_6$ be an Aronhold set of bitangents of C. By Proposition 6.1.1, any three lines are not concurrent. By a linear transformation, and reodering, we may assume

$$\ell_0 = V(t_0), \ell_1 = V(t_1), \ell_2 = V(t_2), \ell_3 = V(t_0 + t_1 + t_2)$$

and the remaining ones are $\ell_{3+i} = V(a_{0i}t_0 + a_{1i}t_1 + a_{2i}t_2), i = 1, 2, 3$.

Theorem 6.1.5. *There exist linear forms u_0, u_1, u_2 such that C can be given by the equation*

$$\sqrt{t_0u_0} + \sqrt{t_1u_1} + \sqrt{t_2u_2} = 0.$$

The forms u_i can be found from equations

$$\begin{aligned} u_0 + u_1 + u_2 + x_0 + x_1 + x_2 &= 0, \\ \frac{u_0}{a_{01}} + \frac{u_1}{a_{11}} + \frac{u_2}{a_{21}} + k_1a_{01}x_0 + k_1a_{11}x_1 + k_1a_{21}x_2 &= 0, \\ \frac{u_0}{a_{02}} + \frac{u_1}{a_{12}} + \frac{u_2}{a_{22}} + k_2a_{02}x_0 + k_2a_{12}x_1 + k_2a_{22}x_2 &= 0, \\ \frac{u_0}{a_{03}} + \frac{u_1}{a_{13}} + \frac{u_2}{a_{23}} + k_3a_{03}x_0 + k_3a_{13}x_1 + k_3a_{23}x_2 &= 0, \end{aligned}$$

where k_1, k_2, k_3 can be found from solving the following linear equations:

$$\begin{pmatrix} \frac{1}{a_{01}} & \frac{1}{a_{02}} & \frac{1}{a_{03}} \\ \frac{1}{a_{11}} & \frac{1}{a_{12}} & \frac{1}{a_{13}} \\ \frac{1}{a_{21}} & \frac{1}{a_{22}} & \frac{1}{a_{23}} \end{pmatrix} \cdot \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix},$$

$$\begin{pmatrix}\lambda_0 a_{01} & \lambda_1 a_{11} & \lambda_2 a_{21}\\ \lambda_0 a_{02} & \lambda_1 a_{12} & \lambda_2 a_{22}\\ \lambda_0 a_{03} & \lambda_1 a_{13} & \lambda_2 a_{23}\end{pmatrix} \cdot \begin{pmatrix}k_1\\ k_2\\ k_3\end{pmatrix} = \begin{pmatrix}-1\\ -1\\ -1\end{pmatrix}.$$

The equations of the remaining 21 bitangents are:

- $u_0 = 0,\ u_1 = 0,\ u_2 = 0,$
- $x_0 + x_1 + u_2 = 0,\ x_0 + x_2 + u_1 = 0,\ x_1 + x_2 + u_0 = 0,$
- *9 of type* $\frac{u_0}{a_{01}} + k_i a_{1i} x_1 + k_i a_{2i} x_2 = 0,\ i = 1, 2, 3,$
- $\frac{u_0}{1-k_i a_{1i} a_{2i}} + \frac{u_1}{1-k_i a_{0i} a_{2i}} + \frac{u_2}{1-k_i a_{01} a_{1i}} = 0,\ i = 1, 2, 3,$
- $\frac{u_0}{a_{0i}(1-k_i a_{1i} a_{2i})} + \frac{u_1}{a_{1i}(1-k_i a_{0i} a_{2i})} + \frac{u_2}{a_{2i}(1-k_i a_{01} a_{1i})} = 0,\ i = 1, 2, 3.$

Proof. Applying Proposition 6.1.3 we can find three Steiner complexes partitioned in pairs

$$\begin{gathered}(\ell_1, \xi_2), (\ell_2, \xi_1), (\ell_3, \xi_{30}), \ldots, (\ell_6, \xi_{60}),\\ (\ell_2, \xi_0), (\ell_0, \xi_2), (\ell_3, \xi_{31}), \ldots, (\ell_6, \xi_{61}),\\ (\ell_0, \xi_1), (\ell_1, \xi_0), (\ell_3, \xi_{32}), \ldots, (\ell_6, \xi_{62}).\end{gathered} \tag{6.4}$$

In the following we often identify a bitangent with the corresponding odd theta characterstic. We also write

$$\ell_i = V(l_i),\ \xi_i = V(u_i),\ \xi_{ij} = V(l_{ij})$$

for some linear forms l_i, u_i, l_{ij}.

We use here that the intersection of two Steiner complexes cannot consist of five tangents. Now we have

$$\ell_1 - \xi_2 = \ell_2 - \xi_1,\ \ell_2 - \xi_0 = \ell_0 - \xi_1,\ \ell_0 - \xi_2 = \ell_1 - \xi_0.$$

This implies that $\ell_0 - \xi_0 = \ell_1 - \xi_1 = \ell_2 - \xi_2$, i.e. the pairs $(\ell_0, \xi_0), (\ell_1, \xi_1), (\ell_2, \xi_2)$ belong to the same Steiner complex Σ. One easily checks that

$$\langle \ell_0 - \xi_0, \ell_0 - \xi_1\rangle = \langle \ell_1 - \xi_1, \ell_1 - \xi_2\rangle = \langle \ell_2 - \xi_2, \ell_2 - \xi_0\rangle = 0,$$

and hence Σ is syzygetic to the three complexes (6.4) and therefore it does not contain $\ell_i, i \geq 3$.

Now we use the three pairs $(\ell_0, \xi_0), (\ell_1, \xi_1), (\ell_2, \xi_2)$ to write C in the form (6.3)

$$\sqrt{x_0 u_0} + \sqrt{x_1 u_1} + \sqrt{x_2 u_2} = 0.$$

By (6.1), we can introduce the quadratic forms:

$$\begin{aligned} q_1 &= -x_0 u_0 + x_1 u_1 + x_2 u_3,\\ q_2 &= x_0 u_0 - x_1 u_1 + x_2 u_3,\\ q_3 &= x_0 u_0 + x_1 u_1 - x_2 u_3,\end{aligned} \tag{6.5}$$

such that

$$C = V(4x_0x_1u_0u_1 - q_3^2) = V(-4x_0x_2u_0u_2 - q_1^2) = V(-4x_1x_2u_1u_2 - q_3^2). \quad (6.6)$$

Now we use the first Steiner complex from (6.4) to do the same by using the first three pairs. We obtain

$$C = V(4x_1u_2l_3l_{30} - q^2),$$

where q is a quadratic form. As in the proof of formula (6.3), we find that

$$q_1 - q = 2\lambda_1 x_1 u_2,\ q_1 + q = \frac{2(x_2u_2 - l_3l_{30})}{\lambda_1}.$$

From this we get

$$q_1 = \lambda_1 x_1 u_2 + \frac{x_2u_1 - l_3l_{30}}{\lambda_1} = -x_0u_0 + x_1u_1 + x_2u_3.$$

This gives

$$\begin{aligned} l_3l_{30} &= x_2u_1 - \lambda_1(-x_0u_0 + x_1u_1 + x_2u_3) + \lambda_1^2 x_1u_2, \\ l_3l_{31} &= x_2u_1 - \lambda_1(x_0u_0 - x_1u_1 + x_2u_3) + \lambda_1^2 x_2u_0, \\ l_3l_{32} &= x_2u_1 - \lambda_1(x_0u_0 + x_1u_1 - x_2u_3) + \lambda_1^2 x_0u_1. \end{aligned} \quad (6.7)$$

The last two equations give

$$l_3(\frac{l_{31}}{\lambda_2} + \frac{l_{32}}{\lambda_3}) = x_0(-2u_0 + \lambda_3 u_1 + \frac{u_2}{\lambda_3}) + u_0(\lambda_2 x_2 + \frac{x_1}{\lambda_3}). \quad (6.8)$$

The lines ℓ_3, ℓ_0, and ξ_0 belong to the third Steiner complex (6.4), and by Lemma 6.1.2 form an azygetic triad. By Proposition 6.1.1, they cannot be concurrent. This implies that the line $V(\lambda_2x_2 + \frac{u_1}{\lambda_3})$ passes through the intersection point of the lines ξ_0 and ℓ_3. This gives a linear dependence between the linear functions $l_3 = a_0x_0 + a_1x_1 + a_2x_2$, $l_0 = x_0$ and $\lambda_2x_2 + \frac{x_1}{\lambda_3}$ (we can assume that $a_0 = a_1 = a_2 = 1$ but will do it later). This can happen only if

$$\lambda_2 = c_1a_2, \frac{1}{\lambda_3} = c_1a_1,$$

for some constant c_1. Now $\lambda_2x_2 + \frac{1}{\lambda_3}x_1 = c_1(a_2x_2 + a_1x_1) = c_1(l_3 - a_0x_0)$, and we can rewrite (6.8) in the form

$$c_1l_3(\frac{l_{31}}{\lambda_2} + \frac{l_{32}}{\lambda_3} - c_1u_0) = x_0(-c_1(2 + a_0c_1)u_0 + \frac{u_1}{a_1} + \frac{u_2}{a_2}).$$

This implies that

$$\frac{l_{31}}{\lambda_2} + \frac{l_{32}}{\lambda_3} = c_1u_0 + \frac{k_1}{c_1}x_0, \quad (6.9)$$

$$k_1l_3 = -c_1(2 + c_1a_0)u_0 + \frac{u_1}{a_1} + \frac{u_2}{a_2}, \quad (6.10)$$

for some constant k_1. Similarly, we get

$$k_2l_3 = -c_2(2+c_2a_1)u_1 + \frac{u_0}{a_0} + \frac{u_2}{a_2},$$

$$k_3l_3 = -c_3(2+c_3a_2)u_2 + \frac{u_1}{a_0} + \frac{u_2}{a_1}.$$

It is easy to see that this implies that

$$k_1 = k_2 = k_3 = k, c_1 = -a_0, c_2 = -a_1, c_3 = -a_2.$$

The equations (6.9) and (6.10) become

$$\frac{l_{31}}{\alpha_2} + \frac{l_{32}}{\lambda_3} = -a_0u_0 - \frac{k}{a_0}x_0, \tag{6.11}$$

$$kl_3 = \frac{u_0}{a_0} + \frac{u_1}{a_1} + \frac{u_2}{a_2}. \tag{6.12}$$

At this point, we can scale the coordinates to assume $a_1 = a_2 = a_2 = 1, k = -1$, and obtain our first equation

$$x_0 + x_1 + x_2 + u_0 + u_1 + u_2 = 0.$$

Replacing l_{30} with l_{40}, l_{50}, l_{60}, and repeating the argument we obtain the remaining three equations relating u_0, u_1, u_2 with x_0, x_1, x_2.

It remains to find the constants k_1, k_2, k_3. We have found 4 linear equations relating 6 linear functions $x_0, x_1, x_2, u_0, u_1, u_2$. Since three of them form a basis in the space of linear functions, there must be one relation. We may assume that the first equation is a linear combination of the last three with some coefficients $\lambda_1, \lambda_2, \lambda_3$. This leads to the system of linear equations from the statement of the theorem.

Finally, we have to find the equations of the bitangents. The equations (6.6) show that the lines ξ_0, ξ_1, ξ_2 are bitangents. The equation (6.11) and similar equations

$$\frac{l_{32}}{\alpha_3} + \frac{l_{30}}{\lambda_1} = -a_1u_1 - \frac{k}{a_1}x_1,$$

$$\frac{l_{30}}{\alpha_1} + \frac{l_{31}}{\lambda_2} = -a_2u_2 - \frac{k}{a_2}x_2,$$

after adding up, give

$$\frac{l_{30}}{\lambda_1} + \frac{l_{31}}{\lambda_2} + \frac{l_{32}}{\lambda_3} = -k(a_0x_0 + a_1x_1 + a_2x_2),$$

and then

$$\frac{l_{30}}{\lambda_1} = \frac{u_0}{a_0} - k(a_1x_1 + a_2x_2),$$

$$\frac{l_{31}}{\lambda_1} = \frac{u_1}{a_1} - k(a_0x_0 + a_2x_2),$$

$$\frac{l_{32}}{\lambda_1} = \frac{u_2}{a_2} - k(a_0x_0 + a_1x_1).$$

After our normalization $k = -1, a_0 = a_1 = a_2 = 1$, we get three equations of the second type. Similarly, we get the expressions for l_{4i}, l_{5i}, l_{6i} which are the nine equations of the third type.

Let us use the Aronhold set $(\ell_0, \ldots, \ell_6)$ to index bitangents by subsets of $[0,7]$ of cardinality 2. As we explained at the end of the previous section, the bitangents ξ_0, ξ_1, ξ_2 correspond to the subsets $(12), (02), (01)$. The bitangents $\xi_{3k}, \xi_{4k}, \xi_{5k}, \xi_{6k}$ correspond to the subsets $(k3), (k4), (k5), (k6), k = 0, 1, 2$. What is left are the bitangents corresponding to the subsets $(45), (46), (56), (34), (35), (36)$. The first three look like $(12), (02), (01)$, both of type $\triangle$. The second three look like $\xi_{4k}, \xi_{5k}, \xi_{6k}$, both of type $\bigvee$. To find the equations of bitangents of type $\triangle$, we interchange the roles of the lines ℓ_0, ℓ_1, ℓ_2 with the lines ℓ_4, ℓ_5, ℓ_6. Our lines will be the new lines analogous to the lines ξ_0, ξ_1, ξ_2. Solving the system, we find their equations. To find the equations of the triple of bitangents of type, we delete ℓ_3 from the original Aronhold set, and consider the Steiner complex containing the remaining lines as we did in (6.4). The lines making the pairs with ℓ_4, ℓ_5, ℓ_6 will be our lines. We find their equations following as we found the equations for $\xi_{4k}, \xi_{5k}, \xi_{6k}$. □

Remark 6.1.3. We will see later in Chapter 10 that any seven lines in general linear position can be realized as an Aronhold set for a plane quartic curve. Another way to see it can be found in [423], p. 447.

6.2 Quadratic determinant equations

6.2.1 Hesse-Coble-Roth construction

Let C be a nonsingular plane quartic. Let $a \in \mathrm{Pic}^0(C) \setminus \{0\}$. Consider the natural bilinear map

$$\mu : H^0(C, \mathcal{O}_C(K_C + a)) \times H^0(C, \mathcal{O}_C(K_C - a)) \to H^0(C, \mathcal{O}_C(2K_C))$$

defined by the tensor multiplication of the sections. The associated map of complete linear systems

$$\varphi : |K_C + a| \times |K_C - a| \to |2K_C| = |\mathcal{O}_{\mathbb{P}^2}(2)| \cong \mathbb{P}^5. \tag{6.13}$$

assigns to a pair of divisors $D \in |K_C + a|$ and $D' \in |K_C - a|$ the divisor $D + D' \in |2K_C|$ cut out by a unique conic which we denote by $\langle D, D' \rangle$. If we choose a basis (s_1, s_2) of $H^0(C, \mathcal{O}_C(K_C + a))$ and a basis (s_1', s_2') of $H^0(C, \mathcal{O}_C(K_C - a))$, then the map μ is given by

$$(\lambda s_1 + \mu s_2, \lambda' s_1' + \mu' s_2') \mapsto \lambda\lambda' a_{11} + \lambda\mu' a_{12} + \lambda'\mu a_{21} + \mu\mu' a_{22}, \tag{6.14}$$

where $a_{11}, a_{12}, a_{21}, a_{22} \in H^0(C, \mathcal{O}_C(2K_C)) \cong H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ are identified with homogeneous polynomials of degree 2 in variables t_0, t_1, t_2. Consider the variety

$$W = \{(D_1, D_2, x) \in |K_C + a| \times |K_C - a| \times \mathbb{P}^2 : x \in \langle D_1, D_2 \rangle\}. \tag{6.15}$$

If we identify $|K_C + a|$ and $|K_C - a|$ with $\mathbb{P}^1$, we see that

$$W \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$$

is a hypersurface defined by the multi-homogeneous equation

$$\lambda\lambda' a_{11}(t) + \lambda\mu' a_{12}(t) + \lambda'\mu a_{21}(t) + \mu\mu' a_{22}(t) = 0 \tag{6.16}$$

of multi-degree $(1, 1, 2)$.

Consider the projections

$$p_1 : W \to \mathbb{P}^1 \times \mathbb{P}^1, \quad p_2 : W \to \mathbb{P}^2. \tag{6.17}$$

The fibre of p_1 over a point $([\lambda, \mu], [\lambda', \mu'])$ is isomorphic (under p_2) to a conic. It is singular if and only if the discriminant of the conic (6.16) is equal to zero. It is easy to see that this is a bihomogeneous polynomial in the variables $(\lambda, \mu), (\lambda', \mu')$ of bidegree $(3, 3)$. Thus the locus Δ_1 of points $[\lambda, \mu], [\lambda', \mu']$ such that p_1^{-1} is a reducible conic is a curve in $\mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(3, 3)$.

The fibre of p_2 over a point $x \in \mathbb{P}^2$ is isomorphic (under the first projection) to a curve of bidegree $(1, 1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. Under the Segre isomorphism between $\mathbb{P}^1 \times \mathbb{P}^1$ and a quadric in $\mathbb{P}^3$, such a curve is isomorphic to a conic. This conic is reducible if and only if the equation

$$\lambda\lambda' a_{11}(t) + \lambda\mu' a_{12}(t) + \lambda'\mu a_{21}(t) + \mu\mu' a_{22}(t) = 0$$

is a "cross" on $\mathbb{P}^1 \times \mathbb{P}^1$ (i.e. the union of two lines belonging to different rulings). We can rewrite the equation in the form

$$(\lambda, \mu) \cdot \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix} \cdot \begin{pmatrix} \lambda' \\ \mu' \end{pmatrix}.$$

It defines a reducible curve if and only if there exists (λ_0, μ_0) such that, after plugging in $\lambda = \lambda_0, \mu = b\mu_0$, any (λ', μ') will satisfy the equation. The condition for this is of course

$$\det \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix} = 0. \tag{6.18}$$

This defines a homogeneous equation of degree 4 in variables t_0, t_1, t_2. It is not identically equal to zero otherwise the entries a_{11}, a_{21} must have a common linear factor. The corresponding conics cut out the divisors $D_1 + D_1', D_2 + D_1'$, where $D_i = \text{div}(s_i), D_i' = \text{div}(s_i')$. Their common points form a divisor $D_1' \in |K_C + a|$. Since $a \neq 0$, D_1' cannot be cut out by a line. Thus (6.18) defines a quartic curve. It must coincide with our curve C. To see this it is enough to show that each point of C satisfies the equation. Let $x \in C$, then we choose a unique $D \in |K_C + a|$ containing x and take any $D' \in |K_C - a|$. We obtain a subset of the conic $p_2^{-1}(x)$ isomorphic to $\mathbb{P}^1$. This shows that $p_2^{-1}(x)$ is a reducible conic.

Conversely, suppose C is given by a determinantal equation as above. For every $x \in C$ we have the left and the right kernel of the corresponding matrix. These are one-dimensional vector spaces. The corresponding maps $\phi_i : C \to \mathbb{P}^1, i = 1, 2,$ are defined by quadratic polynomials $(-a_{21}(t), a_{11}(t))$ and $(-a_{12}(t), a_{11}(t))$, respectively. Note that the common zeros of both coordinates belong to the curve C . Thus the linear system defined by the two conics has four base points on C and hence ϕ_i is given by a linear system V_i of degree 4. We may assume that V_i is contained in $|K_C + d_i|$ for some divisor classes d_i of degree 0. It is easy to see that the base loci of the two linear systems add up to the zeros of a_{11}. This immediately implies that $d_1 + d_2 = 0$.

Thus we have proved:

Theorem 6.2.1. *An equation of a nonsingular plane quartic C can be written in the form*

$$\begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix} = 0,$$

where a_i's are homogeneous forms of degree 2. Let X be the set of matrices A with quadratic forms as its entries such that $C = V(\det A)$ modulo the equivalence relation defined by $A \sim B$ if $A = CBC'$ for some constant invertible matrices. The set X is bijective to the set $\mathrm{Pic}^0(C) \setminus \{0\}$.

Remark 6.2.1. The previous Theorem agrees with the general theory developed in Chapter 5. To define a quadratic determinant one considers the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-2)^2 \to \mathcal{O}_{\mathbb{P}^2}^2 \to i_*(\mathcal{M}) \to 0.$$

We have

$$h^0(\mathcal{M}) = 2, \quad h^0(\mathcal{M}(-1)) = h^1(\mathcal{M}) = 0, \quad h^1(\mathcal{M}(-1)) = 1.$$

By Riemann-Roch, $\deg(\mathcal{M}) = 4$, hence $\mathcal{M} = \mathcal{O}_C(K_C + a)$, for some $a \in \mathrm{Pic}^0(C)$. Since $h^0(\mathcal{M}(-1)) = 0$, we obtain $a \neq 0$.

Remark 6.2.2. We assume that the reader is familiar with the theory of 3-folds. The variety W which was introduced in (6.15) is a Fano 3-fold. Its canonical sheaf is equal to

$$\omega_W \cong \mathcal{O}_{\mathbb{P}^1}(-1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(-1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(-1).$$

If we use the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^{11}$, then ω_W can be identified with $\mathcal{O}_W(-1)$. The variety W admits two structures of a conic bundle. They are induced by the projections $p_1 : W \to \mathbb{P}^1 \times \mathbb{P}^1$ and $p_2 : W \to \mathbb{P}^2$. The degeneration locus of the first map is a curve Δ_1 of arithmetic genus 4, and the degeneration locus Δ_2 of the second map is the curve C of genus 3. Note that each curve has a double cover defined by considering the irreducible components of the fibres. The double cover over Δ_2 splits since each component corresponds to one of the two rulings of $\mathbb{P}^1 \times \mathbb{P}^1$. The double cover $\tilde{\Delta}_1 \to \Delta_1$ does not split. For a general C, the curve Δ_1 is nonsingular and the double cover over it is unramified. One shows that the intermediate Jacobian variety of W is isomorphic to the Prym variety of the cover $\tilde{\Delta}_1 \to \Delta_1$. It is aslo isomorphic to the Prym variety of the trivial cover over Δ_2 which is the Jacobian of C. Thus we obtain that the intermediate Jacobian of W is isomorphic to Jac(C).

Remark 6.2.3. Let

$$V = H^0(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2, \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(2)).$$

It is a vector space of dimension 24. Let U be an open subset in $|V|$ which consists of sections (6.16) such that the corresponding determinant (6.18) defines a nonsingular quartic curve. The group $G = \mathrm{SL}(2) \times \mathrm{SL}(2)$ acts naturally on U and the orbit space is isomorphic to the space $\mathcal{P}ic_4^0 \setminus \{\text{zero section}\}$. Let W be the 3-fold (6.17) defined by a section from U. The projection $W \to \mathbb{P}^1 \times \mathbb{P}^1$ defines a curve Δ_1 of bidegree $(3,3)$ parameterizing singular fibres. It comes with a double cover defined by choosing a component of a reducible fibre. In this way we see that U/G is birationally isomorphic to the space of nonsingular curves of bidegree $(3,3)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ together with an unramified double cover. If we further act by $G' = \mathrm{SL}(3)$ the orbit space is birationally isomorphic to the universal Jacobian space over $\mathcal{M}_3$ and, on the other hand, to the moduli space $\mathcal{R}_4$ of curves of genus 4 together with a nontrivial 2-torsion divisor class. It was proven by F. Catanese that the latter space is a rational variety.

6.2.2 Symmetric quadratic determinants

Assume now that $a = \epsilon$ is a 2-torsion divisor class. Then $H^0(C, \mathcal{O}_C(K_C + \epsilon)) = H^0(C, \mathcal{O}_C(K_C - \epsilon))$ and the bilinear map μ is symmetric.

The determinantal equation of C corresponding to ϵ must be given by a symmetric quadratic determinant

$$\begin{vmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{vmatrix} = q_{11}q_{22} - q_{12}^2. \tag{6.19}$$

Thus we obtain the following.

Theorem 6.2.2. *An equation of a nonsingular plane quartic can be written in the form*

$$\begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = 0,$$

where a_1, a_2, a_3 are homogeneous forms of degree 2. Let X be the set of symmetric 2×2 matrices A with quadratic forms as its entries such that $C = V(\det A)$ modulo the equivalence relation defined by $A \sim B$ if $A = CBC'$ for some constant nonsingular matrices. The set X is bijective to the set $\mathrm{Pic}(C)[2] \setminus \{0\}$.

Since $\varphi(D_1, D_2) = \varphi(D_2, D_1)$, the map ϕ factors through a linear map

$$\bar{\varphi} : \mathbb{P}^2 \to |\mathcal{O}_{\mathbb{P}^2}(2)|.$$

Here we identify $|K_C + \epsilon|$ with $\mathbb{P}^1$ and the symmetric square $\mathbb{P}^1 \times \mathbb{P}^1/\mathfrak{S}_2$ with $\mathbb{P}^2$ (a set of k unordered points in $\mathbb{P}^1$ is a positive divisor of degree k, i.e. an element of $|\mathcal{O}_{\mathbb{P}^1}(k)| \cong \mathbb{P}^k$). Explicitly, we view $\mathbb{P}^2$ as $\mathbb{P}(S^2H^0(C, \mathcal{O}_C(K_C + \epsilon)))$. The corresponding linear map $S^2H^0(C, \mathcal{O}_C(K_C + \epsilon)) \to S^2H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ defines a regular map

$$\phi : \mathbb{P}^1 = |K_C + \epsilon| \to |\mathcal{O}_{\mathbb{P}^2}(2)|$$

which is quadratic. Explicitly,

$$\phi((t_0,t_1)) = V(t_0^2a_{11}(t) + 2t_0t_1a_{12}(t) + t_1^2a_{22}(t)).$$

Let $L(\epsilon)$ be a net of conics equal to the image of the map $\bar{\varphi}$. By choosing a basis (s_0, s_1) of $H^0(C, K_C + \epsilon)$ we may assume that $L(\epsilon)$ is spanned by the conics

$$V(a_{11}) = \langle 2D_1\rangle, \quad V(a_{12}) = \langle D_1, D_2\rangle, \quad V(a_{22}) = \langle 2D_2\rangle,$$

where $D_1 = \text{div}(s_0), D_2 = \text{div}(s_1)$. In particular, we see that $L(\epsilon)$ has no base points (since $|K_C + \epsilon|$ has no base points).

The set $B(\epsilon)$ of singular conics in $L(\epsilon)$ is a plane cubic isomorphic to a plane section of the discriminant hypersurface $\mathcal{D}_2(2)$. Its preimage under the map $\mathbb{P}^1\times\mathbb{P}^1 \to \mathbb{P}^2$ is the degeneration curve Δ_1 of the conic bundle $W \to \mathbb{P}^1 \times \mathbb{P}^1$ from (6.17).

Remark 6.2.4. We can view ϕ as the composition of the Veronese map $v_2 : \mathbb{P}^1 \to \mathbb{P}^2$ and the map $\bar{\varphi}$. Let $v_2(\mathbb{P}^1)$ be the Veronese curve. The preimage of $\mathcal{D}_2(2)$ under the map ϕ is the locus of zeros of a binary sextic. It corresponds to the intersection scheme of the cubic $B(\epsilon)$ and the conic $v_2(\mathbb{P}^1)$. One can show that the Jacobian variety of the genus 2 curve corresponding to this binary sextic is isomorphic to the Prym variety of the pair (C, ϵ). The curve C can be defined as the locus of point $x \in \mathbb{P}^2$ such that, viewed as hyperplanes in $|\mathcal{O}_{\mathbb{P}^2}(2)|$, the preimage $\phi^{-1}(x)$ is a degenerate binary quadratic form.

We apply Hesse's determinantal identity for bordered determinants from Lemma 4.1.7 and its interpretation to obtain the identity

$$\begin{vmatrix} a_{11} & a_{12} & u_0\\ a_{21} & a_{22} & u_1\\ u_0 & u_1 & 0\end{vmatrix} \times \begin{vmatrix} a_{11} & a_{12} & v_0\\ a_{21} & a_{22} & v_1\\ v_0 & v_1 & 0\end{vmatrix} - \begin{vmatrix} a_{11} & a_{12} & u_0\\ a_{21} & a_{22} & u_1\\ v_0 & v_1 & 0\end{vmatrix}^2 = |A|U(u,v;t).$$

Here $[u_0, u_1], [v_0, v_1]$ are coordinates in each copy of $|K_C + \epsilon|$ and $U = 0$ is the family of conics parametrized by $|K_C + \epsilon| \times |K_C + \epsilon|$.

Lemma 6.2.3. *The cubic curve $B(\epsilon)$ is nonsingular if and only if the linear system $|K_C + \epsilon|$ does not contain a divisor of the form $2a + 2b$.*

Proof. The plane section $L(\epsilon)\cap\mathcal{D}_2(2)$ is singular if and only if $L(\epsilon)$ contains a singular point of $\mathcal{D}_2(2)$ represented by a double line, or if it is tangent to $\mathcal{D}_2(2)$ at a nonsingular point. We proved in Chapter 2, section 2.1.2 that the tangent hypersurface of $\mathcal{D}_2(2)$ at a nonsingular point represented by a reducible conic Q is equal to the space of conics passing through the singular point q of Q. If L is contained in the tangent hyperplane, then all conics from $L(\epsilon)$ pass through q. But we have seen already that L is base point free. This shows that $L(\epsilon)$ intersects transversally the nonsingular locus of $\mathcal{D}_2(2)$.

In particular, $B(\epsilon)$ is singular if and only if $L(\epsilon)$ contains a double line. Assume that this happens. Then we get two divisors $D_1, D_2 \in |K_C + \epsilon|$ such that $D_1 + D_2 = 2A$, where $A = a_1+a_2+a_3+a_4$ is cut out by a line ℓ. Let $D_1 = p_1+p_2+p_3+p_4, D_2 = q_1+q_2+q_3+q_4$. Then the equality of divisors (not the divisor classes)

$$p_1+p_2+p_3+p_4+q_1+q_2+q_3+q_4 = 2(a_1+a_2+a_3+a_4)$$

implies that either D_1 and D_2 share a point x, or $D_1 = 2p_1+2p_2$, $D_2 = 2q_1+2q_2$. The first case is impossible, since $|K_C+\epsilon-x|$ is of dimension 0. The second case happens if and only if $|K_C+\epsilon|$ contains a divisor $D_1 = 2a+2b$. The converse is also true. For each such divisor the line $\langle a,b\rangle$ defines a residual pair of points c,d such that $D_2 = 2c+2d \in |K_C+\epsilon|$ and $\varphi(D_1,D_2)$ is a double line. □

Let
$$I = \{(x,\ell) \in B(\epsilon)\times \check{\mathbb{P}}^2 : \ell \subset \varphi(x)\}. \tag{6.20}$$
The first projection $p_1 : I \to B(\epsilon)$ is a double cover ramified at singular points of $B(\epsilon)$. The image $\tilde{B}(\epsilon)$ of the second projection is locus of lines in $\mathbb{P}^2$ which are irreducible components of reducible conics from $L(\epsilon)$. It is a plane curve of some degree d in the dual plane or the whole plane.

Lemma 6.2.4. *The curve $\tilde{B}(\epsilon) \subset \check{\mathbb{P}}^2$ parameterizing irreducible components of reducible conics from the linear system $L(\epsilon)$ is a plane cubic. If $B(\epsilon)$ is nonsingular, then $\tilde{B}(\epsilon)$ is also nonsingular and is isomorphic to an unramified double cover of $B(\epsilon)$.*

Proof. Let us see that $d = \deg(\tilde{B}(\epsilon)) = 3$. A line in the dual plane is the pencil of lines in the original plane. Thus d is equal to the number of line components of reducible conics in L which pass through a general point q in $\mathbb{P}^2$. Since q is a general point, we may assume that q is not a singular point of any reducible conic from $L(\epsilon)$. Then there are d different reducible conics passing through q.

We know that $L(\epsilon)$ has no base points. Then q must be a base point of a pencil of conics in $L(\epsilon)$. Note that a general pencil of conics in $L(\epsilon)$ has 4 distinct base points. To see this we consider the regular map $\mathbb{P}^2 \to |L(\epsilon)|^*$ defined by the linear system $|L(\epsilon)|$. Its degree is equal to 4, hence it general fibre consists of 4 distinct points. It is easy to check that a pencil of conics with 4 distinct base points contains 3 reducible conics. This shows that $d = 3$. If $B(\epsilon)$ is nonsingular, its double cover $p_1 : I \to B(\epsilon)$ is unramified, hence I is an elliptic curve. Its image $\tilde{B}(\epsilon) = p_2(I)$ in $\check{\mathbb{P}}^2$ is a plane cubic.

Note that two reducible conics $f(D_1,D_2)$ and $f(D_3,D_4)$ in $B(\epsilon)$ share a common irreducible component if and only if D_1+D_2 is cut out by two lines ℓ and ℓ' and D_3+D_4 is cut out by two lines ℓ and ℓ''. Let A be the divisor on C cut out by ℓ. We know that no two divisors from $|K_C+\epsilon|$ share a common point. Also no divisor is cut out by a line. This easily implies that $D_i \cap \ell$ consists of one point for each $i = 1,\ldots,4$. Since $D_1+D_2 \geq A$, $D_3+D_4 \geq A$, we see that ℓ contains at least 2 ramification points of the map $C \to \mathbb{P}^1$ defined by the linear system $|K_C+\epsilon|$. Since we have only finitely many such points, we see that there are only finitely many such lines ℓ. In particular, the second projection $p_2 : I \to \tilde{B}(\epsilon)$ is an isomorphism over a dense Zariski subset of $\tilde{B}(\epsilon)$.

If $B(\epsilon)$ is nonsingular, then $p_2 : I \to \tilde{B}(\epsilon)$ is a birational map of an elliptic curve to a cubic. Obviously, this cubic must be nonsingular. □

Theorem 6.2.5. *Let $S = \{(\ell_1, \ell_1'), \ldots, (\ell_6, \ell_6')\}$ be a Steiner complex of 12 bitangents associated to a 2-torsion divisor class ϵ. Then the 12 bitangents, considered as points in the dual plane, lie on the cubic curve $\tilde{B}(\epsilon)$. If we assume that $|K_C + \epsilon|$ does not contain a divisor of the form $2p + 2q$, then the cubic curve is nonsingular.*

Proof. Let $(\vartheta_i, \vartheta_i')$ be a pair of odd theta characteristics corresponding to a pair (ℓ_i, ℓ_i') of bitangents from S. They define a divisor $D = \vartheta_i + \vartheta_i' \in |K_C + \epsilon|$ which is cut out by two lines. Thus $f(D, D) \in B(\epsilon)$ and the bitangents ℓ_i, ℓ_i' belong to $\tilde{B}(\epsilon)$. The rest of the assertions follow from the previous lemmas. □

Remark 6.2.5. Let S_1, S_2, S_3 be a syzygetic (azygetic) triad of Steiner complexes. They define three cubic curves $\tilde{B}(\epsilon), \tilde{B}(\eta), \tilde{B}(\eta + \epsilon)$ which have 4 (resp. 6) points in common.

Remark 6.2.6. The cubic $\tilde{B}(\epsilon)$ has at most ordinary nodes as singularities. We know that the projection $p_1 : I \to B(\epsilon)$ is a double cover unramified outside singular points of $B(\epsilon)$ corresponding to double lines. If $B(\epsilon)$ is an irreducible cuspidal cubic, the complement of the cusp is isomorphic to $\mathbb{C}$ and hence does not admit nontrivial unramified covers. If $B(\epsilon)$ is the union of a conic and a line touching it at some point, then, again the complement of the singular point is the disjoint union of two copies of $\mathbb{C}$ and hence does not admit an unramified cover. Finally, if $B(\epsilon)$ is the union of 3 concurrent lines, then the complement to the singular point is the disjoint union of three copies of $\mathbb{C}$, no unramified covers again. Thus $B(\epsilon)$ is nonsingular or a nodal cubic. It is easy to see that its cover I is again nonsingular or a nodal curve of arithmetic genus 1. The second projection $p_2 : I \to \tilde{B}(\epsilon)$ is an isomorphism over the complement of finitely many points. It is easy to see that the image of a nodal curve is a nodal curve.

Remark 6.2.7. Let Δ_1 be a curve of bidegree $(3, 3)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ parameterizing singular fibres of the projection $p_1 : W \to \mathbb{P}^1 \times \mathbb{P}^1$. Let $\pi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1/S_2 = \mathbb{P}^2$ be the quotient map. The curve Δ_1 is equal to $\pi^{-1}(B(\epsilon))$. The cover $\pi|_{\Delta_1} : \Delta_1 \to B(\epsilon)$ is a double cover ramified along the points where Δ_1 intersects the diagonal. It consists of pairs (D, D), where $D \in |K_C + \epsilon|$ such that $2D$ is cut out by a reducible conic. It is easy to see that this conic must be the union of two bitangents which form one of 6 pairs of bitangents from the Steiner complex associated to ϵ. The branch locus of $\pi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$ is a conic C. It can be identified with the Veronese curve $v_2(\mathbb{P}^1)$ which we discussed in Remark 6.2.4. The cubic $B(\epsilon)$ intersects it transversally at 6 points. If it is nonsingular, Δ_1 is a nonsingular curve of genus 4. If $B(\epsilon)$ is an irreducible cubic with a node (by the previous remark it cannot have a cusp), Δ_1 is an irreducible curve of arithmetic genus 2 with two nodes. If $B(\epsilon)$ is the union of a conic and a line intersecting each other transversally, then Δ_1 is the union of a nonsingular elliptic curve of bidegree $(2, 2)$ and a nonsingular rational curve of bidegree $(1, 1)$ which intersect each other transversally. If $B(\epsilon)$ is the union of three lines, then Δ_1 is the union of three nonsingular rational curves of bidegree $(1, 1)$, each pair intersect transversally.

Remark 6.2.8. Let

$$V = H^0(\mathbb{P}^1 \times \mathbb{P}^2, \mathcal{O}_{\mathbb{P}^1}(2) \boxtimes \mathcal{O}_{\mathbb{P}^2}(2)).$$

It is a vector space of dimension 18. Let U be an open subset in $|V|$ which consists of sections $t_0^2 a_{11}(t) + 2t_0 t_1 a_{12}(t) + t_1^2 a_{22}(t)$ such that the corresponding determinant

(6.19) defines a nonsingular quartic curve. The group $G = \mathrm{SL}(2)$ acts naturally on U via its action on $\mathbb{P}^1$ and the orbit space X is a cover of degree 63 of the space $|\mathcal{O}_{\mathbb{P}^2}(4)|^{\mathrm{ns}}$ of nonsingular plane quartics. The fibre over C_4 is naturally identified with the set of nonzero 2-torsion divisor classes on C_4. Since X is obviously irreducible and of dimension 14, we obtain that X is an irreducible unramified finite cover of degree 63 of $|\mathcal{O}_{\mathbb{P}^2}(4)|^{\mathrm{ns}}$. Let Z be the closed subset of U of sections such that the linear system of quadrics spanned by $a_{11}(t), a_{12}(t), a_{22}(t)$ contains a double line. Its image in $|\mathcal{O}_{\mathbb{P}^2}(4)|^{\mathrm{ns}}$ is a closed set. Thus a general quartic satisfies the assumption of Lemma 6.2.3 for any ϵ. I do not know whether there exists a nonsingular quartic which does not satisfy these assumptions for any ϵ.

If we further act on X by $G' = \mathrm{SL}(3)$ via its natural action on $\mathbb{P}^2$ we obtain the orbit space birationally isomorphic to the space $\mathcal{R}_3$ of isomorphism classes of genus 3 curves together with a nontrivial divisor class of order 2. This space is known to be rational [140], [244]. This space is also birationally isomorphic to the space of bielliptic curves of genus 4 (see Exercise 6.11).

Let $f = q_1q_3 - q_2^2$ be an expression of f as a symmetric determinant. Consider a quadratic pencil of conics

$$q(\lambda, \mu) := \lambda^2 q_1 + 2\lambda\mu q_2 + \mu^2 q_3 = 0. \tag{6.21}$$

Then the condition that $Q(\lambda, \mu)$ is tangent to C is the vanishing of the discriminant $D = -q_1q_3 + q_2^2$ on C. Since it is identically vanishes on C, we see that every conic from the pencil is tangent to our quartic C. Thus we obtain

Corollary 6.2.6. *A nonsingular plane quartic can be in 63 different ways represented as an evolute of a quadratic pencil of conics.*

Remark 6.2.9. A quadratic pencil of conics (6.21) can be thought as a subvariety X of $\mathbb{P}^1 \times \mathbb{P}^2$ given by a bi-homogeneous equation of bidegree $(2, 2)$. The projection to $\mathbb{P}^1$ is a conic bundle with 6 degenerate fibres corresponding to six pairs of bitangents in the Steiner complex corresponding to the pencil. The projection to $\mathbb{P}^2$ is a double cover branched along the quartic C. Later on we will identify X with the Del Pezzo surface of degree 2 associated to a nonsingular plane quartic.

We refer to [359] for a refined analysis of the theory in the case when the quartic curve is singular.

6.3 Even theta characteristics

6.3.1 Contact cubics

We specialize the results from section 4.1.3. A nonsingular plane quartic has $36 = 2^2(2^3+1)$ even theta characteristic. None of them vanishes since C is not hyperelliptic. For any even theta characteristic ϑ the linear system $|K_C + \vartheta|$ defines a symmetric determinant expression for C. Let $\mathbb{P}^2 = |E|$ and $V^\vee = H^0(C, \mathcal{O}_C(K_C + \vartheta))$. Recall that the symmetric determinantal expression for C corresponding to ϑ defines a linear map

$$E \to S^2V^\vee \subset \mathrm{Hom}(V, V^\vee)$$

which, after projectivization, defines a linear map of projective spaces

$$s : \mathbb{P}^2 \to |\mathcal{O}_{\mathbb{P}^3}(2)|,$$

where $\mathbb{P}^3 = |V| = |K_C + \vartheta|^*$. The image of s is a net of quadrics $\mathcal{N}$ in $\mathbb{P}^3$ whose locus of singular quadrics is equal to C. The set of singular points of quadrics from $\mathcal{N}$ is a sextic model S of C, the image of C under a map given by the linear system $|K_C + \vartheta|$.

The preimage under s of a hyperplane cuts out a divisor $D \in |K_C+\vartheta|$. The divisor $2D \in |3K_C|$ is cut out by a unique cubic. This cubic is called a *contact cubic*. When we vary D in $|K_C + \vartheta|$ we get a 3-dimensional variety of contact cubics isomorphic to $\mathbb{P}^3$. Thus we obtain 36 irreducible families of contact cubics.

Explicitly, ϑ defines a symmetric determinantal representation $C = V(|A|)$, where $A = (l_{ij})$ is a 4×4 symmetric matrix of linear forms in coordinates $[t_0, t_1, t_2]$ in $\mathbb{P}^2$. Contact cubics in the algebraic system corresponding to ϑ are parametrized by coordinates $[u_0, u_1, u_2, u_3]$ in $\mathbb{P}^3 = |K_C + \vartheta|^*$ and are given by the equation

$$\begin{vmatrix} l_{11} & l_{12} & l_{13} & l_{14} & u_0 \\ l_{21} & l_{22} & l_{23} & l_{24} & u_1 \\ l_{31} & l_{32} & l_{33} & l_{34} & u_2 \\ l_{41} & l_{42} & l_{43} & l_{44} & u_3 \\ u_0 & u_1 & u_2 & u_3 & 0 \end{vmatrix} = 0. \tag{6.22}$$

We can also interpret the determinantal identity from Lemma 4.1.7 which we write in the self-explanatory form

$$\begin{vmatrix} A & \mathbf{u} \\ \mathbf{u} & 0 \end{vmatrix} \times \begin{vmatrix} A & \mathbf{v} \\ \mathbf{v} & 0 \end{vmatrix} - \begin{vmatrix} A & \mathbf{u} \\ \mathbf{v} & 0 \end{vmatrix}^2 = |A|U. \tag{6.23}$$

It shows that two contact cubics cut out on C a set of 12 points that lie on a cubic curve.

Remark 6.3.1. Consider the set of nets of quadrics in $\mathbb{P}^3$ as the Grassmannian $G(3, 10)$ of 3-dimensional subspaces in $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))$. Let U be an open subset defining nets $\mathcal{N}$ of conics such that the locus of singular conics defines a nonsingular plane quartic curve $C \subset \mathcal{N}$ together with an even theta characteristic ϑ. The group $\mathrm{SL}(4)$ acts $G(3, 10)$ via its natural action in $\mathbb{P}^3$. The orbit space is birationally isomorphic to the unramified cover of degree 36 of $\mathcal{M}_3$ parameterizing isomorphism classes of pairs (C, ϑ), where C is a nonsingular non-hyperelliptic curve of genus 3 and ϑ is an even theta characteristic. We will show later that this space is birationally isomorphic to $\mathcal{M}_3$.

6.3.2 Cayley octads

The image of s is a net $\mathcal{N}$ (i.e. two-dimensional linear system) of quadrics. Take a basis Q_1, Q_2, Q_3 of $\mathcal{N}$. The base locus of $\mathcal{N}$ is the complete intersection of these quadrics. One expects to get 8 distinct points. Let us see that this is indeed true.

Proposition 6.3.1. *The set of base points of the net of quadrics $\mathcal{N}$ consists of 8 distinct points, no three of which are collinear, no four are coplanar.*

Proof. Suppose three points are on a line ℓ. This includes the case when two points coincide. This implies that ℓ is contained in all quadrics from $\mathcal{N}$. Take a point $x \in \ell$. For any quadric $Q \in \mathcal{N}$, the tangent plane of Q at x contains the line ℓ. Thus the tangent planes form a pencil of planes through ℓ. Since $\mathcal{N}$ is a net, there must be a quadric which is singular at x. Thus each point of ℓ is a singular point of some quadric from $\mathcal{N}$. However, the set of singular points of quadrics from $\mathcal{N}$ is equal to the nonsingular sextic S. This shows that no three points are collinear.

Suppose that 4 points lie in a plane π. Restricting quadrics from $\mathcal{N}$ to π defines a linear system of conics through 4 points no three of which are collinear. It is of dimension 1. Thus, there exists a quadric in $\mathcal{N}$ which contains π. However, since C is nonsingular all quadrics in $\mathcal{N}$ are of corank ≤ 1. $\square$

Definition 6.1. *A set of 8 distinct points in $\mathbb{P}^3$ which is a complete intersection of 3 quadrics is called a* Cayley octad*.*

From now on we assume that a cayley octad satisfies the properties from Theorem 6.3.1.

Let $\varphi : C \to S \subset \mathbb{P}^3$ be the map defined by the linear system $|K_C + \vartheta|$. Its image is a sextic model of C given by the right kernel of the matrix defining the determinantal equation.

Theorem 6.3.2. *Let $q_1, \ldots, q_8$ be the Cayley octad defined by the net of quadrics $\mathcal{N}$. Each line $\overline{q_i, q_j}$ intersects S at two points $\varphi(p_i), \varphi(p_j)$. The line $\langle p_i, p_j \rangle$ is a bitangent of C.*

Proof. Fix a point q on $\ell_{ij} = \langle q_i, q_j \rangle$ different from q_i, q_j. Each quadric from $\mathcal{N}$ vanishing at q has 3 common points with ℓ_{ij}. Hence it contains ℓ_{ij}. Since vanishing at a point is one linear condition on the coefficients of a quadric, we obtain a pencil of quadrics in $\mathcal{N}$ such that ℓ_{ij} is contained in its set of base points. Two quadrics intersect along a curve of degree 4. Thus the base locus of the pencil is a reducible curve of degree 4 which contains a line component. The residual curve is a twisted cubic. Take a nonsingular quadric in the pencil. Then the cubic is a curve of bidegree $(2,1)$ and a line is a curve of bidegree $(0,1)$. Thus they intersect at 2 points x, y (not necessary distinct). Any two nonsingular quadrics from the pencil do not intersect at these points transversally. Hence they have a common tangent plane. For each point, an appropriate linear combination of these quadrics will be singular at this point. The pencil does not have any other singular quadrics. Indeed, a singular point of such a quadric must lie on ℓ_{ij} and hence define a singular point of the base locus. So it must be one of the two singular points x, y. No two quadrics from the same pencil share a singular point since the set C of singular quadrics does not contain a line. This shows that $x, y \in S$ and the pencil of quadrics is equal to the image of a line ℓ in $\mathbb{P}^2$ under the map s. The line ℓ intersects C at two points p_i, p_j such that $\varphi(p_i) = x, \varphi(p_j) = y$. Thus ℓ is a bitangent. $\square$

We can also see all even theta characteristics.

Theorem 6.3.3. *Let $q_1, \ldots, q_8$ be the Cayley octad associated to an even theta characteristic ϑ. Let ϑ_{ij} be the odd theta characteristic corresponding to the lines $\langle q_i, q_j\rangle$. Then any even theta characteristic different from ϑ can be represented by the divisor class*

$$\vartheta_{i,jkl} = \vartheta_{ij} + \vartheta_{ik} + \vartheta_{il} - K_C$$

for some distinct i, j, k, l.

Proof. Suppose that $\vartheta_{i,jkl}$ is an odd theta characteristic ϑ_{mn}. Consider the plane π which contains the points q_i, q_j, q_k. It intersects S at six points corresponding to the theta characteristics $\vartheta_{ij}, \vartheta_{ik}, \vartheta_{jk}$. Since the planes cut out divisors from $|K_C + \vartheta|$, we obtain

$$\vartheta_{ij} + \vartheta_{ik} + \vartheta_{jk} \sim K_C + \vartheta$$

This implies that

$$\vartheta_{jk} + \vartheta_{il} + \vartheta_{mn} \sim K_C + \vartheta.$$

Hence $\langle q_j, q_k\rangle$ and $\langle q_i, q_l\rangle$ lie in a plane π'. The intersection point of the lines $\langle q_j, q_k\rangle$ and $\langle q_i, q_l\rangle$ is a base point of two pencils in $\mathcal{N}$ and hence is a base point of $\mathcal{N}$. However, it does not belong to the Cayley octad. This contradiction proves the assertion. □

Remark 6.3.2. Note that

$$\vartheta_{i,jkl} = \vartheta_{j,ikl} = \vartheta_{k,ijl} = \vartheta_{l,ijk}.$$

Thus $\vartheta_{i,jkl}$ depends only on the choice of a subset of four elements in $\{1, \ldots, 8\}$. Also it is easy to check that the complementary sets define the same theta characteristic. This shows that we get $35 = \binom{8}{4}/2$ different even theta characteristics. Together with $\vartheta = \vartheta_\emptyset$ we obtain 36 even theta characteristics. Observe now that the notations ϑ_{ij} for odd thetas and $\vartheta_{i,jkl}, \vartheta_\emptyset$ agrees with the notation we used for odd even theta characteristics on curves of genus 3. For example, any set $\vartheta_{18}, \ldots, \vartheta_{78}$ defines an Aronhold set. Or, a syzygetic tetrad corresponds to four chords forming a spatial quadrangle, for example $\overline{p_1, p_3}, \overline{p_2, p_4}, \overline{p_2, p_3}, \overline{p_1, p_4}$.

Here is another application of Cayley octads.

Proposition 6.3.4. *There are 10080 azygetic hexads of bitangents of C such that their 12 contact points lie on a cubic.*

Proof. Let ℓ_1, ℓ_2, ℓ_3 be an azygetic triad of bitangents. The corresponding odd theta characteristics add up to $K_C + \vartheta$, where ϑ is an even theta characteristic. Let O be the Cayley octad corresponding to the net of quadrics for which C is the Hessian curve and $S \subset \mathbb{P}^3 = |K_C + \vartheta|^*$ be the corresponding sextic model of C. We know that the restriction map

$$|\mathcal{O}_{\mathbb{P}^3}(2)| \to |\mathcal{O}_S(2)| = |\mathcal{O}_C(3K_C)| = |\mathcal{O}_{\mathbb{P}^2}(3)|$$

is a bijection. We also know that the double planes in $|\mathcal{O}_{\mathbb{P}^3}(2)|$ are mapped to contact cubics corresponding to ϑ. The cubic curve $\ell_1 + \ell_2 + \ell_3$ is one of them. Using the interpretation of bitangents as chords of the Cayley octad given in Theorem 6.3.2, we see that the union of the three chords corresponding to ℓ_1, ℓ_2, ℓ_3 cut out on S six coplanar points.This means that the three chords span a plane in $\mathbb{P}^3$. Obviously, the chords must be of the form $\overline{q_i, q_j}, \overline{q_i, q_k}, \overline{q_j, q_k}$, where $1 \le i < j < k \le 8$. The number of such triples is $\binom{8}{3} = 56$. So, incidentally, we see that the number of points $[u_0, u_1, u_2, u_3]$ such that the determinant (6.22) represents the union of three lines is equal to 56. Fixing such a triple of chords, we can find $\binom{5}{3} = 10$ triples disjoint from the fixed one. The sum of the six corresponding odd theta characteristics is equal to $3K$ and hence the contact points are on a cubic. We can also see it by using the determinantal identity (6.23). Thus any even ϑ contributes $(56 \times 10)/2 = 280$ hexads from the assertion of the proposition. The total number is equal to $36 \cdot 280 = 10080$. □

In Salmon's book [356] one can find possible types of such hexads.

- 280 of type $(12, 23, 31, 45, 56, 64)$;
- 168 of type $(12, 34, 35, 36, 37, 38)$;
- 560 of type $(12, 13, 14, 56, 57, 58)$.

Recall that the three types correspond to three orbits of the permutation group $\mathfrak{S}_8$ on the set of azygetic hexads whose contact points are on a cubic. Note that not any azygetic hexad has this property. For example, a subset of an Aronhold set does not have this property.

For completeness sake, let us give the number of not azygetic hexads whose contact points are on a cubic. Each such is the union of two syzygetic hexads with a unique common bitangent. Using the classification of syzygetic tetrads octads one can find the number of such pairs. It is equal to 5040. Here is the list.

- 840 of type $(12, 23, 13, 14, 45, 15)$;
- 1680 of type $(12, 23, 34, 45, 56, 16)$;
- 2520 of type $(12, 34, 35, 36, 67, 68)$.

6.3.3 Seven points in the plane

Let $p_1, \ldots, p_7$ be seven points in the projective plane. We assume that the points satisfy the following condition:

(*) no three of the points are collinear and no six lie on a conic.

Consider the linear system $\mathcal{N}$ of cubic curves through these points. The conditions on the points imply that each member of $\mathcal{N}$ is an irreducible cubic. A subpencil in $\mathcal{N}$ has two base points outside the base locus of $\mathcal{N}$. The line spanned by these points (or the common tangent if these points coincide) is a point in the dual plane $\check{\mathbb{P}}^2$. This allows us to identify the net $\mathcal{N}$ with the plane, where our seven points lie. This is a special property of Laguerre nets which discussed in Example 7.1.3.

Proposition 6.3.5. *The linear system $\mathcal{N}$ is of dimension 2. The rational map $\mathcal{N} = \mathbb{P}^2 - \to \mathbb{P}^2 = \mathcal{N}^\vee$ is of degree 2. It extends to a regular finite map of degree 2 $\phi : S \to \mathbb{P}^2$, where S is the blow-up of the seven points. The branch curve of ϕ is a nonsingular plane quartic C. The ramification curve W is the proper transform of a curve of degree 6 with double points at each p_i. Conversely, given a nonsingular plane quartic C, the double cover of $\mathbb{P}^2$ ramified over C is a nonsingular surface isomorphic to the blow-up of 7 points $p_1, \ldots, p_7$ in the plane satisfying (*).*

We postpone the proof of this proposition until Chapter 8. The surface S is a Del Pezzo surface of degree 2 .

Let $\sigma : S \to \mathbb{P}^2$ be the blowing up map. The curves $E_i = \sigma^{-1}(p_i)$ are exceptional curves of the first kind, (-1)-curves for short. We will often identify $\mathcal{N}$ with its proper transform in S equal to $|-K_S| = |-\sigma^*(K_{\mathbb{P}^2}) - E_1 - \ldots - E_7|$.

The preimage of a line ℓ in $\mathcal{N}^\vee$ is a nonsingular member of $\mathcal{N}$ if and only if ℓ intersects transversally C. In this case it is a double cover of ℓ branched over $\ell \cap C$. The preimage of a tangent line is a singular member, the singular points lie over the contact points. Thus, the preimage of a general tangent line is an irreducible cubic curve with a singular point at $\sigma(W)$. The preimage of a bitangent is a member of $|-K_S|$ with singular points (they may coincide if the bitangent is a flex bitangent). It is easy to see that its image in the plane is either an irreducible cubic F_i with a double point at p_i or the union of a line $\overline{p_i, p_j}$ and the conic Q_{ij} passing through the point $p_k, k \neq i, j$. In this way we can account for all 28 = 7+21 bitangents. If we denote the bitangents corresponding to F_i by ℓ_{i8} and the bitangents corresponding to $\overline{p_i, p_j} + Q_{ij}$ by ℓ_{ij} we can even accommodate the notation of bitangents by subsets of cardinality 2 of $[1, 8]$. We will see below that this notation agrees with the previous notation. In particular, the bitangents corresponding to the curves F_i's form an Aronhold set.

Note that quartic curve C does not determine the point $p_1, \ldots, p_7$ uniquely. There are many ways to define the blowing morphism $\sigma : S \to \mathbb{P}^2$. However, if we fix an Aronhold set of bitangents there is only one way, up to composition with a projective tranformation, to blow-down seven disjoint (-1)-curves such that the corresponding cubic curves F_i are mapped to the bitangents from the Aronhold set. We will see this later in Chapter 8. Thus a choice of an Aronhold set is equivalent to a choice of a set of seven points $p_1, \ldots, p_7$ defining C.

Consider the universal family

$$\mathcal{X} = \{(s, F) \in S \times \mathcal{N} : s \in F\}.$$

The fibre of the first projection $\pi_1 : \mathcal{X} \to S$ over a point $s \in S$ can be identified, via the second projection, with the pencil $\mathcal{N}(s) \subset \mathcal{N}$ of curves passing through the point s. The second fibration $\pi_2 : \mathcal{X} \to L$ is an elliptic fibration, its fibres isomorphic to the corresponding members of $\mathcal{N}$. It has 7 rational sections $\mathcal{E}_i$ defined by the (-1)-curves E_i's. The projection $\pi_2 : \mathcal{E}_i \to L$ is an isomorphism over $L \setminus \{F_i\}$. The fibre over $\{F_i\}$ is identified, via π_1, with E_i. Thus each section $\mathcal{E}_i$ is isomorphic to the ruled surface $\mathbf{F}_1$. The restriction of the projection π_1 to $\mathcal{E}_i$ is the map $\mathcal{E}_i \to E_i$ defined by the ruling of $\mathbf{F}_1$.

There is another natural rational section $\mathcal{E}_8$ defined as follows. It is easy to see that any g_2^1 on a nonsingular cubic curve F is obtained by projection from a point $p \in F$ to a line. The point p is the intersection point of F with the line spanned by any divisor from the g_2^1. It was called by Sylvester the *coresidual point* of F (see [356], p. 134). Take a general curve $F \in \mathcal{N}$ and restrict $\mathcal{N}$ to F. This defines a g_2^1 on F parametrized by the image of F under the double cover $\phi : S \to \mathcal{N}^\vee$. Let $\mathfrak{c}_F$ be the corresponding coresidual point. If F is an irreducible curve from $\mathcal{N}$ with singular point s not equal to one of the base points, the $\mathfrak{c}_F$ is defined as follows. The nonsingular curves in $\mathcal{N}$ passing through s have a common tangent line ℓ_s at s. The coresidual point $\mathfrak{c}_F$ is equal to the third intersection point of ℓ_s with F. If F is a reducible member equal to the union of the line through two base points and the conic through the remaining base points, then the coresidual point $\mathfrak{c}_F$ is the point on the conic component equal to the intersection of the two lines ℓ_s corresponding to singular points of F. Finally, if $F = F_i$ with singular point $s = p_i$, then we consider the pencil of curves from $\mathcal{N}$ which are tangent to F_i at one of the branches. The coresidual point is equal in to the base pointp_i. The the correspondence $F \mapsto \mathfrak{c}_F$ allows one to identify the original plane $\mathbb{P}^2$ with the plane $\mathcal{N}$. It is easy to see that this identification coincides with the identification defined by the property of a Laguerre net.

Let $\mathcal{E}_8 \subset \mathcal{X}$ be the closure of the set of points $(\sigma^{-1}(\mathfrak{c}_F), F)$. The projection $\pi_2 : \mathcal{E}_8 \to \mathcal{N}$ is the blow-up of the seven points corresponding to the curves F_i. The restriction of the projection π_1 to $\mathcal{E}_8$ is an isomorphism onto S. Thus $\mathcal{E}_8$ defines a rational section of π_2 and a regular section of π_1. The section $\mathcal{E}_8$ intersects the section $\mathcal{E}_i$ along the exceptional curve E_i identified with $\pi_2^{-1}(\{F_i\}) \cap \mathcal{E}_8$. The first seven sections are disjoint.

Proposition 6.3.6. *The linear system $H = |\pi_1^*(\sigma^*\mathcal{O}_{\mathbb{P}^2}(1)) \otimes \mathcal{O}_{\mathcal{X}}(\mathcal{E}_8))|$ has the base locus equal to $\mathcal{E}_8 \cap (\mathcal{E}_1 + \cdots + \mathcal{E}_7)$. Let $\tau : \mathcal{X}' \to \mathcal{X}$ be the blow-up of the base locus. The proper transform of H on $\mathcal{X}'$ defines a birational morphism $\alpha : \mathcal{X}' \to \mathbb{P}^3$. The proper transform of each divisor $\mathcal{E}_i, i = 1, \ldots, 8$ blows down to a point q_i. The exceptional divisor of τ is blown down to the union of lines $\overline{q_i, q_8}$. The rational map $\pi_2 \circ \tau \circ \alpha^{-1} : \mathbb{P}^3 \to \mathcal{N}$ is given by the linear system of quadrics through the points $q_1, \ldots, q_8$.*

It follows from this proposition that the set of points $q_1, \ldots, q_8$ is a Cayley octad. Conversely, let $q_1, \ldots, q_8$ be a Cayley octad O. The net $\mathcal{Q}$ of quadrics through O defines a rational map $\mathbb{P}^3 - \to \mathbb{P}^2 = \mathcal{Q}^\vee$ whose general fibre is a quartic elliptic curve. Projecting from q_8, we obtain a net $\mathcal{N}$ of cubic curves through the points $p_1, \ldots, p_7$ equal to the projections of $q_1, \ldots, q_7$. We can identify the linear systems $\mathcal{N}$ and $\mathcal{Q}$. The linear system $\mathcal{Q}$ parametrizes quartic curves of arithmetic genus 1 through the Cayley octad O. The singular members of $\mathcal{Q}$ and $\mathcal{N}$ are parametrized by the dual curve $C^\vee$ of C. One can match the interpretation of bitangents via singular members of the net of cubic curves $\mathcal{N}$ and the net of quartic space curves $\mathcal{Q}$. The line $\overline{q_i, q_j}$ defines a pencil of quadrics that contains the line. Its base locus is a member of the net $\mathcal{Q}$ and consists of the union of the line and a rational cubic curve intersecting the line at two points. If $(i, j) = (i, 8)$, projecting from this point we get a member of L equal to the singular cubic curve F_i. Otherwise, it is projected to the union of the line $\overline{p_i, p_j}$ and a conic Q_{ij}. This shows that the notation of bitangents via the Cayley octad, or via the Aronhold set, or via the set of seven points in the plane agree.

Remark 6.3.3. We know, via the determinantal representation of a quartic plane curve that a choice of an even theta characteristic defines the projective equivalence class of a Cayley octad from which C is reconstructed as the Hessian curve of the corresponding net of quadrics. On the other hand, a choice of an Aronhold set defines a choice of the projective equivalence class of a set of 7 points in the plane from which C is reconstructed as the branch curve of the double cover defined by the net of cubic curves through the seven points. Proposition 6.3.6 shows that an Aronhold set defines a Cayley octad. However, this Cayley octad comes with a choice of one of its points. Conversely, a Cayley octad together with a choice of its points defines an Aronhold set via the projection from this point. This shows that the moduli space $\mathcal{M}_3^{\mathrm{ev}}$ of curves of genus 3 together with an even theta characteristic admits a rational map of degree 8 onto the moduli space of projective equivalence classes of sets of unordered 7 points in $\mathbb{P}^2$.

6.4 Polar polygons

6.4.1 Clebsch and Lüroth quartics

Since 5 general points in $\mathbb{P}(E^\vee)$ lie on a singular quartic (a double conic), a general quartic does not admit a polar 5-polyhedron (*polar pentagon*) although the count of constants suggests that this is possible. This remarkable fact was first discovered by J. Lüroth in 1868. Suppose a quartic C admits a polar pentagon $\{[l_1],\ldots,[l_5]\}$. Let $Q = V(q)$ be a conic in $\mathbb{P}(E^\vee)$ passing through the points $[l_1],\ldots,[l_5]$. Then $q \in \mathrm{AP}_2(f)$. The space $\mathrm{AP}_2(f) \neq \{0\}$ if and only if $\det \mathrm{Cat}_2(f) = 0$. Thus the set of quartics admitting a polar pentagon is the locus of the catalecticant invariant on the space $\mathbb{P}(S^4E^\vee)$. It is a polynomial of degree 6 in the coefficients of a homogeneous form of degree 4.

Definition 6.2. *A plane quartic admitting a polar pentagon is called a* Clebsch quartic*.*

A Clebsch quartic $C = V(f)$ is called *nondegenerate* if $\dim \mathrm{AP}_2(f) = 1$. Thus the polar pentagon of a nondegenerate Clebsch quartic lies on a unique conic. We call it the *apolar conic*. The apolar conic is reducible if and only if the corresponding operator is the product of two linear operators. This means that the second polar $P_{ab}(f) = 0$ for some points $a, b \in |E|$.

Proposition 6.4.1. *Let $f \in S^4E^\vee$ be such that the second polar $D_{ab}(f) = 0$ for some $a, b \in |E|$. Then, in appropriate coordinate system*

$$f = f_3(t_0,t_1)t_0 + f_4(t_1,t_2), \quad a \neq b,$$

$$f = f_3(t_1,t_2)t_0 + f_4(t_1,t_2), \quad a = b.$$

In particular, $D_{aa}(f) = 0$ if and only if $V(f)$ has a triple point.

Proof. Suppose $a \neq b$. Choose coordinates such that $a = [1,0,0], b = [0,0,1]$ and write

$$f = \sum_{i=0}^{4} f_i(t_1,t_2)t_0^{4-i}.$$

Then $D_{aa}(f) = \frac{\partial^2}{\partial t_0^2} = 0$, $D_{ab}(f) = \frac{\partial^2}{\partial t_2 \partial t_0}(f) = 0$. Now the assertions easily follow. $\square$

We will assume that the apolar conic of a nondegenerate Clebsch quartic is irreducible.

Let $\{[l_1], \ldots, [l_5]\}$ be a polar pentagon of f such that $f = l_1^4 + \cdots + l_5^4$. For any $1 \leq i < j \leq 5$, let $a_{ij} = [l_i] \cap [l_j] \in |E|$. We can identify a_{ij} with a linear operator $\psi_{ij} \in E$ (defined up to a constant factor). Obviously, $D_{\psi_{ij}}(f)$ coincides with the first polar $D_{a_{ij}}(f)$. Applying ψ_{ij} we obtain

$$D_{\psi_{ij}}(f) = D_{\psi_{ij}}(l_1^4 + \cdots + l_5^4) = 4 \sum_{k \neq i,j} \psi_{ij}(l_k) l_k^3.$$

Thus $[l_k], k \neq i, j$, form a polar triangle of $P_{a_{ij}}(f)$. Since the associated conic is irreducible no three points among the $[l_k]$'s are linearly dependent. Thus $P_{a_{ij}}(V(f))$ is a Fermat cubic.

Lemma 6.4.2. *Let $f \in S^4E^\vee$. Assume that $D_{ab}(f) \neq 0$ for any $a, b \in |E|$. Let S be the locus of points $a \in |E|$ such that the first polar of $V(f)$ is isomorphic to a Fermat cubic or belongs to the closure of its orbit. Then S is a plane quartic.*

Proof. Let $I_4 : S^3E^\vee \to \mathbb{C}$ be the Aronhold invariant vanishing on the locus of Fermat cubics (see Remark 3.2.4). It is a polynomial of degree 4 in coefficients of a cubic.

Compose I_4 with the polarization map $E \times S^4E^\vee \to S^3E^\vee$, $(\psi, f) \mapsto D_\psi(f)$. We get a bihomogeneous map of degree $(4, 4)$ $E \times S^4E^\vee \to \mathbb{C}$. It defines a degree 4 homogeneous map

$$S : S^4E^\vee \to S^4E^\vee. \tag{6.24}$$

This map is called the *Clebsch quartic covariant*. It assigns to a quartic form in three variables another quartic form in three variables. By construction, this map does not depend on the choice of coordinates. Thus it is a *covariant* of quartics, i.e. a $\mathrm{GL}(E)$-equivariant map from $S^4E^\vee$ to some $S^dE^\vee$. By definition, the locus of $v \in E$ such that $S(f)(v) = 0$ is the set of vectors $v \in E$ such that $I_4(P_a(f)) = 0$, i.e., $V(D_v(f))$ belongs to the closure of the orbit of a Fermat cubic. $\square$

Example 6.4.1. Assume that the equation of f is given in the form

$$f = at_0^4 + bt_1^4 + ct_2^4 + 6dt_1^2t_2^2 + 6et_0^2t_1^2 + 6ht_0^2t_1^2.$$

Then the explicit formula for the Clebsch covariant gives

$$S(f) = a't_0^4 + b't_1^4 + c't_2^4 + 6d't_1^2t_2^2 + 6e't_0^2t_1^2 + 6h't_0^2t_1^2,$$

where

$$\begin{aligned} a' &= 6e^2h^2, \\ b' &= 6h^2f^2, \\ c' &= 6f^2g^2, \\ d' &= bceh - f(be^2 + ch^2) - ehd^2, \\ e' &= acdh - e(ch^2 + ad^2) - dhe^2, \\ h' &= abde - h(ad^2 + be^2) - deh^2. \end{aligned}$$

For a general f the formula for S is too long.

Note that the Clebsch covariant S defines a rational map

$$S : \mathbb{P}(S^4E^\vee) - \to \mathbb{P}(S^4E^\vee). \tag{6.25}$$

Note that the map is not defined on the closed subset of quartics $V(f)$ such that $V(P_a(f))$ belongs to the closure of the orbit of a Fermat cubic for any $a \in |E|$.

Proposition 6.4.3. *The map S is not defined on $V(f)$ if and only if $V(f)$ is a Clebsch quartic admitting a reducible apolar conic.*

We refer for a proof to [132].

For any quartic curve C satisfying the assumption of the previous proposition, the curve $S(C) := V(S(f))$ will be called the *Clebsch covariant quartic* associated to C. We will show that for a general quartic C the Clebsch quartic $S(C)$ comes with a certain non-effective theta characteristic ϑ such that the pair $(S(C), \vartheta)$ is Scorza general and the Scorza quartic associated to (C, ϑ) is equal to C.

If C is a nondegenerate Clebsch quartic, then, as we explained in above, the vertices of its polar pentagon must belong to the Clebsch covariant quartic $S(C)$. This gives

Proposition 6.4.4. *Let $C = V(f)$ be a nondegenerate Clebsch quartic. Then each polar pentagon of C is inscribed in the quartic curve $V(S(f))$.*

Lemma 6.4.5. *A quartic curve C circumscribing a pentagon defined by 5 lines $[l_i]$ can be written in the form $C = V(g)$, where*

$$g = l_1 \cdots l_5 \sum_{i=1} \frac{a_i}{l_i}$$

for some $a_i \in \mathbb{C}$.

Proof. Consider the linear system of quartics passing through 10 vertices of a pentagon. The expected dimension of this linear system is equal to 4. Suppose it is larger than 4. Since each side of the pentagon contains 4 vertices, requiring that a quartic vanishes at some additional point on the side forces the quartic contain the side. Since we have 5 sides, we will be able to find a quartic containing the union of 5 lines, obviously a contradiction. Now consider the linear system of quartics whose equation can be wriitten as in the assertion of the lemma. The equations have 5 parameters and it is easy to see that the polynomials $l_1 \cdots l_5/l_i, i = 1, \ldots, 5$, are linearly independent. $\square$

Definition 6.3. *A plane quartic circumscribing a pentagon is called a* Lüroth quartic*.*

Thus we see that for any Clebsch quartic C the quartic $S(C)$ is a Lüroth quartic. One can prove that any Lüroth quartic is obtained in this way from a unique Clebsch quartic (see [132]). Since the locus of Clebsch quartics is a hypersurface (of degree 6) in the space of all quartics, the locus of Lüroth quartics is also a hypersurface. Its degree is equal to 54 ([286]) and the number is equal to one of the coefficients of the Donaldson polynomial for the projective plane (see [267]).

Let $C = V(f)$ be a general Clebsch quartic. Consider the map

$$c : \mathrm{VSP}(f;5)^o \to \mathbb{P}(S^2E) \tag{6.26}$$

defined by assigning to $\{\ell_1, \ldots, \ell_5\} \in VSP(f;5)^o$ the unique conic passing through these points in the dual plane. This conic is apolar to C. The fibres of this map are polar pentagons of f inscribed in the apolar conic. We know that the closure of the set of Clebsch quartics is defined by one polynomial in coefficients of quartic, the catalecticant invariant. Thus the varierty of Clebsch quartics is of dimension 13. Consider the map $(E^\vee)^5 \to \mathbb{P}(S^4E^\vee)$ defined by $(l_1, \ldots, l_5) \mapsto V(l_1^4 + \cdots + l_4^4)$. The image of this map is the variety of Clebsch quartics. A general fibre must be of dimension $15 - 13 = 2$. However, scaling the l_i by the same factor, defines the same quartic. Thus the dimesnion of the space of all polar pentagons of a general Clebsch quartic is equal to 1. Over an open subset of the Clebsch locus, the fibres of c are irreducible one-dimensional varietes.

Proposition 6.4.6. *Let $C = V(f)$ be a nondegenerate Clebsch quartic and Q be its apolar conic. Consider any polar pentagon of C as a set of 5 points on Q (the dual of its sides). Then* $\mathrm{VSP}(f;5)^o$ *is an open non-empty subset of a linear pencil on Q of degree 5.*

Proof. Consider the correspondence

$$X = \{(x, \{\ell_1, \ldots, \ell_5\}) \in Q \times \mathrm{VSP}(f;5)^o : x = [l_i] \text{ for some } i = 1, \ldots, 5\}.$$

Let us look at the fibres of the projection to Q. Suppose we have two polar pentagons of f with the same side $[l]$. We can write

$$f - l^4 = l_1^4 + \cdots + l_4^4,$$

$$f - \lambda l^4 = m_1^4 + \cdots + m_4^4.$$

For any $\psi \in S^2E$ such that $\psi([l_i]) = 0, i = 1, \ldots, 4$, we get $D_\psi(f) = 12\psi(l)l^2$. Similarly, for any $\psi' \in S^2E$ such that $\psi'([m_i]) = 0, i = 1, \ldots, 4$, we get $D_{\psi'}(f) = 12\lambda\psi'(l)l^2$. This implies that $\psi(l)\psi' - \psi'(l)\psi = 0$ defines an apolar conic to C. Since C was a general Clebsch quartic, there is only one apolar conic. The set of $V(\psi)$'s is a pencil with base points $V(l_i)$, the set of $V(\psi')$ is a pencil with base points $V(l_i)$. This gives a contradiction unless the two pencils coincide. But then their base points coincide and the two pentagons are equal. This shows that the projection to Q is a one-to-one map. In particular, X is an irreducible curve.

Now it is easy to finish the proof. The set of degree 5 positive divisors on $Q \cong \mathbb{P}^1$ is the projective space $|\mathcal{O}_{\mathbb{P}^1}(5)|$. The closure $\mathcal{P}$ of our curve of polar pentagons lies in this space. All divisors containing one fixed point in their support form a hyperplane. Thus the polar pentagons containing one common side $[l]$ correspond to a hyperplane section of $\mathcal{P}$. Since we know that there is only one such pentagon and we take $[l]$ in an open Zariski subset of Q, we see that the curve is of degree 1, i.e. a line. So our curve is contained in one-dimensional linear system of divisors of degree 5. $\square$

6.4.2 The Scorza quartic

Here we assume that $C = V(f)$ is not projectively equivalent to the quartics from Proposition 6.4.1. Let us study the map (6.25) in more detail. Let $S = S(C)$. For any $a \in S$, the first polar $P_a(f)$ defines a Fermat curve (or its degeneration). As we saw in the proof of Lemma 3.2.7, these curves are characterized by the property that there exists a point b such that the first polar is a double line. This defines a correspondence

$$R_C = \{(a,b) \in S \times S : P_b(P_a(C)) \text{ is a double line}\}.$$

Proposition 6.4.7. *Let $C = V(f)$ be a general plane quartic. Then $S = S(C)$ is a nonsingular curve and there exists a non-effective theta characteristic ϑ with $d(\vartheta) = 1$ on S such that R_C coincides with the Scorza correspondence R_ϑ on S.*

Proof. Take linear forms l_i in general position and consider the Clebsch quartic $C = V(\sum l_i^4)$. Then $S(C)$ is the Lüroth quartic given by the equation from Lemma 6.4.5. one can directly check that it is nonsingular. Thus the image of the Scorza map contains a nonsingular curve, and hence for general C the curve $S(C)$ is nonsingular. The variety of nonsingular Lüroth quartics is an open subset in a hypersurface in the space of quartics. The image of this open subset in the moduli space $\mathcal{M}_3$ of curves of genus 3 is of codimension 1. Taking C general enough we may assume that $S(C)$ does not admit a non-constant map to curves of genus 1 or 2. The moduli space of curves that admit such maps is of higher codimension in $\mathcal{M}_3$. Thus we may assume that the image of the Scorza map contains an open subset U of nonsingular curves that do not admit non-constant maps to curves of genus 1 or 2.

Assume that $S(C) \in U$. Applying Proposition 5.5.6, it suffices to check that R_C is symmetric, of type $(3,3)$ and has valence -1. The symmetry of R_C is obvious. We have a map from S to the closure $\mathcal{F}$ of the Fermat locus defined by $a \mapsto V(P_a(f))$. For any curve in $\mathcal{F}$, except the union of three lines, the set of points such that the first polar is a double line is finite. It is equal to the set of double points of the Hessian curve and consists of 3 points for Fermat curves, one point for cuspidal cubics and 2 points for the unions of a conic and a line. If C is general enough the image of S in $\mathcal{F}$ does not intersect the locus of the unions of three lines (which is of codimension 2). Thus we see that each projection from R_C to S is a finite map of degree 3. Thus R_C is of type $(3,3)$.

For any general point $x \in S$, the first polar $P_x(C)$ is projectively equivalent to a Fermat cubic. The divisor $R_C(x)$ consists of the three vertices of its unique polar triangle. For any $y \in R_C(x)$, the side $H = [l]$ opposite to y is defined by $P_y(P_x(C)) = P_x(P_y(C)) = V(l^2)$. It is a common side of the polar triangles of $P_x(C)$ and $P_y(C)$. We have $H \cap S = y_1 + y_2 + x_1 + x_2$, where $R_C(x) = \{y, y_1, y_2\}$ and $R_C(y) = \{x, x_1, x_2\}$. This gives

$$y_1 + y_2 + x_1 + x_2 = (R_C(x) - x) + (R_C(y) - y) \in |K_S|.$$

Consider a map $\alpha : S \to \mathrm{Pic}^2(S)$ given by $x \to [R(x) - x]$. Assume α is not constant. If we replace in the previous formula y with y_1 or y_2, we obtain that $\alpha(y) = \alpha(y_1) = \alpha(y_2) = K_S - \alpha(x)$. Thus $\alpha : S \to \alpha(S)$ is a map of degree ≥ 3. It defines a finite map of degree ≥ 3 from S to the normalization W of $\alpha(S)$. Since a rational curve does not admit non-constant maps to an abelian variety, we obtain that W is of positive genus. By Hurwitz formula, the genus of W is less or equal than 2. By assumption, S does not admit a non-constant map to W. Hence R_C has valence $v = -1$. Using the Cayley-Brill formula, we obtain that R_C has no united points. In particular, $P_{aa}(C)$ is never a double line. Thus R_C satisfies all the assumption of Proposition 5.5.6.

It remains to verify that $d(\vartheta) = 1$. We may assume that C is nondegenerate in the sense of section 1.4.1. This means that the polarity map $\psi \mapsto P_\psi(C)$ is bijective. It follows from the definition of the correspondence $R_C = R_\vartheta$ that the curve $\Gamma(\vartheta)$ is the locus of lines $[l]$ such that $P_{ab}(C)$ is the double line $V(l^2)$ for some $a, b \in S(C)$. This implies that the map $R_\vartheta \to \Gamma(\vartheta)$ is of degree 2, hence $d(\vartheta) = 1$.

□

Example 6.4.2. Let $C = V(f)$ be a nondegenerate Clebsch quartic and $S = S(C)$ be the Lüroth quartic. It follows from Proposition 6.4.4 that each polar pentagon is inscribed in S. If we take two vertices x, y of a pentagon, then $P_{x,y}(C)$ is a double line representing one of the sides of the pentagon. This means that the apolar conic of C is the curve $\Gamma(\vartheta)$, hence $d(\vartheta) = 3$. The theta characteristic ϑ on a Lüroth quartic obtained in this way is called the *pentagonal theta characteristic*. We do not know any other example of a theta characteristic with $d(\vartheta) \neq 1$.

Recall from Proposition 6.4.6 that the polar pentagons of C are parametrized by $\mathbb{P}^1$. Two pentagons cannot have a common vertex x since the three sides not containing x are equal to the irreducible components of the Hessian of $P_x(C)$ and other two sides are reconstructed from the vertices of the triangle formed by the Hessian. Assigning to $x \in S$ the unique polar pentagon with vertex x we obtain a regular map $\varphi : S \to \mathbb{P}^1$ of degree 10.

Proposition 6.4.8. *Let C be a general plane quartic such that the associated pair $(S(C), \vartheta)$ is Scorza general. Then the Scorza quartic associated to the pair $(S(C), \vartheta)$ is equal to C.*

Proof. Let C' be the Scorza quartic associated to $(S(C), \vartheta)$. It follows immediately from (6.15) in the proof of Theorem 5.5.10 that for any point $(x, y) \in R_\vartheta$ the second polar $P_{x,y}(C')$ is a double line. This shows that $P_x(C')$ is a Fermat cubic, and hence $S(C) = S(C')$. Let $\mathcal{Q}^{\mathrm{ev}}$ be the variety parameterizing Scorza general pairs (C, ϑ). Assigning to a pair the Scorza quartic curve, we define a map $\mathcal{Q}^{\mathrm{ev}} \to |\mathcal{O}_{\mathbb{P}^2}(4)|$. The Scorza map S defines a rational map $S : |\mathcal{O}_{\mathbb{P}^2}(4)| \to \mathcal{TC}_4^{\mathrm{ev}}$. By Proposition 5.2.2 the variety $\mathcal{TC}_4^{\mathrm{ev}}$ is irreducible. As we have just showed this map admits a rational section. Since both varieties are irreducible varieties of the same dimension, the section is dominant and injective, hence birational. This shows that the Scorza map is injective on an open Zariski subset and the assertion is proved. □

Passing to the rational quotients by $\mathrm{PGL}(3)$, we obtain

Corollary 6.4.9. *Let $\mathcal{M}_3^{\mathrm{ev}}$ be the moduli space of curves of genus g together with an even theta characteristic. The birational map $S : |\mathcal{O}_{\mathbb{P}^2}(4)| \to \mathcal{Q}^{ev}$ has the inverse defined by assigning to a pair (C, ϑ) the Scorza quartic. It induces a birational isomorphism*

$$\mathcal{M}_3 \cong \mathcal{M}_3^{\mathrm{ev}}.$$

The composition of this map with the forgetting map $\mathcal{M}_3^{\mathrm{ev}} \to \mathcal{M}_3$ is a rational self-map of $\mathcal{M}_3$ of degree 36.

Remark 6.4.1. The Corollary generalizes to genus 3 the fact that the map from the space of plane cubics $|\mathcal{O}_{\mathbb{P}^2}(3)|$ to itself defined by the Hessian is a birational map to the cover $|\mathcal{O}_{\mathbb{P}^2}(3)|^{\mathrm{ev}}$, formed by pairs (X,ϵ), where ϵ is a non-trivial 2-torsion point (an even characteristic in this case). Note that the Hessian covariant is defined similarly to the Clebsch invariant. We compose the polarization map $V \times S^3E^\vee \to S^2E^\vee$ with the discriminant invariant $S^2E^\vee \to \mathbb{C}$.

6.4.3 Polar hexagons

A general quartic admits a polar 6-polyhedron (*polar hexagon*). It follows from Proposition 1.3.12 that the variety $\mathrm{VSP}(f;6)$ is a smooth irreducible 3-fold.

Let us see how to construct polar hexagons of f explicitly. Let

$$f = l_1^4 + \cdots + l_6^4,$$

where $\ell_i = V(l_i)$. We know that each pair $l_i, l_j, i \neq j$, is conjugate with respect to f, i.e.,

$$\Omega_{\check{f}}(l_i^2, l_j^2) = \check{f}(l_i^2 l_j^2) = 0.$$

Let ψ_i and $\psi_j \in S^2E$ be the anti-polars of l_i and l_j with respect to f, i.e.

$$D_{\psi_i}(f) = l_i^2, \quad D_{\psi_j}(f) = l_j^2.$$

It follows from (1.55) that

$$\psi_i([l_j]) = \psi_j([l_i]) = 0.$$

Assume that ψ_i is irreducible. Then the map

$$S^2E \times S^2E \to S^4E, \quad (\alpha,\beta) \mapsto \alpha\psi_i + \beta\psi_j$$

has one-dimensional kernel spanned by $(\beta,-\alpha)$. This easily implies that the dimension of the linear space L of quartic forms $\alpha\psi_i + \beta\psi_j, \alpha,\beta \in S^2E$, is equal to 9. Thus L coincides with $I_Z(4)$. Note that any form from L vanishes on l_i, l_j and common zeros of α and β. This shows that $Z' = \{\ell_i,\ell_j\} \cup V(\alpha) \cap V(\beta)$ is a polar hexagon of f. By Proposition 1.4.7 it must coincide with Z. This shows that the points $\ell_k, k \neq i,j$ are reconstructed from the points ℓ_i,ℓ_j. It also suggests the following construction of polar hexagons of f.

Start with any $[l] \in \mathbb{P}(E^\vee)$ such that its anti-polar ψ is irreducible and does not vanish at l. This is an open condition on $[l]$. Note that the latter condition means that $[l]$ does not belong to the quartic $V(\check{f}) \subset \mathbb{P}(E^\vee)$. Let $[l'] \in V(\psi)$ and let ψ' be the anti-polar of l'. For any $\alpha,\alpha' \in S^2E$ with $\alpha([l]) = \alpha'([l']) = 0$, we have

$$D_{\alpha\psi+\alpha'\psi'}(f) = D_\alpha(D_\psi(f)) + D_{\alpha'}(D_{\psi'}(f)) = D_\alpha(l^2) + D_{\alpha'}(l'^2) = 0.$$

This shows that the linear space L of quartic forms $\alpha\psi + \alpha'\psi'$ is contained in $\mathrm{AP}_4(f)$. As before we compute its dimension to find that it is equal to 9. Thus L coincide with

$I_Z(4)$, where $Z = \{[l], [l']\} \cup (V(\alpha) \cap V(\alpha'))$. By Proposition 1.3.4, Z is a generalized polar hexagon of f (an ordinary one if $V(\alpha)$ intersects $V(\alpha')$ transversally). Note that this confirms the dimension of $\mathrm{VSP}(f;6)$. We can choose $[l]$ in ∞^2 ways, and then choose $[l']$ in ∞^1 ways.

Remark 6.4.2. Consider the variety

$$\widetilde{\mathrm{VSP}}(F;6) = \{([l], Z) \in \mathbb{P}(E^\vee) \times \mathrm{VSP}(f;6) : \{[l]\} \subset Z\}.$$

The projection to the second factor is a degree 6 map. The general fibre over a point $[l]$ is isomorphic to the anti-polar conic $V(\psi)$ of $[l]$.

6.4.4 A Fano model of VSP$(f;6)$

Recall that each $Z \in \mathrm{VSP}(f;6)$ defines a subspace $I_Z(3) \subset \mathrm{AP}_3(f)$. Its dual space $I_Z(3)^\perp \subset W = S^3E^\vee/\mathrm{ap}_f^1(E)$ is an isotropic subspace with respect to Mukai's 2-forms.

Lemma 6.4.10. *Let f be a nondegenerate quartic form and $Z \in \mathrm{VSP}(f;6)$. Then*

$$\dim I_Z(3) = 4.$$

Proof. Counting constants based on the exact sequence (1.45) shows that $\dim I_Z(3) \geq 10 - 6 = 4$. Assume $\dim I_Z(3) > 4$. Let Z_1 be a closed subscheme of Z of length 5. Again counting constant we get $I_{Z_1}(2) \neq \{0\}$. Let C be a conic from the linear system $|I_{Z_1}(2)|$. Obviously, $C \notin I_Z(2)$ since otherwise $\mathrm{AP}_2(f) \neq \{0\}$ contradicting the nondegeneracy assumption on f. Choose a 0-dimensional scheme Z_0 of length 2 such that each irreducible component of C contains a subscheme of $Z' = Z_0 \cup Z_1$ of length ≥ 4. It is always possible. Now, counting constants, we see that $\dim I_{Z'}(3) > 4-2 = 2$. By Bézout's Theorem, all cubics K from $|I_{Z'}(3)|$ are of the form $C + \ell$, where ℓ is a line. Thus the residual lines form a linear system of lines of dimension ≥ 2, hence each line is realized as a component of some cubic K. However, $|I_{Z'}(3)| \subset |I_Z(3)$ and therefore all lines pass through the point in $Z \setminus Z_1$. This is obviously impossible. $\square$

Applying the previous Lemma we obtain a well-defined map

$$\mu : \mathrm{VSP}(f;6) \to G(3, W) \cong G(3,7), \quad Z \mapsto I_Z(3)^\perp.$$

By Proposition 1.4.7, this map is injective. Its image is contained in the locus of subspaces which are isotropic with respect to Mukai's forms.

Theorem 6.4.11. *(S. Mukai) Let $f \in S^4E^\vee$ be a general quartic form in 3 variables. Then the map*

$$\mu : \mathrm{VSP}(f;6) \to G(3,7)$$

is an isomorphism onto a smooth subvariety X equal to the locus of common zeros of a 3-dimensional space of sections of the vector bundle $\bigwedge^2 \mathcal{S}$, where $\mathcal{S}$ is the tautological vector bundle over the Grassmannian. The canonical class of X is equal to $-H$, where H is a hyperplane section of X in the Plücker embedding.

Proof. We refer for the proof to the original paper of Mukai [290], or to [137], where some details of Mukai's proof are provided. □

Recall that a *Fano variety* of dimension n is a smooth projective variety X with ample $-K_X$. If $\mathrm{Pic}(X) \cong \mathbb{Z}$ and $-K_X = mH$, where H is an ample generator of the Picard group, then X is said to be of *index* m. The *degree* of X is the self-intersection number H^n . The number $g = \frac{1}{2}H^n + 1$ is called the *genus*.

Remark 6.4.3. The variety X_2 was omitted in the original classification of Fano varieties with the Picard number 1 due to Gino Fano. It was discovered by V. Iskovskikh. It has the same Betti numbers as the $\mathbb{P}^3$. It was proven by Mukai that every such variety arises as a smooth projective model of $W(f;6)$ for a unique quartic for $V(f)$.

Remark 6.4.4. Another approach to Mukai's description of $\mathrm{VSP}(f;6)$ for a general plane quartic $V(f)$ is due to K. Ranestad and F.-O. Schreyer [325]. It allows them also to extend Theorem 6.4.11 to other 2 cases where $n = 2$ and $\mathrm{wrk}(f) = \binom{2+k}{k}$ $(k = 3, 4)$.

$$\mathrm{VSP}(f;10) \subset G(4,9) \text{ is a K3 surface of degree 38 in } \mathbb{P}^{20},\ k = 3,$$

$$\mathrm{VSP}(f;15)^o \subset G(5,11) \text{ is a set of 16 points},\ k = 4.$$

Although these descriptions were certainly known to Mukai, he did not provide the details of his proofs.

Remark 6.4.5. We refer to [278] for the beautiful geometry of the variety $\mathrm{VSP}(f;6)$, where $V(f)$ is the Klein quartic.

6.5 Automorphisms of plane quartic curves

6.5.1 Automorphisms of finite order

Since an automorphism of a nonsingular plane quartic curve C leaves K_C invariant, it is defined by a projective transformation.

Lemma 6.5.1. *Let σ be an automorphism of order $n > 1$ of a nonsingular plane quartic $C = V(f)$. Then one can choose coordinates in such a way that a generator of the cyclic group (σ) is represented by a diagonal matrix* $\mathrm{diag}[1, \zeta_n^a, \zeta_n^b]$, *where ζ_n is a primitive n-th root of unity, and f is given in the following list.*

(i) $(n = 2), (a, b) = (0, 1)$,

$$t_2^4 + t_2^2 g_2(t_0, t_1) + g_4(t_0, t_1);$$

(ii) $(n = 3), (a, b) = (0, 1)$,

$$t_2^3 g_1(t_0, t_1) + g_4(t_0, t_1);$$

(iii) $(n = 3), (a, b) = (1, 2)$,

$$f = t_0^4 + \alpha t_0^2 t_1 t_2 + t_0 t_1^3 + t_0 t_2^3 + \beta t_1^2 t_2^2;$$

(iv) $(n=4),(a,b)=(0,1)$,
$$t_2^4+g_4(t_0,t_1);$$

(v) $(n=4),(a,b)=(1,2)$,
$$t_0^4+t_1^4+t_2^4+\alpha t_0^2t_2^2+\beta t_0t_1^2t_2;$$

(vi) $(n=6),(a,b)=(3,2)$,
$$t_0^4+t_1^4+\alpha t_0^2t_1^2+t_0t_2^3;$$

(vii) $(n=7),(a,b)=(3,1)$,
$$t_0^3t_1+t_1^3t_2+t_0t_2^3;$$

(viii) $(n=8),(a,b)=(3,7)$,
$$t_0^4+t_1^3t_2+t_1t_2^3;$$

(ix) $(n=9),(a,b)=(3,2)$,
$$t_0^4+t_0t_1^3+t_2^3t_1;$$

(x) $(n=12),(a,b)=(3,4)$,
$$f=t_0^4+t_1^4+t_0t_2^3.$$

Here the subscripts in polynomials g_i indicate their degree.

Proof. Let us first choose coordinates such that σ acts by the formula

$$\sigma:[x_0,x_1,x_2]\mapsto[x_0,\zeta_n^ax_1,\zeta_n^bx_2].$$

We will often use that f is of degree ≥ 3 in each variable. This follows from the assumption that f is nonsingular.

Case 1: $ab=0$, say $a=0$.
Write f as a polynomial in t_2.

$$f=\alpha t_2^4+t_2^3g_1(t_0,t_1)+t_2^2g_2(t_0,t_1)+t_2g_3(t_0,t_1)+g_4(t_0,t_1). \tag{6.27}$$

If $\alpha\neq 0$, we must have $4b=0\mod n$. This implies that $n=2$ or 4. In the first case $g_1=g_3=0$, and we get case (i). If $n=4$, we must have $g_1=g_2=g_3=0$, and we get case (iv).

If $\alpha=0$, then $3b=0\mod n$. This implies that $n=3$ and $g_2=g_3=0$. This gives case (ii).

Case 2: $ab\neq 0$. Note that the case when $a=b\neq 0$ is reduced to Case 1 by scaling the matrix of the transformation and permuting the variables. In particular, $n>2$. Let $p_1=[1,0,0],p_2=[0,1,0],p_3=[0,0,1]$ be the reference points.

Case 2a: All reference points lie on C.

This implies that the degree of f in each variable is equal to 3. We can write f in the form

$$f = t_0^3 a_1(t_1,t_2) + t_1^3 b_1(t_0,t_2) + t_2^3 c_1(t_0,t_1)$$
$$+ t_0^2 a_2(t_1,t_2) + t_1^2 b_2(t_0,t_2) + t_2^2 c_2(t_0,t_1),$$

where a_i, b_i, c_i are homogeneous forms of degree i. Since f is invariant, it is clear that any t_i cannot enter in two different coefficients a_1, b_1, c_1. Without loss of generality, we may assume that

$$f = t_0^3 t_1 + t_1^3 t_2 + t_2^3 t_0 + t_0^2 a_2(t_2,t_3) + t_1^2 b_2(t_0,t_2) + t_2^2 c_2(t_0,t_1).$$

Now we have $a = 3a + b = 3b \mod n$. This easily implies $n = 7$ and we can take a generator of (g) such that $(a,b) = (3,1)$. By checking the eigenvalues of other monomials, we verify that no other monomials enters in f. This is case (vii).

Case 2b: Only two reference points lie on the curve.

By normalizing the matrix and permuting the coordinates we may assume that $p_1 = [1,0,0]$ does not lie on C. Then we can write

$$f = t_0^4 + t_0^2 g_2(t_1,t_2) + t_0 g_3(t_1,t_2) + g_4(t_1,t_2),$$

where t_1^4, t_2^4 do not enter in g_4.

Without loss of generality, we may assume that $t_1^3 t_2$ enters in g_4. This gives $3a + b = 0 \mod n$. Suppose $t_1 t_2^3$ enters in g_4. Then $a + 3b = 0 \mod n$. This gives $n = 8$, and we may take a generator of $\langle\sigma\rangle$ corresponding to $(a,b) = (3,7)$. This is case (viii). If $t_1 t_2^3$ does not enter in g_4, then t_2^3 enters in g_3. This gives $3b = 0 \mod n$. Together with $3a + b = 0 \mod n$ this gives $n = 3$ and we take g with $(a,b) = (1,2)$ or $n = 9$ and $(a,b) = (3,2)$. These are cases (iii) and (ix).

Case 2c: Only one reference point lies on the curve.

By normalizing the matrix and permuting the coordinates we may assume that $p_1 = [1,0,0], p_2 = [0,1,0]$ do not lie on C. Then we can write

$$f = t_0^4 + t_1^4 + t_0^2 g_2(t_1,t_2) + t_0 g_3(t_1,t_2) + g_4(t_1,t_2),$$

where t_1^4, t_2^4 do not enter in g_4. This immediately gives $4a = 0 \mod n$. Suppose t_2^3 enters in g_3. Then $3b = 0 \mod n$, hence $n = 6$ or $n = 12$. It is easy to see that

$$f = t_0^3 + t_1^3 + \alpha t_0^2 t_1^2 + t_0 t_2^3.$$

If $n = 6$, then $(a,b) = (3,2)$ and α may be different from 0. This is case (vi). If $n = 12$, then $(a,b) = (3,4)$ and $\alpha = 0$. This is case (x).

Case 2d: None of the reference point lies on the curve.

In this case we may assume that

$$f = t_0^4 + t_1^4 + t_2^4 + t_0^2 g_2(t_1,t_2) + t_0 g_3(t_1,t_2) + \alpha t_1^3 t_2 + \beta t_1 t_2^3.$$

Obviously, $4a = 4b = 0 \mod n$. If $n = 2$, we are in case (i). If $n = 4$ and $(a,b) = (1,1)$, multiplying all coordinates by ζ_4^3, we are reduced to Case 1. If $n = 4$ and $(a,b) = (1,2)$, then we get case (v). The case $(1,3)$ is reduced to the case $(1,2)$ if we multiply the coordinates by ζ_4. $\square$

6.5.2 Automorphism groups

Recall some standard terminology from the theory of linear groups. Let G be a subgroup of the general linear group $\mathrm{GL}(V)$ of a finite-dimensional complex vector space. The group G is called *intransitive* if the representation of G in $\mathrm{GL}(V)$ is reducible. Otherwise, it is called *transitive*. The group G is called *imprimitive* if G contains a proper intransitive normal subgroup G'. In this case V decomposes into a direct sum of G'-invariant proper subspaces, and elements from G permute them. Finally, G is *primitive* if V is an irreducible representation.

We employ the notation from [90]: a cyclic group of order n is denoted by n, the semi-direct product $A \rtimes B$ is denoted by $A : B$, a central extension of a group A with kernel B is denoted by $B.A$.

Theorem 6.5.2. *The following is the list of all possible groups of automorphisms of a nonsingular plane quartic.*

Type	Order	Structure	Equation	Parameters
I	168	$L_2(7)$	$t_0^3t_1 + t_1^3t_2 + t_2^3t_0$	
II	96	$4^2 : \mathfrak{S}_3$	$t_0^4 + t_1^4 + t_2^4$	
III	48	$4.\mathfrak{A}_4$	$t_2^4 + t_1^4 \pm 2\sqrt{-3}t_0^2t_1^2 + t_0^4$	
IV	24	$\mathfrak{S}_4$	$t_0^4 + t_1^4 + t_2^4 + a(t_0^2t_1^2 + t_0^2t_2^2 + t_1^2t_2^2)$	$a \neq \frac{-1\pm\sqrt{-7}}{2}$
V	16	4.2^2	$t_2^4 + t_1^4 + at_0^2t_1^2 + t_0^4$	$a \neq 0, \pm 2\sqrt{-3}, \pm 6$
VI	9	9	$t_0^4 + t_0t_1^3 + t_1t_2^3$	
VII	8	D_8	$t_2^4 + t_0^4 + t_1^4 + at_0^2t_1^2 + bt_2^2t_0t_1$	$a, b \neq 0$
VIII	6	6	$t_0^4 + at_0^2t_1^2 + t_1^4 + t_1t_2^3$	$a \neq 0$
IX	6	$\mathfrak{S}_3$	$t_0^4 + t_0(t_1^3 + t_2^3) + at_0^2t_1t_2 + bt_1^2t_2^2$	$b \neq 0$
X	4	2^2	$t_2^4 + t_0^4 + t_1^4 + at_2^2t_0^2 + bt_0^2t_1^2 + ct_0^2t_1^2$	$(a-b)(b-c)(a-c) \neq 0$
XI	3	3	$t_2^3g_1(t_0, t_1) + g_4(t_0, t_1)$	
XII	2	2	$t_2^4 + t_2^2g_2(t_0, t_1) + t_0^4 + at_0^2t_1^2 + t_1^4$	

Table 6.1: Automorphisms of plane quartics

Proof. Case 1. Let G be an intransitive group realized as a group of automorphisms of a nonsingular plane quartic. Since in our case $\dim V = 3$, V must be the direct sum of one-dimensional subspaces V_i, or a one-dimensional subspace V_1 and a 2-dimensional subspace V_2.

Case 1a: $V = V_1 \oplus V_2 \oplus V_3$.

Choose coordinates (t_0, t_1, t_2) such that V_1 is spanned by $(1, 0, 0)$ and so on. Let $\sigma \in G$ be an element of order n. Assume $n > 4$, i.e. $n = 6, 7, 8, 9$ or 12. It is clear that two elements of different orders > 4 cannot belong to G since otherwise G contains an element of order > 12. If $n = 8$, the equation of type (viii) from Lemma 6.5.1 can be transformed by a linear change of variables t_1, t_2 to the equation of a surface of type II. If $n = 12$, then a linear change of variables t_0, t_2 transforms the equation to one of type II (use that $t_0^4 + t_0t_2^3$ can be transformed to the form $u_0^4 + u_2^4$). If $n = 6, 7, 9$, we get that G is a cyclic group for a general curve with equation (vi), (vii), (ix) from Lemma

6.5.1. This gives the rows of types VIII, I, and VI, respectively. Assume G contains an element σ' of order $m \leq 4$. Again, if m does not divide n, we get an element of order > 12 unless $m = 4, n = 6$. It is easy to check that in this case G is cyclic of order 12. If m divides n, we easily check that G is cyclic of order n.

Assume $n = 4$. If σ has two equal eigenvalues, then the equation can be reduced to type (iv) from Lemma 6.5.1. One can show that the binary quartic g_4 can be reduced by a linear transformation to the form $t_0^4 + t_1^4 + at_0^2t_1^2$. For general a, the group of automorphisms of the curve is generated by the transformations

$$\sigma = \text{diag}[i, i, 1],\ \sigma' = \text{diag}[1, -1, 1],\ \tau : [t_0, t_1, t_2] \mapsto [t_1, t_0, t_2].$$

The element σ generates the center and the quotient by the center is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. This gives the group of Type V from the table. If σ has distinct eigenvalues, then the equation can be reduced to the form (v) from the Lemma. It has an additional automorphism of order 4 equal to τ from above. The group is isomorphic to the dihedral group D_8.

Assume $n = 3$. If G contains an element of order $\neq 3$, we are in one of the previous cases. If σ has two equal eigenvalues, then the equation of C can be reduced to type (ii) from Lemma 6.5.1, where we may assume $g_1 = t_0$. It is clear that $G = \langle \sigma \rangle$. This gives Type XI. If g has distinct eigenvalues, we get equation of type (iii). It has additional symmetry defined by switching the variables t_1, t_2. This gives the permutation group $\mathfrak{S}_3$ of Type IX.

Finally if $n = 2$ we get Types XII if equation (i) from Lemma 6.5.1 has no additional symmetry of order 2. If we have additional symmetry of order 2, we may assume that it is given by $\text{diag}[1, -1, -1]$. This implies that g_2 does not contain the monomial t_0t_1 and the coefficient g_4 does not contain the monomials $t_0^3t_1, t_0t_1^3$. But then we get an additional symmetry defined by switching the variables t_0 and t_1. After a linear transformation of variables, this case is reduced to Type X.

Case 1b: $V = V_1 \oplus V_2, \dim V_2 = 2$, where V_2 is an irreducible representation of G. In particular, the image of G in $\text{GL}(V_2)$ is not abelian.

Choose coordinates such that $(1, 0, 0) \in V_1$ and V_2 is spanned by $(0, 1, 0)$ and $(0, 0, 1)$. We have a natural homomorphism

$$\rho : G \to \text{GL}(V_1) \times \text{GL}(V_2) \cong \mathbb{C}^\vee \times \text{GL}(2).$$

Since we are interested in projective representations, we may assume that the projection of $\rho(G)$ to $\text{GL}(V_1)$ is trivial and identify G with a subgroup G' of $\text{GL}(V_2)$.

Write

$$f = \alpha t_0^4 + t_0^3 g_1(t_1, t_2) + t_0^2 g_2(t_1, t_2) + t_0 g_3(t_1, t_2) + g_4(t_1, t_2).$$

Since V_2 is irreducible, $g_1 = 0$. Since $V(f)$ is nonsingular, this implies that $\alpha \neq 0$.

Assume $g_2 \neq 0$. Then g_2 must be G'-invariant. Since G' is not abelian, this easily implies that, after a linear change of variables, we may assume that $g_2 = at_1t_2$ and G' is generated by the transformations

$$\sigma_1 : [t_1, t_2] \mapsto [\zeta_n t_1, \zeta_n^{-1} t_2],\ \sigma_2 : [t_0, t_1] \mapsto [t_2, t_1], \quad \zeta_n = e^{2\pi i/n}. \tag{6.28}$$

They form the group isomorphic to the dihedral group D_{2n} of order $2n$. Since $g_3 \neq 0$, this group acts on the locus $V(g_3)$ of zeros of g_3 in $\mathbb{P}(V_2)$. Since G' is non-abelian, we obtain that $n = 3$ and $G' \cong D_6 \cong \mathfrak{S}_3$. It is easy to see that $g_3 = a(t_1^3 + t_2^3)$. Finally, if $g_4 \neq 0$, its set of zeros must consist of two points taken with multiplicity 2. We must get $g_4 = bt_0^2 t_1^2$. This leads to the row Type IX in the Table 1. If $g_3 = 0$, we get $g_4 = at_1^2 t_2^2$ which shows that the curve is singular in this case.

Assume $g_1 = g_2 = 0$ but $g_3 \neq 0$. Since $V(g_3)$ is invariant and V_2 is irreducible, $V(g_3)$ consists of three points permuted by $G' \cong \mathfrak{S}_3$. We can choose coordinates to assume that $g_3 = t_1^3 + t_2^3$, where G' acts as in (6.28) with $n = 3$. In order that $V(g_4)$ be G'-invariant we must have $g_4 = at_1^2 t_2^2$. So we get Type IX again.

Finally we can consider the case $g_1 = g_2 = g_3 = 0$. The image of G in $\mathrm{PGL}(2)$ must leave $V(g_4)$ invariant. Since the curve is nonsingular, $V(g_4)$ consists of 4 distinct points. After a linear change of variables we may assume that $g_4 = t_0^4 + t_1^4 + at_1^2 t_2^2$. We know from Case 1a that G contains a subgroup G_1 isomorphic to $4 \cdot 2^2$. There are two special values of a when the group is bigger. These are the cases where the elliptic curve defined by the binary quartic is harmonic with $a = \pm 6$ or equianharmonic with $a = \pm 2\sqrt{-3}$ (see Exercise 3.14). The quartic quartic acquires an additional symmetry of order 2 in the first case and of order 3 in the second case.

To see the additional symmetries, we use the following identities:

$$
\begin{aligned}
x^4 + y^4 &= \frac{1}{8}\big((x+y)^4 + (x-y)^4 + 6(x+y)^2(x-y)^2\big), \qquad (6.29)\\
x^4 + y^4 + 2\sqrt{-3}x^2y^2 &= \frac{e^{-\pi i/3}}{4}\big((x+iy)^4 + (x-iy)^4 + 2\sqrt{-3}(x+iy)^2(x-iy)^2\big).
\end{aligned}
$$

Since $g_4(t_0, \sqrt{-1}t_1) = t_0^4 + t_1^4 - at_1^2 t_2^2$, we get similar identities for $a = -6, -2\sqrt{-3}$.

The first identity shows that the form g_4 can be reduced to the form $t_0^4 + t_1^4$. Hence, , the curve is isomorphic to the Fermat quartic. The Hessian of this curve is the union of three lines which are permuted by the group of automorphisms. This easily implies that the group is isomorphic to the extension $4^2 : \mathfrak{S}_3$ of order 96. This is Type II. The second identity shows that, if $a = 2\sqrt{-3}$, then $V(g_4)$ has an additional symmetry of order 3 defined by

$$
(t_0, t_1) \mapsto \frac{1}{1-i}(t_0 + it_1, t_0 - it_1) = \frac{1}{\sqrt{2}}(e^{\pi i/4}t_0 + e^{3\pi i/4}t_1, e^{\pi i/4}t_0 + e^{-\pi i/4}t_1).
$$

It multiplies g_4 by $e^{4\pi i/3}$. The group G of automorphisms of the quartic $V(t_0^4 + t_1^4 + 2\sqrt{-3}t_0^2 t_1^2 + t_2^4)$ is generated by $\sigma_1 : [t_0, t_1, t_2] \mapsto [t_1, t_0, t_2], \sigma_2 : (t_0, t_1, t_2) \mapsto [t_0, -t_1, t_2], \sigma_3 : [t_0, t_1, t_2] \mapsto [t_0, t_1, it_2]$, and

$$
\sigma_4 : [t_0, t_1, t_2] \mapsto [\frac{t_0}{1-i} + \frac{it_1}{1-i}, \frac{t_0}{1-i} - \frac{it_1}{1-i}, e^{\pi i/3}t_2].
$$

The element σ_3 of order 4 generates the center of G. We have

$$
\sigma_1^2 = \sigma_2^2 = 1, (\sigma_1\sigma_2)^2 = \sigma_3^2, \sigma_4^3 = 1, \sigma_1\sigma_4\sigma_1^{-1} = \sigma_4\sigma_3^{-1}.
$$

Thus $G \cong 4.\mathfrak{A}_4$. This is Type III.

Observe that the group $4.\mathfrak{A}_4$ contains 4 elements of order 12. One can verify (see Exercise 6.12) that the polynomial $t_0^4 + t_1^4 + 2\sqrt{-3}t_0^2 t_1^2$ can be reduced to the polynomial

$(t_0^3 + t_1^3)t_1$ by a linear change of variables. This gives the equation of the curve from case (x) of Lemma 6.5.1.

Case 3: The group G contains a normal transitive imprimitive subgroup H. The group H contains a subgroup from Case 1 and the quotient by this subgroup permutes cyclically the coordinates. It follows from the list in Lemma 6.5.1 that it can happen only if

$$f = t_0^4 + \alpha t_0^2 t_1 t_2 + t_0(t_1^3 + t_2^3) + \beta t_1^2 t_2^2, \tag{6.30}$$

$$f = t_0^3 t_1 + t_1^3 t_2 + t_2^3 t_0, \tag{6.31}$$

$$f = t_0^4 + t_1^4 + t_2^4, \tag{6.32}$$

$$f = t_0^4 + t_1^4 + t_2^4 + a(t_0^2 t_1^2 + t_0^2 t_2^2 + t_1^2 t_2^2). \tag{6.33}$$

In the first curve we have the additional automorphism of order 2 interchanging t_1 and t_2. This gives Type X.

The second curve is the *Klein quartic* which will be discussed in the next section.

The third curve is the *Fermat quartic*. We have seen this curve already in the previous case.

In the forth case $\mathrm{Aut}(C)$ consists of permutations and sign changes of coordinates. It is easy to see that this defines a subgroup G of $\mathrm{Aut}(C)$ of order 24. It acts by permutations on the set of 4 bitangents $V(t_0 \pm t_1 \pm t_2)$ of C. This easily shows that G is isomorphic to the permutation group $\mathfrak{S}_4$ (or the octahedron group). One can show that the full automorphism group of the curve coincides with $\mathfrak{S}_4$ unless $a = \frac{1}{2}(-1 \pm \sqrt{7})$. This is Type IV. In the latter case the curve is isomorphic to the Klein curve (see [70], [169]).

Case 4: G is a simple group.

Here we use the classification of simple non-abelian finite subgroups of $\mathrm{PGL}(3)$ (see [31]). There are only two transitive simple groups. One is the group G of order 168 isomorphic to the group of automorphisms of the Klein quartic. It contains an element σ of order 7 and element of order 3 from the normalizer of the group $\langle\sigma\rangle$. Thus G contains a imprimitive subgroup of order divisible by 7. It follows from the previous classification that C must be as in case (x) with $\alpha = 0$, so it is the Klein quartic. This is Type I.

The other group is the *Valentiner group* of order 360 isomorphic to the alternating group $\mathfrak{A}_6$. It is known that latter group does not admit a 3-dimensional linear representation (a certain central extension of degree 3 does). Since any automorphism group of a plane quartic acts on the 3-dimensional linear space $H^0(C, \omega_C)$ the Valentiner group cannot be realized as an automorphism group of a plane quartic.

□

6.5.3 The Klein quartic

Recall the following well-known result of A. Hurwitz (see [206], Chapter IV, Exercise 2.5).

Theorem 6.5.3. *Let X be a nonsingular connected projective curve of genus $g > 1$. Then*

$$\#\mathrm{Aut}(X) \le 84(g-1).$$

For $g = 3$, the bound gives $\#\mathrm{Aut}(X) \le 168$ and it is achieved for the Klein quartic

$$C = V(t_0^3 t_1 + t_1^3 t_2 + t_2^3 t_0). \tag{6.34}$$

Recall that we know that its group of automorphisms contains an element S of order 7 acting by the formula

$$S : [t_0, t_1, t_2] \mapsto [t_0\epsilon, \epsilon^2 t_1, \epsilon^4 t_2], \quad \epsilon = e^{2\pi i/7},$$

where we scaled the action to represent the transformation by a matrix from $\mathrm{SL}(3)$. Another obvious symmetry is an automorphism G_2 of order 3 given by cyclic permutation U of the coordinates. It is easy to check that

$$U^{-1}SU = S^2, \tag{6.35}$$

so that the subgroup generated by S, U is a group of order 21 isomorphic to the semi-direct product 7 : 3.

By a direct computation one checks that the following unimodular matrix defines an automorphism T of C of order 2:

$$\frac{i}{\sqrt{7}} \begin{pmatrix} \epsilon - \epsilon^6 & \epsilon^2 - \epsilon^5 & \epsilon^4 - \epsilon^3 \\ \epsilon^2 - \epsilon^5 & \epsilon^4 - \epsilon^3 & \epsilon - \epsilon^6 \\ \epsilon^4 - \epsilon^3 & \epsilon - \epsilon^6 & \epsilon^2 - \epsilon^5 \end{pmatrix}. \tag{6.36}$$

We have

$$T^{-1}UT = U^2, \tag{6.37}$$

so that the subgroup generated by U, T is the dihedral group of order 6. One checks that the 49 products $S^a T S^b$ are all distinct. In particular, the cyclic subgroup (S) is not normal in the group G generated by S, T, U. Since the order of G is divisible by $2 \cdot 3 \cdot 7 = 42$, we see that $\#G = 42, 84, 126$, or 168. It follows from Sylow's Theorem that the subgroup (S) must be normal in the first three cases, so $\#G = 168$, and by Hurwitz's Theorem

$$\mathrm{Aut}(C) = G = \langle S, U, T \rangle.$$

Lemma 6.5.4. *The group $G = \mathrm{Aut}(C)$ is a simple group of order 168.*

Proof. Suppose H is a nontrivial normal subgroup of G. Assume that its order is divisible by 7. Since its Sylow 7-subgroup cannot be normal in H, we see that H contains all Sylow 7-subgroups of G. By Sylow's Theorem, their number is equal to 8. This shows that $\#H = 56$ or 84. In the first case H contains a Sylow 2-subgroup of order 8. Since H is normal, all its conjugates are in H, and, in particular, $T \in H$. The quotient group G/H is of order 3. It follows from (6.37) that the coset of U must be trivial. Since 3 does not divide 56, we get a contradiction. In the second case, H contains S, T, U and hence coincide with G. So, we have shown that H cannot contain an element of order 7. Suppose it contains an element of order 3. Since all such elements are conjugate, H contains U. It follows from (6.35), that the coset of S in G/H is trivial, hence $S \in H$ contradicting the assumption. It remains to consider the case when H is a 2-subgroup. Then $\#G/H = 2^a \cdot 3 \cdot 7$, with $a \leq 2$. It follows from Sylow's Theorem that the image of the Sylow 7-subgroup in G/H is normal. Thus its preimage in G is normal. This contradiction finishes the proof that G is simple.

□

Remark 6.5.1. One can show that

$$G \cong \mathrm{PSL}_2(\mathbb{F}_7) \cong \mathrm{PSL}_3(\mathbb{F}_2).$$

The first isomorphism has a natural construction via the theory of automorphic functions. The Klein curve is isomorphic to a compactification of the modular curve $X(7)$ corresponding to the principal congruence subgroup of full level 7. The second isomorphism has a natural construction via considering a model of the Klein curve over a finite field of 2 elements (see [159]). When can see an explicit action of G on 28 bitangents via the geometry of the projective line $\mathbb{P}^1(\mathbb{F}_7)$ (see [100], [237]).

The group $\mathrm{Aut}(C)$ has 3 orbits on C with non-trivial stabilizers of orders $2, 3, 7$. They are of cardinality $84, 56$ and 24, respectively.

The orbit of cardinality 24 consists of inflection points of C. We know that a cyclic group of order 7 is normalized by an element of order 3. Thus the orbit is equal to the union of 8 sets each consisting of an orbit of a group of order 3. An example of such a group is the vertices of the triangle formed by the inflection tangent lines

$$t_0 + t_1 + t_2 = 0,\ t_0 + \eta_3 t_1 + \eta_3^2 t_2 = 0,\ t_0 + \eta_3^2 t_1 + \eta_3 t_2 = 0.$$

This can be directly checked. From this it follows that the inflection points form the set of vertices of 8 triangles. We know that the inflection points are the intersection points of C and its Hessian given by the equation

$$\mathrm{He}(f) = 5t_0^2t_1^2t_2^2 - t_0t_1^5 - t_0^5t_2 - t_1t_2^5 = 0.$$

The orbit of cardinality 56 consists of the tangency points of 28 bitangents of C. An example of an element of order 3 is a cyclic permutation of coordinates. It has 2 fixed points $[1, \eta_3, \eta_3^2]$ and $[1, \eta_3^2, \eta_3]$ on C. They lie on the bitangent with equation

$$4t_0 + (3\eta_2^2 + 1)t_1 + (3\eta_3 + 1)t_2 = 0.$$

Define a polynomial of degree 14 by

$$\Psi = \det \begin{pmatrix} \frac{\partial^2 f}{\partial t_0^2} & \frac{\partial^2 f}{\partial t_0 t_1} & \frac{\partial^2 f}{\partial t_0 t_2} & \frac{\partial \psi}{\partial t_0} \\ \frac{\partial^2 f}{\partial t_1 t_0} & \frac{\partial^2 f}{\partial t_1^2} & \frac{\partial^2 f}{\partial t_1 t_2} & \frac{\partial \psi}{\partial t_1} \\ \frac{\partial^2 f}{\partial t_2 t_0} & \frac{\partial^2 f}{\partial t_2 t_1} & \frac{\partial^2 f}{\partial t_2^2} & \frac{\partial \psi}{\partial t_2} \\ \frac{\partial \psi}{\partial t_0} & \frac{\partial \psi}{\partial t_1} & \frac{\partial \psi}{\partial t_2} & 0 \end{pmatrix}.$$

One checks that it is invariant with respect to G and does not contain f as a factor. Hence it cuts out in $V(f)$ a G-invariant positive divisor of degree 56. It must consists of a G-orbit of cardinality 56.

One can compute it explicitly (see [423], p. 524) to find that

$$\Psi = t_0^{14} + t_1^{14} + t_2^{14} - 34t_0t_1t_2(t_0^{10}t_2 + \ldots) - 250t_0t_1t_2(t_0^3t_1^8 + \ldots) +$$
$$375t_0^2t_1^2t_2^2(t_0^6t_2^2 + \ldots) + 18(t_0^7t_1^7 + \ldots) - 126t_0^3t_1^3t_2^3(t_0^3t_1^2 + \ldots).$$

Here the dots mean monomials obtained from the first one by permutation of variables.

The orbit of cardinality 84 is equal to the union of 21 sets, each consisting of 4 intersection points of C with the line of fixed points of a transformation of order 2. An example of such a point is

$$((\epsilon^4 - \epsilon^3)(\epsilon - \epsilon^6)\epsilon^4, (\epsilon^2 - \epsilon^5)(\epsilon - \epsilon^6)\epsilon, (\epsilon^4 - \epsilon^3)(\epsilon^2 - \epsilon^5)\epsilon^2).$$

Similarly to the above, one considers the Jacobian determinant $\Xi = J(f, g, h)$ of the polynomials f, g, h. It is a G-invariant polynomial of degree 21. Its zeros on $V(f)$ give the orbit of 84 points. One can compute Ξ explicitly (see [189], p. 372) to find that

$$\Xi = t_0^{21} + t_1^{21} + t_2^{21} - 7t_0t_1t_2(t_0^{17}t_2 + \ldots) + 217t_0t_1t_2(t_0^3t_1^{15} \ldots) -$$
$$308t_0^2t_1^2t_2^2(t_0^{13}t_2^2 + \ldots) - 57(t_0^{14}t_2^7 + \ldots) - 289(t_0^7t_1^{14} + \ldots) +$$
$$4018t_0^3t_1^3t_2^3(t_0^2t_1^{10} + \ldots) + 637t_0^3t_1^3t_2^3(t_0^9t_1^3 + \ldots) +$$
$$1638t_0t_1t_2(t_0^{10}t_1^8 + \ldots) - 6279t_0^2t_1^2t_2^2(t_0^6t_1^9 + \ldots) +$$
$$7007t_0^5t_1^5t_2^5(t_0t_1^5 + \ldots) - 10010t_0^4t_1^4t_2^4(t_0^5t_1^4 + \ldots) + 3432t_0^7t_1^7t_2^7.$$

Remark 6.5.2. The polynomial of degree 14 that cuts out 56 contact points of bitangents on the Fermat quartic looks very simple (see [15], vol. VI, p. 13):

$$\Xi = (t_0t_1t_2)^2(t_0^8 + t_1^8 + t_2^8). \tag{6.38}$$

A set of 12 bitangents can be easily seen by factoring the polynomial $t_i^4 + t_j^4, 0 \le i < j \le 2$. The remaining 16 bitangents can be derived from the identity

$$(t_0+t_1+t_2)(t_0+t_1-t_2)(t_0-t_1+t_2)(t_0-t_1-t_2)+2(t_0^4+t_1^4+t_2^4) = (t_0^2+t_1^2+t_2)^2,$$

and other similar identities obtained by multiplying the coordinates by forth roots of 1. An example of a plane quartic with 28 real bitangents can be found in [154].

.

Exercises

6.1 Show that two syzygetic tetrads of bitangents cannot have two common bitangents.

6.2 Let $C_t = V(tf + q^2)$ be a family of plane quartics over $\mathbb{C}$ depending on a parameter t. Assume that $V(f)$ is nonsingular and $V(f)$ and $V(q)$ intersect transversally at 8 points $p_1, \ldots, p_8$.

Show that C_t is nonsingular for all t in some open neighborhood of 0 in usual topology and the limit of 28 bitangents when $t \to 0$ is equal the set of 28 lines $\overline{p_i, p_j}$.

6.3 Show that the locus of nonsingular quartics which admit a flex bitangent is a hypersurface in the space of all nonsingular quartics.

6.4 Consider the Fermat quartic $V(t_0^4 + t_1^4 + t_2^4)$. Find all bitangents and all Steiner complexes. Show that it admits 12 flex bitangents.

6.5 An open problem: what is the maximal possible number of flex bitangents on a nonsingular quartic?

6.6 Show that a nonsingular plane quartic C admits 63 irreducible one-parameter families of conics which are tangent to C at 4 points.

6.7 Let $S = \{(\ell_1, \ell_1'), \ldots, (\ell_6, \ell_6')\}$ be a Steiner complex of 12 bitangents. Prove that the six intersection points $\ell_i \cap \ell_i'$ lie on a conic and all $\binom{28}{2} = 378$ intersection points of bitangents lie on 63 conics. [Hint: the conic is the Veronese curve from Remark 6.2.4].

6.8 Find all possible types of azygetic hexads of bitangents. Which types are contained in a Steiner complex?

6.9 Show that the pencil of conics passing through the four contact points of two bitangents contains five members each passing through the contact points of a pair of bitangents.

6.10 Show that the linear system $L(\epsilon)$ of conics associated to a nonzero 2-torsion divisor class is equal to the linear system of first polars of the cubic $B(\epsilon)$.

6.11 Show that a choice of $\epsilon \in \mathrm{Jac}(C)[2] \setminus \{0\}$ defines a conic Q and a cubic B such that C is equal to the locus of points x such that the polar $P_x(B)$ is touching Q.

6.12 Let $C = V(a_{11}a_{22} - a_{12}^2)$ be a representation of a nonsingular quartic C as a symmetric quadratic determinant corresponding to a choice of a 2-torsion divisor class ϵ. Let $\tilde{C}$ be the unramified double cover of C corresponding to ϵ. Show that $\tilde{C}$ is isomorphic to a canonical curve of genus 5 given by the equations

$$a_{11}[t_0, t_1, t_2] - t_3^2 = a_{12}[t_0, t_1, t_2] - t_3 t_4 = a_{22}[t_0, t_1, t_2] - t_4^2 = 0$$

in $\mathbb{P}^4$.

6.13 A nonsingular curve is called *bielliptic* if it admits a double cover to an elliptic curve. Show that the moduli space of bielliptic curves of genus 4 is birationally isomorphic to the moduli space of isomorphism classes of genus 3 curves together with a nonzero 2-torsion divisor class.

6.14 Show that the curves $V(t_0^4 + t_1^4 + t_2^4 + 2\sqrt{-3}t_1^2t_2^2)$ and $V(t_0^4 + t_2^4 + t_0t_0^3 + t_0t_1^3)$ are isomorphic.

6.15 A plane quartic $C = V(f)$ is called a *Caporali quartic* if $\mathrm{VSP}(f;4)^o \neq \emptyset$.

(i) What is the dimension of the locus of the Caporali quartics?

(ii) Show that the Clebsch covariant quartic $S(C)$ is reducible.

(iii) Find the intersection of the loci of Fermat quartics and Caporali quartics.

([40]).

6.16 Let q be a nondegenerate quadratic form in 3 variables. Show that $W(q^2; 6)^o$ is a homogeneous space for the group $\mathrm{PSL}(2, \mathbb{C})$.

6.17 Let C be a hyperelliptic curve of genus g. Show that the graph of the hyperelliptic involution has valence 2.

6.18 Let $f = t_0^3 t_1 + t_1^3 t_2 + t_2^3 t_0$. Show that $V(S(f)) = V(f)$.

6.19 Show that the binary form $f = t_0(t_0 + 2t_1)^2$ does not admit nondegenerate polar 2-th polyhedron.

6.20 Show that the locus of lines $\ell = V(l)$ such that the anti-polar conic of l^2 with respect to a quartic curve $V(f)$ is reducible is a plane curve of degree 6 in the dual plane.

6.21 Show that the Clebsch covariant of the Fermat quartic C is equal to C.

6.22 Classify automorphism groups of irreducible singular plane quartics.

6.23 For each nonsingular plane quartic curve C with automorphism group G describe the ramification scheme of the cover $C \to C/G$.

6.24 Let C be the Klein quartic. For any subgroup H of Aut(C) determine the genus of H and the ramification scheme of the cover $C \to C/H$.

6.25 Analyze the action of the automorphism group of the Klein quartic C on the set of even theta characteristics. Show that there is only one which is invariant with respect to the whole group. Find the corresponding determinantal representation of C.

6.26 Let C be a general plane quartic. A triangle of lines is called a *biscribed triangle* of C if each side is a tangent line and each vertex is on C.

(i) Show that for any biscribed triangle there exists a unique contact cubic which is tangent to C at the vertices of the triangle and at the tangency points of its sides.

(ii) Show that the contact cubic defined by a biscribed triangle corresponds to an even theta characteristic on C. Using this show that there are $288 = 8 \cdot 36$ biscribed triangles.

6.27 Show that a smooth plane quartic admits an automorphism of order 2 if and only if among its 28 bitangents four form a syzygetic set of bitangents intersecting at one point.

6.28 Show that the dual curve of a general nonsingular plane quartic is a curve of degree 12 with 24 cusps and 28 nodes.

6.29 Consider the curve S (resp. T) in the dual plane equal to the closure of the locus of lines that intersects a nonsingular quartic at 4 points with equianharmonic(resp. harmonic) cross ratio. Show that C (resp. C') is of degree 4 (resp. 6) and intersects the dual quartic at its 24 cusps.

6.30 Let $C = l^4 + l_1^4 + l_2^4 + l_3^4 + l_4^4$ be a Clebsch quartic. Show that the pentagonal theta characteristic on the Lüroth quartic $S = S(C)$ defines the representation of S as the determinant of the 4×4 matrix with diagonal entries $l + l_i$ and off-diagonal entries equal to l (communicated by B. van Geemen).

6.31 Show that

(i) Let C and K be general conic and a cubic. Show that the set of points a such that $P_a(C)$ is tangent to $P_a(K)$ is a Lüroth quartic.

(ii) Show that the set of polar lines $P_a(C)$ which coincide with polar lines $P_a(K)$ is equal to the set of seven Aronhold bitangents of the Lüroth quartic ([19]).

6.32 Show that the set of 28 bitangents of the Klein quartic contains 21 subsets of four concurrent bitangents and each bitangent has 3 concurrency points.

6.33 Show that there exists a curve of degree 14 that cuts out the 56 contact points of 28 bitangents of a plane quartic curve ([211]). Show that in the case of Klein's quartic this curve can be chosen to be invariant with respect to the Klein group of order 168.

Historical Notes

The fact that a general plane quartic curve has 28 bitangents was first proved in 1850 by C. Jacobi [234] although the number was apparently known to J. Poncelet. The proof used Plücker formulas and so did not apply to any nonsingular curve. Using contact cubics, Hesse extended this result to arbitrary nonsingular quartics [213].

The first systematic study of the configuration of bitangents began by O. Hesse [213],[214] and J. Steiner [396]. Although the Steiner's paper does not contain proofs. They considered azygetic and syzygetic sets, Steiner complexes of bitangents although the terminology was introduced later by Frobenius [170]. Hesse's approach used the relationship between bitangents and Cayley octads. The notion of a Steiner group of bitangents was introduced by A. Cayley in [62]. Weber [422] changed it to a Steiner complex in order not to be confused with the terminology of group theory.

The fact that the equation of a nonsingular quartic could be brought to the form (6.1) was first noticed by J. Plücker [320]. Equation (6.3) arising from a Steiner complex appears first in Hesse's paper [214], §9. The determinantal identity for bordered determinants (6.23) appears in [213]. The number of hexads of bitangents with contact points on a cubic curve was first computed by Hesse [213] and Salmon [356].

The equation of a quartic as a quadratic determinant appeared first in Plücker [318], p. 228 and in Hesse [214], §10. Both of them knew that it can be done in 63 different ways. Hesse also proves that the 12 lines of a Steiner complex, consider as points in the dual plane, lie on a cubic. More details appear in Roth's paper [344] and Coble's book [87].

Using his determinantal identity Hesse showed that a linear symmetric determinantal representation of a plane curve of degree d defines a $d-1$-dimensional family of contact curves of degree $d-1$. However, he acknowledges in [213] that he did not prove that general curve of degree $d>4$ admits such a representation. This was proved much later by Dixon [125]. For quartic curves Hesse proves the existence of a determinantal representation in 36 differerent ways.

The relationship between seven points in the projective plane and bitangents of a plane quartic was first given by S. Aronhold [10]. The fact that Hesse's construction and Aronhold' construction are equivalent via the projection from one point of a Cayley octad was first noticed by A. Dixon [126].

The relation of bitangents to theta functions with odd characteristics goes back to B. Riemann [337] and Weber [422] and was developed later by A. Clebsch [77] and G. Frobenius [170], [172]. In particular, Frobenius found a relationship between the sets of seven points or Cayley octads with theta functions of genus 3. Coble's book [87] has a nice exposition of Frobenius's work. The equations of bitangents presented in Theorem 6.1.5 were first found by Riemann, with more details explained by Weber.

The theory of covariants and contravariants of plane quartics was initiated by A. Clebsch in his fundamental paper about plane quartic curves [75]. In this paper he introduces his covariant quartic $S(C)$, the catalecticant invariant and shows that its vanishing is necessary for writing the equation of a quartic as a sum of five powers of linear forms. Much later G. Scorza [369] proved that the rational map S on the space of quartics is of degree 36 and related this number with the number of even theta characteristics. The interpretation of the apolar conic as the parameter space of inscribed pentagons was given by G. Lüroth [274].

The groups of automorphisms of nonsingular plane quartic curves were classified by S. Kantor [241] and A. Wiman [426]. The first two curves from our table were studied earlier by F. Klein [249] and W. Dyck [147]. Of course, the Klein curve is the most famous of those and appears often in the modern literature (see, for example, [388]).

The classical literature about plane quartics is enormous. We refer to Ciani's paper [71] for a nice survey of classical results, as well as to his own contributions to the study of plane quartics which are assembled in [72]. Other surveys can be found in [308] and [161].

Chapter 7

Planar Cremona transformations

7.1 Homaloidal linear systems

7.1.1 Linear systems and their base schemes

Here we recall some known definitions from the theory of linear systems and rational maps (see [265]). Let X be a nonsingular irreducible variety of dimension n and $\mathfrak{a}$ be a sheaf of ideals on X. A *resolution* of $\mathfrak{a}$ is a projective birational morphism $\pi : Y \to X$ of nonsingular varieties such that $\pi^{-1}(\mathfrak{a}) := \mathfrak{a} \cdot \mathcal{O}_Y \cong \mathcal{O}_Y(-F)$ for some effective divisor F' on Y. Using the universal property of blow-up of ideals, we see that a resolution of $\mathfrak{a}$ is a resolution of singularities of the normalization of the blow-up of $\mathfrak{a}$.

Let $\nu : B(\mathfrak{a}) \to X$ be the normalization of the blow-up of the ideal $\mathfrak{a}$ so that $\nu^{-1}(\mathfrak{a}) = \mathcal{O}_Y(-F')$ for some effective divisor. The ideal $\bar{\mathfrak{a}} = \sigma_*\mathcal{O}_{B(\mathfrak{a})}(-F')$ is equal to the *integral closure* of the ideal $\mathfrak{a}$. In the case when X is affine, it consists of regular functions on X such that, considered as rational functions on $B(\mathfrak{a})$, they belong to the space $H^0(B(\mathfrak{a}), \mathcal{O}_{B(\mathfrak{a})}(-F'))$. By the universal property of the blow-up, π factors: $\pi = \nu \circ \pi' : Y \to B(\mathfrak{a}) \to X$, hence $\sigma_*\mathcal{O}_{B(\mathfrak{a})}(-F') = \pi_*\mathcal{O}_Y(-F') = \bar{\mathfrak{a}}$.

We will be applying this to the case when $\mathfrak{a}$ is equal to the base ideal of a linear system.

To fix the notation, let us remind the definition. Let $\mathcal{L}$ be an invertible sheaf on X and $V \subset H^0(X, \mathcal{L})$ be a linear subspace of positive dimension. The projective space $|V|$ is identified with the set of divisors D_s of zeros of sections s from $V \setminus \{0\}$. In case $V = H^0(X, \mathcal{L})$ we employ the notation $|\mathcal{L}|$ or $|D|$, where $\mathcal{L} \cong \mathcal{O}_X(D)$. The set of divisors $D_s, s \in V \setminus \{0\}$, is called the *linear system* defined by the subspace V, in the case $|V| = |\mathcal{L}|$, it is called a *complete linear system*. We assume that $|V|$ has no *fixed component* (i.e. effective divisor $F \neq 0$ such that V is contained in the image of the natural map $H^0(X, \mathcal{L}(-E)) \to H^0(X, \mathcal{L})$).

The evaluation map

$$\mathrm{ev} : V \otimes \mathcal{O}_X \to \mathcal{L} \tag{7.1}$$

defines a map of sheaves

$$V \otimes \mathcal{L}^{-1} \to \mathcal{O}_X.$$

Its image is an ideal sheaf $\mathfrak{b}(|V|)$ in $\mathcal{O}_X$ which is called the *base ideal* of $|V|$. The closed subscheme $\mathrm{Bs}(|V|)$ of X defined by this ideal is called the *base scheme* of $|V|$ and its support is the *base locus* of $|V|$. We have

$$\mathrm{Bs}(|V|) = \cap_{D \in |V|} D,$$

where each D is identified with a closed subscheme of X. Let $s_0, \ldots, s_m$ be a basis of V and $D_i = D_{s_i} \in |V|$ be the corresponding divisors. Then

$$\mathrm{Bs}(|V|) = \bigcap_{i=0}^{m} D_i.$$

By definition,

$$V \subset H^0(X, \mathcal{L} \otimes \mathfrak{b}(|V|)). \tag{7.2}$$

Let $\nu : B \to X$ be the normalization of the blow-up of the ideal sheaf $\mathfrak{b}(|V|)$ and $\nu^{-1}(\mathfrak{b}(|V|)) = \mathcal{O}_B(-F')$. Applying the projection formula, we get

$$\sigma_*(\sigma^*\mathcal{L}(-F')) = \mathcal{L} \otimes \overline{\mathfrak{b}(|V|)},$$

hence

$$H^0(X, \mathcal{L} \otimes \overline{\mathfrak{b}(|V|)}) = H^0(B, \sigma^*\mathcal{L}(-F')).$$

Combining with (7.2), we obtain an injective linear map

$$\sigma^* : V \hookrightarrow H^0(B, \sigma^*\mathcal{L}(-F')).$$

Let $\pi : Y \to X$ be a resolution of $\mathfrak{b}(|V|)$ (also called a resolution of $|V|$). Since π factors through σ, and the direct image of $\mathcal{O}_Y(-F)$ in B is equal to $\mathcal{O}_B(-F')$, we obtain an inclusion

$$\pi^* : V \to H^0(Y, \pi^*\mathcal{L}(-F)). \tag{7.3}$$

Let

$$\phi_{|V|} : X - \to \mathbb{P}(V)$$

be the rational map defined by the linear system $|V|$. The rational map $\phi_{|V|}$ is given by assigning to a point $x \in X \setminus \mathrm{Bs}(|V|)$ the hyperplane in $|V|$ of sections vanishing at x. A choice of a basis in V defines projective coordinates in $\mathbb{P}(V)$ and the explicit formula

$$\phi_{|V|}(x) = [s_0(x), \ldots, s_m(x)]. \tag{7.4}$$

The rational map $\phi_{|V|}$ is regular if and only if the base locus of $|V|$ is empty, or equivalently, the evaluation map (7.1) is surjective. In this case $|V|$ is called *base point free*. It is called *very ample* if its base locus is empty and $\phi_{|V|}$ defines a closed embedding. Let $\pi : Y \to X$ be a resolution of $|V|$, then $|\pi^*(V)|$ is a base-free linear system in the complete linear system $|\sigma^*(\mathcal{L})(-F))|$. Let $\sigma : Y \to \mathbb{P}(V^\vee)$ be the corresponding regular map. We obtain a commutative diagram

$$\begin{tikzcd} & Y \arrow[dl, "\pi"'] \arrow[dr, "\sigma"] & \\ X \arrow[rr, dashed, "\phi_{|V|}"] & & \mathbb{P}(V). \end{tikzcd} \tag{7.5}$$

It follows from the definition of a rational map defined by a base-free linear system that $\sigma^*\mathcal{O}_{\mathbb{P}(V)} = (\pi^*\mathcal{L})(-F))$ and $\sigma^*(H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(1)))$ coincides with the image of V in $H^0(Y, (\pi^*\mathcal{L})(-F))$. If we assume that the base ideal $\mathfrak{b}(|V|)$ is integrally closed, then we can identify the complete linear system $|(\pi^*\mathcal{L})(-F)|$ with $|\mathcal{L}\otimes \overline{\mathfrak{b}(|V|)}|$, where we identify $H^0(X, \mathcal{L} \otimes \overline{\mathfrak{b}(|V|)})$ with a subspace of $H^0(X, \mathcal{L})$ via the inclusion of sheaves $\overline{\mathfrak{b}(|V|)} \to \mathcal{O}_X$.

Note that one can also define the proper inverse image $f^{-1}(|V|)$ of a linear system $|V'| \subset \mathcal{L}'$ on X' under a rational map $f : X - \to X'$. We consider f as a regular map $f : \mathrm{dom}(f) \to X'$ and use that any invertible sheaf of $\mathrm{dom}(f)$ and its section can be uniquely extended to an invertible sheaf and its section on X.

For any rational map $\phi : X - \to X'$, a commutative diagram

$$\begin{tikzcd} & Y \arrow[dl, "\pi"'] \arrow[dr, "\sigma"] & \\ X \arrow[rr, dashed, "\phi"] & & X', \end{tikzcd} \tag{7.6}$$

where π is a birational projective morphism and σ is a morphism, is called a *resolution of indeterminacy points* of f. We will always assume that Y is normal and π is an isomorphism over $\mathrm{dom}(f)$. Thus a resolution of $\mathfrak{b}(|V|)$ defines a resolution of the rational map $\phi_{|V|}$.

Consider a resolution of indeterminacy points of the rational map $\phi = \phi_{|V|} : X - \to \mathbb{P}(V)$. Then $\pi^{-1}(|V|)$ defines a regular map σ, hence $\pi^{-1}(\mathfrak{b}(|V|))$ is an invertible sheaf. By the universal property of the blow-up, the map π factors through the blow-up of $\mathfrak{b}(|V|)$. Note that the pair (π, σ) defines a regular map from Y to the normalization $\bar{\gamma}_\phi$ of the *graph* Γ_ϕ of f, the Zariski closure of the graph of the map $\phi : \mathrm{dom}(\phi) \to X'$ in $X \times X'$. It always defines a resolution of indeterminacy points of ϕ. When $\phi = \phi_{|V|}$, the graph $\Gamma_{\phi_{|V|}}$ is isomorphic to the blow-up of $\mathfrak{b}(|V|)$.

To define a rational map $\phi : X \to X'$ of projective varieties, we choose a very ample sheaf $\mathcal{L}'$ on X' which defines a closed embedding $\iota : X' \hookrightarrow \mathbb{P}^n = |H^0(X', \mathcal{L}')^\vee|$. Let $|V| = \phi^{-1}(|\mathcal{L}'|) \subset |\phi^{-1}(\mathcal{L}')|$. Then $\phi_{|V|} = \iota \circ \phi : X - \to X' \hookrightarrow \mathbb{P}^n$.

Proposition 7.1.1. *Assume that the image X' of $\phi_{|V|}$ is linearly normal in $\mathbb{P}(V)$ and the map $\phi_{|V|} : X - \to X'$ is of degree 1. Then the map*

$$\pi^* : V \to H^0(Y, (\pi^*\mathcal{L})(-F))$$

is bijective. In particular, the base ideal $\mathfrak{b} = \mathfrak{b}(|V|)$ is integrally closed and

$$|V| = |H^0(Y, \mathcal{L} \otimes \mathfrak{b})|.$$

Proof. Recall that a projective subvariety $Z \subset \mathbb{P}^n$ is called *linearly normal* if it is a normal variety and the canonical restriction map $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \to H^0(Z, \mathcal{O}_Z(1))$ is bijective. Let $\sigma : Y \to X' \subset \mathbb{P}(V)$ be the map given by the linear system $|\pi^*(V)| \subset |\pi^*\mathcal{L}(-F)|$. We know that $\pi^{-1}(|V|) = \sigma^*(|\mathcal{O}_{\mathbb{P}(V)}(1)|)$, so it suffices to prove that the linear system $\sigma^*(|\mathcal{O}_{\mathbb{P}(V)}(1)|)$ is complete. By Zariski's Main Theorem ([206], Chapter 3, §11), $\sigma_*\mathcal{O}_Y = \iota_*\mathcal{O}_{X'}$, where $\iota : X' \hookrightarrow \mathbb{P}(V)$ is the closed embedding. We have

$$H^0(Y, \sigma^*\mathcal{O}_{\mathbb{P}(V)}(1)) = H^0(\mathbb{P}(V), \sigma_*(\sigma^*\mathcal{O}_{\mathbb{P}(V)}(1)))$$

$$= H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(1) \otimes \sigma_*\mathcal{O}_Y) = H^0(X', \mathcal{O}_{X'}(1)).$$

By definition of ϕ, we have $|V| = \phi^{-1}(|\mathcal{O}_{\mathbb{P}(V)}(1)|)$, hence

$$\pi^*(|V|) = \pi^*(\phi^{-1}(|\mathcal{O}_{\mathbb{P}(V)}(1)|)) = \sigma^*(|\mathcal{O}_{\mathbb{P}(V)}(1)|).$$

Since X' is linearly normal, the restriction map

$$H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(1)) \to H^0(X', \mathcal{O}_{X'}(1))$$

is bijective. Hence $\sigma^*(|\mathcal{O}_{\mathbb{P}(V)}(1)|) = |\sigma^*\mathcal{O}_{\mathbb{P}(V)}(1)|$. This proves the assertion. □

Suppose $\phi = \phi_{|V|}$ is of finite degree $\deg\phi = [\mathbb{C}(X) : f^*(\mathbb{C}(X'))]$, where $X' = \phi(X) = \sigma(Y)$ is the image of the rational map f. Consider a resolution of indeterminacy points of f. Let $\pi^{-1}(\mathfrak{b}(|V|)) = \mathcal{O}_Y(-F)$ for some effective divisor F. It follows from the intersection theory on algebraic varieties (see [173], Chapter 4, §4) that

$$\deg f \deg X' = \int_Y (\pi^*(D) - F)^n,$$

where $D \in |V|$ and $\deg X'$ is the degree of X' in the projective space $\mathbb{P}(V^\vee)$.

In particular, f is a birational map if and only if

$$\int_Y (\pi^*(D) - F)^{\dim X} = \deg X'. \tag{7.7}$$

7.1.2 Exceptional configurations

From now on we assume that X is a nonsingular surface and $|V|$ is a linear system without fixed components defining a rational map $f : X \to \mathbb{P}^m$. Let $\pi : Y \to X, \sigma : Y \to \mathbb{P}^m$ be its resolution of indeterminacy points. Resolving singularities of Y we assume that Y is a nonsingular surface. We will assume that Y is a minimal resolution of singularities.

We know (see [206], Chapter V, §5) that any birational morphism $\pi : Y \to X$ of nonsingular projective surfaces can be factored into a composition of blow-ups with centers at closed points. Let

$$\pi : Y = Y_N \xrightarrow{\pi_N} Y_{N-1} \xrightarrow{\pi_{N-1}} \ldots \xrightarrow{\pi_2} Y_1 \xrightarrow{\pi_1} Y_0 = X \tag{7.8}$$

be such a factorization. Here $\pi_i : Y_i \to Y_{i-1}$ is the blow-up of a point $x_i \in Y_{i-1}$. Let

$$E_i = \pi_i^{-1}(x_i), \quad \mathcal{E}_i = (\pi_{i+1} \circ \ldots \circ \pi_N)^*(E_i). \tag{7.9}$$

The divisors $\mathcal{E}_i$ are called the *exceptional configurations* of the birational morphism $\pi : Y \to X$. Note that $\mathcal{E}_i$ should be considered as an effective divisor, not necessary reduced.

For any effective divisor $D \neq 0$ on X let $\mathrm{mult}_{x_i} D$ be defined inductively in the following way. We set $\mathrm{mult}_{x_1} D$ to be the usual multiplicity of D at x_1. It is defined as the largest integer m such that the local equation of D at x_1 belongs to the m-th power of the maximal ideal $\mathfrak{m}_{X,x_1}$. Suppose $\mathrm{mult}_{x_i} D$ is defined. We take the proper inverse transform $\pi_i^{-1}(D)$ of D in X_i and define $\mathrm{mult}_{x_{i+1}}(D) = \mathrm{mult}_{x_{i+1}} \pi_i^{-1}(D)$. It follows from the definition that

$$\pi^{-1}(D) = \pi^*(D) - \sum_{i=1}^N m_i \mathcal{E}_i,$$

where $m_i = \mathrm{mult}_{x_i} D$. Now suppose $\pi : Y \to X$ is a resolution of indeterminacy points of a rational map f defined by a linear system $|V| \subset |\mathcal{L}|$. Let

$$m_i = \min_{D \in |V|} \mathrm{mult}_{x_i} D, \ i = 1, \ldots, N.$$

If $D_0, \ldots, D_t$ are divisors corresponding to a basis of V, then

$$m_i = \min\{\mathrm{mult}_{x_i} D_0, \ldots, \mathrm{mult}_{x_i} D_t\}, \ i = 1, \ldots, N.$$

It is clear that

$$\pi^{-1}(|V|) = \pi^*(|V|) - \sum_{i=1}^N m_i \mathcal{E}_i. \tag{7.10}$$

Let $F = \sum_{i=1}^N m_i \mathcal{E}_i$, then $\pi^{-1}(|V|)$ is contained in the linear system $|\pi^*(\mathcal{L})(-F)|$. Let $\mathfrak{b} = \mathfrak{b}(|V|)$. The ideal sheaf $\pi^{-1}(\mathfrak{b}) = \mathfrak{b} \cdot \mathcal{O}_Y$ is the base locus of $\pi^{-1}(|V|)$ and hence coincides with $\mathcal{O}_Y(-F)$. Applying (7.7), we obtain that $\deg \phi_{|V|} = (\pi^*(D) - F)^2$, where $D \in |\mathcal{L}|$ and we consider $\phi_{|V|}$ as a rational map from X onto its image X'. Suppose that $|\mathcal{L}|$ is base point-free. Then we can choose D such that it does not contain base points of $|V|$. This gives

$$D^2 - F^2 = \deg \phi_{|V|} \deg X'. \tag{7.11}$$

Lemma 7.1.2. *Let $\pi : Y \to X$ be a birational morphism of nonsingular surfaces and $\mathcal{E}_i, i = 1, \ldots, N$, be its exceptional configurations. Then*

$$\mathcal{E}_i \cdot \mathcal{E}_j = -\delta_{ij},$$

$$\mathcal{E}_i \cdot K_Y = -1.$$

Proof. This follows from the standard properties of the intersection theory on surfaces. For any morphism of nonsingular projective surfaces $\phi : X' \to X$ and two divisors D, D' on X, we have

$$\phi^*(D) \cdot \phi^*(D') = \deg(\phi) D \cdot D'. \tag{7.12}$$

Also, if C is a curve such that $\phi(C)$ is a point, we have

$$C \cdot \phi^*(D) = 0. \tag{7.13}$$

Applying (7.12), we have

$$-1 = E_i^2 = (\pi_{i+1} \circ \ldots \circ \pi_N)^*(E_i)^2 = \mathcal{E}_i^2.$$

Assume $i < j$. Applying (7.13) by taking $C = E_j$ and $D = (\pi_{i+1} \circ \ldots \circ \pi_{j-1})^*(E_i)$, we obtain

$$0 = E_j \cdot \pi_j^*(D) = (\pi_{j+1} \circ \ldots \circ \pi_N)^*(E_j) \cdot (\pi_{j+1} \circ \ldots \circ \pi_N)^*(D) = \mathcal{E}_j \cdot \mathcal{E}_i.$$

This proves the first assertion.

To prove the second assertion, we use that

$$K_{Y_{i+1}} = \pi_i^*(K_{Y_i}) + E_i.$$

By induction, this implies that

$$K_Y = \pi^*(K_{Y_0}) + \sum_{i=1}^N \mathcal{E}_i. \tag{7.14}$$

Intersecting with both sides and using (7.13), we get

$$K_Y \cdot \mathcal{E}_j = \Big(\sum_{i=1}^N \mathcal{E}_i\Big) \cdot E_j = \mathcal{E}_j^2 = -1.$$

□

Assume now that $\phi_{|V|} : X - \to X'$ is a birational isomorphism of nonsingular projective algebraic surfaces. By Bertini's Theorem ([206], Chapter II, Theorem 8.18), a general hyperplane section H' of X' is a nonsingular irreducible curve of some genus g. Since $\pi^{-1}(|V|)$ has no base points, by another Bertini's Theorem ([206], Chapter II, Corollary 10.9), its general member H is a nonsingular irreducible curve. Since $H \in |\sigma^*(H')|$, we obtain that H is of genus g and the map $\sigma : H \to \sigma(H)$ is an isomorphism. Using the adjunction formula, we obtain

$$H \cdot K_Y = 2g - 2 - H^2 = H'^2 + H' \cdot K_{X'} - H^2.$$

Write $H = \pi^*(D) - F$ and apply the projection formula, to obtain

$$H \cdot K_Y = D \cdot K_X - F \cdot K_Y.$$

Applying (7.11) and the previous Lemma, we obtain

Proposition 7.1.3. *Suppose $\phi_{|V|} : X - \to X'$ is a birational rational map of nonsingular projective algebraic surfaces. Let $D \in |\mathcal{L}|$. Then*

(i) $D^2 - \sum_{i=1}^N m_i^2 = H'^2 = \deg X'$;

(ii) $D \cdot K_X - \sum_{i=1}^N m_i = H' \cdot K_{X'}$.

7.1.3 The bubble space of a surface

Consider a factorization (7.8) of a birational morphism of nonsingular surfaces. Note that, if the morphism $\pi_1 \circ \cdots \circ \pi_i : Y_i \to X$ is an isomorphism on a Zariski open neighborhood of the point x_{i+1}, the points x_i can be identified with its image in X. Other points are called *infinitely near* points in X. To make this notion more precise one introdices the notion of the *bubble space* of a surface X.

Let $B(X)$ be the category of birational morphisms $\pi : X' \to X$ of nonsingular projective surfaces. Recall that a morphism from $(X' \xrightarrow{\pi'} X)$ to $(X'' \xrightarrow{\pi''} X)$ in this category is a regular map $\phi : X' \to X''$ such that $\pi'' \circ \phi = \pi'$.

Definition 7.1. *The* bubble space X^{bb} *of a nonsingular surface X is the factor set*

$$X^{\text{bb}} = \Big(\bigcup_{(X' \xrightarrow{\pi'} X) \in B(X)} X' \Big)/R,$$

where R is the following equivalence relation: $x' \in X'$ is equivalent to $x'' \in X''$ if the rational map $\pi''^{-1} \circ \pi' : X' - \to X''$ maps isomorphically an open neighborhood of x' to an open neighborhood of x''.

It is clear that for any $\pi : X' \to X$ from $B(X)$ we have an injective map $i_{X'} : X' \to X^{\text{bb}}$. We will identify points of X' with their images. If $\phi : X'' \to X'$ is a morphism in $B(X)$ which is isomorphic in $B(X')$ to the blow-up of a point $x' \in X'$, any point $x'' \in \phi^{-1}(x')$ is called *infinitely near point* to x' of the first order. This is denoted by $x'' \succ x'$. By induction, one defines an infinitely near point of order k, denoted by $x'' \succ_k x'$. This defines a partial order on X^{bb}.

We say that a point $x \in X^{\text{bb}}$ is of height k, if $x \succ_k x_0$ for some $x_0 \in X$. This defines the *height function* on the bubble space

$$\text{ht} : X^{\text{bb}} \to \mathbb{N}.$$

Clearly, $X = \text{ht}^{-1}(0)$. Points of height zero are called *proper points* of the bubble space. They will be identified with points in X.

Let $\mathbb{Z}^{X^{\text{bb}}}$ be the free abelian group generated by the set X^{bb}. Its elements are integer valued functions on X^{bb} with finite support. They added up as functions with values in $\mathbb{Z}$. We write elements of $\mathbb{Z}^{X^{\text{bb}}}$ as finite linear combinations $\sum m(x)x$, where $x \in X^{\text{bb}}$ and $m(x) \in \mathbb{Z}$ (similar to divisors on curves). Here $m(x)$ is the value of the corresponding function at x.

Definition 7.2. *A* bubble cycle *is an element $\eta = \sum m(x)x$ of $\mathbb{Z}^{X^{\text{bb}}}$ satisfying the following additional properties:*

(i) $m(x) \geq 0$ *for any* $x \in X^{\text{bb}}$*;*

(ii) $\sum_{x' \succ x} m_{x'} \leq m_x$.

We denote the subgroup of bubble cycles by $\mathcal{Z}_+(X^{\text{bb}})$.

Clearly, any bubble cycle η can be written in a unique way as a sum of bubble cycles Z_k such that the support of η_k is contained in $\mathrm{ht}^{-1}(k)$.

We can describe a bubble cycle by a weighted graph, called the *Enriques diagram*, by assigning to each point from its support a vertex, and joining two vertices by an ordered edge if one of the points is infinitely near to another point of the first order. The edge points to the vertex of lower height. We weight each vertex by the corresponding multiplicity. It is clear that the Enriques diagram is a tree.

Let $\xi = \sum m_x x$ be a bubble cycle. We order the points from the support of η such that $x_i \succ x_j$ implies $j < i$. We refer to such an order as an *admissible order*. We write $\xi = \sum_{i=1}^N m_i x_i$. Then we represent x_1 by a point on X and define $\pi_1 : X_1 \to X$ to be the blow-up of X with center at x_1. Then x_2 can be represented by a point on X_1 as either infinitely near of order 1 to x_1 or as a point equivalent to a point on X. We blow up x_2. Continuing in this way, we get a sequence of birational morphisms:

$$\pi : Y_\xi = Y_N \xrightarrow{\pi_N} Y_{N-1} \xrightarrow{\pi_{N-1}} \ldots \xrightarrow{\pi_2} Y_1 \xrightarrow{\pi_1} Y_0 = X, \tag{7.15}$$

where $\pi_{i+1} : Y_{i+1} \to Y_i$ is the blow-up of a point $x_i \in Y_{i-1}$. Clearly, the bubble cycle η is equal to the bubble cycle $\sum_{i=1}^N m_i x_i$.

Let $\mathcal{L}$ be an invertible sheaf on X and η be a bubble cycle with an admissible order and (7.15) be the corresponding sequence of blow-ups. Let $\mathcal{E}_i, i = 1, \ldots, N$, be the exceptional configurations. Set

$$|\mathcal{L} - \eta| := \{D \in |\mathcal{L}| : \pi^*(D) - \sum_{i=1}^N m_i \mathcal{E}_i \geq 0\}.$$

This is a linear subsystem of $|\mathcal{L}|$. Its elements D satisfy the following linear conditions. For any $x \in \eta$ with $\mathrm{ht}(x) = 0$ we must have $\mathrm{mult}_x D \geq m(x)$. This condition depends only on the equivalence class of x. Let $y \in \eta$ with $\mathrm{ht}(y) = 1$ and $y \succ x$ for some $x \in \eta$. Then we must have $\mathrm{mult}_y(\phi^*(D) - m(x)E) \geq m_y$, where y is represented by a point on the exceptional curve E of the blow-up $\phi : S' \to X$ with center at x. Then we go to level 2 and so on.

Let $F = \sum_{i=1}^N m_i \mathcal{E}_i$ and $\mathfrak{a}_\eta = \pi_\eta(\mathcal{O}_{Y_\eta}(-F))$. It is an integrally closed ideal sheaf on X equal to the integral closure $\bar{\mathfrak{b}}$ of the base ideal $\mathfrak{b}$ of the linear system $|\mathcal{L} - \eta|$. We have

$$|\mathcal{L} - \eta| = |H^0(X, \mathcal{L} \otimes \mathfrak{a}_\eta)|.$$

The following formula is known as the Hoskin-Deligne formula (see [117], [219]).

Proposition 7.1.4.

$$\mathrm{length}(\mathfrak{a}_\eta) := \dim H^0(X, \mathcal{O}_X/\mathfrak{a}_\eta) = \tfrac{1}{2} \sum_{i=1}^N m_i(m_i + 1).$$

The exact sequence

$$0 \to \mathcal{L} \otimes \mathfrak{a}_\eta \to \mathcal{L} \to \mathcal{L} \otimes \mathcal{O}_X/\mathfrak{a}_\eta \to 0$$

shows that

$$\dim|\mathcal{L}-\eta| \geq \dim|\mathcal{L}| - \text{length}(\mathfrak{a}_\eta). \tag{7.16}$$

Thus the Hoskin-Deligne formula justifies the count of constants, passing through a point with multiplicity m imposes $m(m+1)/2$ conditions.

The following proposition follows from Proposition 7.1.1.

Proposition 7.1.5. *Let $|V|$ be a linear system in $|\mathcal{L}|$ without fixed components. Suppose it defines a birational isomorphism onto a projectively normal surface X' in $\mathbb{P}(V^\vee)$. There exists a unique bubble cycle η such that $|V| = |\mathcal{L}-\eta|$.*

7.1.4 Cremona transformations

A birational map $f : \mathbb{P}^n - \to \mathbb{P}^n$ is called a *Cremona transformation*. The group $\text{Bir}(\mathbb{P}^n)$ of birational transformations of $\mathbb{P}^n$ is denoted by $\text{Cr}(n)$ and is called the *Cremona group*. It is isomorphic to the group of automorphisms of the field of rational functions on $\mathbb{P}^n$ identical on constants. In other words

$$\text{Cr}(n) \cong \text{Aut}_{\mathbb{C}}(\mathbb{C}(z_1,\ldots,z_n)).$$

As any rational map defined on $\mathbb{P}^n$, it is given by an n-dimensional linear system $|V| \subset |\mathcal{O}_{\mathbb{P}^n}(d)|$ for some $d \geq 1$. We assume that the linear system has no fixed component. The number d is called the *degree* of the Cremona transformation. A choice of a basis in V gives an explicit formula:

$$\phi : [x_0,\ldots,x_n] \mapsto [f_0(x_0,\ldots,x_n),\ldots,f_n(x_0,\ldots,x_n)],$$

where $f_i(t_0,\ldots,t_n)$ are homogeneous polynomials of degree d without common factor of positive degre. A linear system $|V|$ defining a Cremona transformation is called a *homaloidal linear system*. Its base ideal $\mathfrak{b}(|V|)$ is the sheaf of ideals associated to the homogeneous ideal generated by the polynomials $f_0,\ldots,f_n$. Its base scheme is the closed subscheme of $\mathbb{P}^n$ defined by the equations

$$f_0 = \ldots = f_n = 0.$$

As we explained above there is a resolution of indeterminacy points of ϕ

$$\begin{tikzcd} & Y \arrow[dl, "\pi"'] \arrow[dr, "\sigma"] & \\ \mathbb{P}^n \arrow[rr, dashed, "\phi"] & & \mathbb{P}^n \end{tikzcd}, \tag{7.17}$$

where Y is a nonsingular and π and σ are birational morphisms. Let E be the exceptional divisor of the resolution defined by the property $\pi^{-1}(\mathfrak{b}(|V|))\cdot\mathcal{O}_Y \cong \mathcal{O}_Y(-E)$. Applying (7.7), we have

$$\sigma^*(H)^n = (\pi^*(dH)-E)^n = 1, \tag{7.18}$$

where H is a hyperplane in $\mathbb{P}^n$.

Applying Proposition 7.1.1, we obtain

Proposition 7.1.6. *The base ideal $\mathfrak{b}$ of a homaloidal linear system is integrally closed and its proper inverse transform on a resolution of indeterminacy points coincides with the complete linear system $|\pi^*\mathcal{O}_{\mathbb{P}^n}(d)(-F)|$, where $\pi^{-1}(\mathfrak{b}) = \mathcal{O}_Y(-F)$.*

Let us specialize the previous Proposition to the case of a Cremona transformation $f : \mathbb{P}^2 - \to \mathbb{P}^2$ of degree d. In this case $\mathcal{L}' \cong \mathcal{O}_{\mathbb{P}^2}(1)$ and $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^2}(d)$. Since $\omega_{\mathbb{P}^2} \cong \mathcal{O}_{\mathbb{P}^2}(-3)$, applying Proposition 7.1.5 gives

Proposition 7.1.7.

$$1 = d^2 - \sum_{i=1}^{N} m_i^2, \tag{7.19}$$

$$3 = 3d - \sum_{i=1}^{N} m_i. \tag{7.20}$$

It follows from Proposition 7.1.6 that a homaloidal linear system is equal to a linear system $|\mathcal{O}_{\mathbb{P}^2}(d) - \eta|$. The bubble cycle η is called the bubble cycle the homaloidal net or of the Cremona transformation it defines.

Theorem 7.1.8. *A bubble cycle $\eta = \sum_{i=1}^N m_i x_i$ on $\mathbb{P}^2$ is equal to the bubble cycle of a homaloidal net of degree d if and only if $|\mathcal{O}_{\mathbb{P}^2}(d) - \eta|$ contains an irreducible divisor and equalities (7.19) and (7.20) hold.*

Proof. We have already proved the necessity of the conditions. Consider the linear system $|V| = |\mathcal{O}_{\mathbb{P}^2}(d) - \eta|$. It follows from the conditions that

$$\tfrac{1}{2}d(d+1) - 2 = \tfrac{1}{2}\sum m_i(m_i+1).$$

Applying the Hoskin-Deligne formula and (7.16), we obtain that $\dim |V| \ge 2$. By assumption, the linear system $|V|$ has no fixed components. Let $\pi : Y = Y_\eta \to \mathbb{P}^2$ and $\pi^{-1}(|V|)$ be the proper transform of $|V|$ on Y_η. It has no base points and defines a regular map $\sigma : Y_\eta \to \mathbb{P}^n, n \ge 2$, which resolves ϕ. By (7.19) and (7.18) this map is birational on its image X' and $\deg X' = 1$. This proves the assertion. □

The vector $(d; m_1, \ldots, m_N)$ is called the *characteristic* of the homaloidal net. It depends on an admissible order of the bubble cycle η.

Of course, not any vector $(d; m_1, \ldots, m_N)$ satisfying equalities (7.19) and (7.20) is realized as the characteristic vector of a homaloidal net. There are other necessary conditions for a vector to be realized as the characteristic $(d; m_1, \ldots, m_N)$ for a homaloidal net. For example, if m_1, m_2 correspond to points of height 0 of largest multiplicity, a line through the points should intersect a general member of the net non-negatively. This gives the inequality

$$d \ge m_1 + m_2.$$

Next we take a conic through 5 points with maximal multiplicities. We get

$$2d \ge m_1 + \cdots + m_5.$$

Then we take cubics through 9 points, quartics through 14 points and so on. The first case which can be ruled out in this way is $(5; 3, 3, 1, 1, 1, 1, 1)$. It satisfies the equalities from the Theorem but does not satisfy the condition $m \geq m_1 + m_2$. We will discuss the description of characteristic vectors later in this chapter.

7.1.5 Nets of isologues and fixed points

Let $\phi : \mathbb{P}^2 - \to \mathbb{P}^2$ be a Cremona transformation. Let p be a point in the plane. Consider the locus of points $C_\phi(p)$ such that $x, \phi(x), p$ are collinear. This locus is called *isologue* of p, the point p is called its *center*. In terms of equations, if ϕ is given by polynomials $(f_0(t), f_1(t), f_2(t))$ of degree d and $p = (a_0, a_1, a_2)$, then

$$C_\phi(p) : \det \begin{pmatrix} a_0 & a_1 & a_2 \\ t_0 & t_1 & t_2 \\ f_0(t) & f_1(t) & f_2(t) \end{pmatrix} = 0. \tag{7.21}$$

It follows immediately that $\deg C_\phi(p) = d + 1$ unless $C_\phi(p) = \mathbb{P}^2$. As we will see later, this happens for special De Jonquières transformations. From now on we assume that this is not the case for any point p. Then $C_\phi(p)$ is a curve of degree $d+1$. It passes through the base points of ϕ (because the last row in the determinant is identical zero for such point) and it passes through the *fixed points* of ϕ, i.e. points $x \in \mathrm{dom}(\phi)$ such that $\phi(x) = x$ (because the last two rows are proportional). Also $C_\phi(p)$ contains its center p (because the first two rows are proportional).

One more observation is that

$$C_\phi(p) = C_{\phi^{-1}}(p).$$

When p varies in the plane we obtain a *net of isologues*. If F is the one-dimensional component of the set of fixed points, then F is a fixed component of the net of isologues.

Remark 7.1.1. It follows from the definition that the isologue curve $C(p)$ is *projectively generated* by the pencil of lines ℓ through p and the pencil of curves $\phi^{-1}(\ell)$. Recall that given two pencils $\mathcal{P}$ and $\mathcal{P}'$ of plane curve of degree d_1 and d_2 and a projective isomorphism $\alpha : \mathcal{P} \to \mathcal{P}'$, the union of points $Q \cap \alpha(Q), Q \in \mathcal{P}$, is a plane curve C. Assuming that the pencils have no common base points, C is a plane curve of degree $d_1 + d_2$. To see this we take a general line ℓ and restrict $\mathcal{P}$ and $\mathcal{P}'$ to it. We obtain two linear series g^1_d and $g^1_{d'}$ on ℓ. The intersection $C \cap \ell$ consists of points common to divisors from g^1_d and $g^1_{d'}$. The number of such points is equal to the intersection of the diagonal of $\mathbb{P}^1 \times \mathbb{P}^1$ with a curve of bidegree (d, d'), hence it is equal to $d + d'$. It follows from the definition that C contains the base points of the both pencils.

Proposition 7.1.9. *Assume that ϕ has no infinitely near base points. Then the multiplicity of a general isologue curve at a base point x of multiplicity m is equal to m.*

Proof. Let u, v be local affine parameters at x. For each homogeneous polynomial $p(t_0, t_1, t_2)$ vanishing at x with multiplicity $\geq m$, let $[p]_k := [p]_k(u, v)$ be the degree

k homogeneous term in the Taylor expansion at x. If $V(f)$ is a general member of the homaloidal net, then $[f]_k = 0$ for $k < m$ and $[f_m] \neq 0$. Let B_m be the space of binary forms of degree m in variables u, v. Consider the linear map

$$\alpha : \mathbb{C}^3 \to B_m,\ (a,b,c) \mapsto [(bt_2 - ct_1)f_0(t) + (ct_0 - at_2)f_1(t) + (at_1 - bt_0)f_2(t)]_m.$$

The map is the composition of the linear map $\mathbb{C}^3 \to \mathbb{C}^3$ defined by $(a,b,c) \mapsto ([bt_2 - ct_1]_0, [ct_0 - at_2]_0, [at_1 - bt_0]_0)$ and the linear map $\mathbb{C}^3 \to B_m$ defined by $(a,b,c) \mapsto [af_0 + bf_1 + cf_2]_m$. The rank of the first map is equal to 2, the kernel is generated by $([t_0]_0, [t_1]_0, [t_2]_0)$. Since no infinitely near point is a base point of the homaloidal net, the rank of the second map is greater or equal than 2. This implies that the map α is not the zero map. Hence there exists an isologue curve of multiplicity equal to m. $\square$

Remark 7.1.2. Coolidge claims in [94], p. 460, that the assertion is true even in the case of infinitely near points. The following example shows that this is wrong. Consider the quadratic transformation defined by $(f_0, f_1, f_2) = (t_0t_2 + t_2^2, t_1t_2 + t_0^2, t_2^2)$. It has one base point $(0,1,0)$ and two infinitely near points, all of multiplicity 1. In affine coordinates $x = t_0/t_1, y = t_2/t_1$ the equations of the curves are $(xy + y^2, x^2 + y, y^2)$. The affine equations of the isologue curves are linear combinations of the minors of the matrix

$$\begin{pmatrix} x & 1 & y \\ xy + y^2 & x^2 + y & y^2 \end{pmatrix}.$$

The minors are $x^3 - y^2, -x^2, -y^3$. We see that the multiplicity of all isologue curves at the base point $(0,1,0)$ are equal to 2.

Corollary 7.1.10. *Assume that the homaloidal net has no infinitely near base points and the net of isologues has no fixed component. Then the number of fixed points of ϕ is equal to $d + 2$, counting with appropriate multiplicities.*

Proof. Take two general points p, q in the plane. In particular, the line $\ell = \overline{p, q}$ does not pass through the base points of the homaloidal net and the fixed points. Also $p \neq C_\phi(q)$ and $q \notin C_\phi(p)$. Consider a point x in the intersection $C_\phi(p) \cap C_\phi(q)$ which is neither a base point nor a fixed point. Then $p, q \in \overline{x, \phi(x)}$, hence $x \in \ell \cap C_\phi(p) \cap C_\phi(q)$. Conversely, if $x \in \ell \cap C_\phi(p)$ and $x \neq p$, then $x, \phi(x), p$ are collinear and, since $q \in \ell$, we get that $x, \phi(x), q$ are collinear. This implies that $x \in C_\phi(q)$. This shows that the base points of the pencil of isologue curves $C_\phi(p), p \in \ell$, consists of base points of the homaloidal net, fixed points and d points on ℓ (counted with multiplicities). The base points of the homaloidal net contribute $\sum_{i=1}^N m_i^2$ to the intersection. Applying Theorem 7.1.8, we obtain that fixed points contribute $d + 2 = (d+1)^2 - d - \sum_{i=1}^N m_i^2$ to the intersection. The multiplicity of a fixed points is the index of intersection of two general isologue curves. $\square$

Note that the Cremona transformation from Remark 7.1.2 has no fixed points.

Remark 7.1.3. The assumption that ϕ has no infinitely near points implies that the graph Γ of ϕ is a nonsingular surface in $\mathbb{P}^2 \times \mathbb{P}^2$ isomorphic to the blow-up of the base scheme of the homaloidal net. Let h_1, h_2 be the preimages of the cohomology classes of lines under the projections. They generate the cohomology ring $H^*(\mathbb{P}^2 \times \mathbb{P}^2, \mathbb{Z})$.

Let $[\Gamma]$ be the cohomology class of Γ and $[\Delta]$ be the cohomology class of the diagonal Δ. Write $[\Gamma] = ah_1^2 + bh_1h_2 + ch_2^2$. Since the preimage of a general point under ϕ is a point, we have $[\Gamma] \cdot h_2^2 = 1$. Replacing ϕ with ϕ^{-1}, we get $[\Gamma] \cdot h_1^2 = 1$. Since a general line intersects the preimage of a general line at d points we get $[\Gamma] \cdot h_1 \cdot h_2 = d$. This gives

$$[\Gamma] = h_1^2 + dh_1h_2 + h_2^2. \tag{7.22}$$

Similarly, we get

$$[\Delta] = h_1^2 + h_1h_2 + h_2^2. \tag{7.23}$$

This implies that

$$[\Gamma] \cdot [\Delta] = d + 2.$$

This confirms the assertion of the previous Corollary. In fact, one can use the argument for another proof of the Corollary if we assume (that follows from the Corollary) that no point in the intersection $\Gamma \cap \Delta$ lies on the exceptional curves of the projections.

The net of isologue curves without fixed components is a special case of a *Laguerre net*. It is an irreducible net of plane curves of degree d generated by the curves $V(f_0), V(f_1), V(f_2)$ such that

$$t_0f_1(t) + t_1f_2(t) + t_2f_(t) = 0. \tag{7.24}$$

Replacing the first row in the determinant defining an isologue curve with x_0, x_1, x_2 we see that the net of isologue curves is a Laguerre net.

Take two general curves $C_\lambda = V(\lambda_0f_0 + \lambda_1f_1 + \lambda_2f_2)$ and $C_\mu = V(\mu_0f_0 + \mu_1f_1 + \mu_2f_2)$ from the net. Let $p = [a_0, a_1, a_2]$ belong to $C_\lambda \cap C_\mu$. Assume that p is not a base point. Then $(f_0(a), f_1(a), f_2(a))$ is a nontrivial solution of the system of linear equations with the matrix of coefficients equal to

$$\begin{pmatrix} \lambda_0 & \lambda_1 & \lambda_2 \\ \mu_0 & \mu_1 & \mu_2 \\ a_0 & a_1 & a_2 \end{pmatrix}.$$

This implies that the points $\lambda = [\lambda_0, \lambda_1, \lambda_2], \mu = [\mu_0, \mu_1, \mu_2], a = [a_0, a_1, a_2]$ are collinear. Thus all intersection points of C_λ and C_μ besides base points lie on the line. Conversely, suppose a non-base point point $a \neq \lambda, \mu$ lies on a line $\overline{\lambda, \mu}$ and belongs to the curve C_λ. Then $(f_0(a), f_1(a), f_2(a))$ is a non-trivial solution of

$$\lambda_0t_0 + \lambda_1t_1 + \lambda_2t_2 = 0,\ a_0t_0 + a_1t_1 + a_2t_2 = 0,$$

hence satisfies the third equation $\mu_0t_0 + \mu_1t_1 + \mu_2t_2 = 0$. This shows that $a \in C_\lambda \cap C_\mu$. Thus we see that the intersection $C_\lambda \cap C_\mu$ consists of $d - 1$ non-base points hence the number of base points counting with multiplicities is equal to $d^2 - d + 1$.

Now let $\mathcal{N}$ be an irreducible net of plane curves of degree d with the property that any two its general members intersect at $d - 1$ collinear points outside the base locus. Let us see that $\mathcal{N}$ is a Laguerre net. We follow the proof from [94], p. 423. Let $V(f_1), V(f_2)$ be two general members intersecting at $d - 1$ points on a line $l = 0$ not passing through base points. Let p_i be the residual point on $V(f_i)$. Choose a general line $l_1 = 0$ passing through p_2 and a general line $l_2 = 0$ passing through p_1. Then $V(l_1f_1)$ and $V(l_2f_2)$ contain the same set of $d+1$ points on the line $l = 0$, hence we can write

$$l_1f_1 + cl_2f_2 = lf_3 \tag{7.25}$$

for some polynomial f_3 of degree d and some constant c. For any base point q of the net, we have $l_1(q)f_1(q) + cl_2(q)f_2(q) = l(q)f_3(q)$. Since $l(q) \neq 0$ and $f_1(q) = f_2(q) = 0$, we obtain that $f_3(q) = 0$. Thus the curve $V(f_3)$ passes through each base point and hence belongs to the net $\mathcal{N}$. This shows that f_1, f_2 and f_3 define a basis of $\mathcal{N}$ satisfying (7.25). Changing the basis of the net, we may assume that $l_1 = t_0, cl_2 = t_1, -l = t_2$ and $t_0f_1 + t_1f_2 + t_2f_3 = 0$ proving that $\mathcal{N}$ is a Laguerre net.

Example 7.1.1. Take a net of cubic curves with 7 base points. Then it is a Laguerre net since two residual intersection points of any two general members are on a line. One can prove this invoking the Hilbert-Burch Theorem 9.3.6. Applying this Theorem we obtain that the homogeneous ideal is generated by the maximal minors of a matrix

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ \phi_1 & \phi_2 & \phi_3 \end{pmatrix},$$

where l_i are linear forms and ϕ_i are quadratic forms. Since the minors f_i, in appropriate order, satisfy the equation $l_1f_1 - l_2f_2 + l_3f_3 = 0$, we obtain that the net is a Laguerre net.

7.2 First examples

7.2.1 Involutorial quadratic transformations

Take $d = 2$. We find $\sum m_i^2 = 1, \sum m_i = 3$. This easily implies $m_1 = m_2 = m_3 = 1, N = 3$. The birational transformation of this type is called a *quadratic transformation*. The homaloidal linear system consists of conics passing through a bubble cycle $x_1 + x_2 + x_3$. We have encountered these transformations in section 4.1.5

Assume x_1, x_2, x_3 are proper points. They are not collinear, since otherwise all conics have a common line component. Let g be a projective transformation which sends the points x_1, x_2, x_3 to the points $p_1 = [0, 0, 1], p_2 = [0, 1, 0], p_3 = [1, 0, 0]$. Then $T \circ \sigma^{-1}$ is given by the linear system of conics through the points p_1, p_2, p_3. We can choose a basis formed by the conics $V(t_1t_2), V(t_0t_2), V(t_0t_1)$. The corresponding Cremona transformation is given by the formula

$$\tau_1 : [t_0, t_1, t_2] \mapsto [t_1t_2, t_0t_2, t_0t_1]. \tag{7.26}$$

In affine coordinates $z_1 = t_1/t_0, z_2 = t_2/t_0$, the transformation is given by

$$(x, y) \mapsto (x^{-1}, y^{-1}). \tag{7.27}$$

Thus any quadratic transformation with no infinitely near base points is equal to $\sigma \circ \tau_1 \circ \sigma'$ for some projective transformations σ, σ'. Note that

$$\tau_1 \circ \tau_1 : [t_0, t_1, t_2] \mapsto [t_0t_2t_0t_1, t_1t_2t_0t_1, t_1t_2t_0t_2) = t_0t_1t_2[t_0, t_1, t_2] = [t_0, t_1, t_2].$$

Thus τ_1 is an involution. However, in general, $\sigma \circ \tau_1 \circ \sigma'$ is not an involution. The transformation τ_1 is called the *standard quadratic transformation*.

Assume now that x_1 and x_2 are proper points and $x_3 \succ_1 x_1$. Again, after a linear change of variables, we may assume that $x_1 = [0,0,1], x_2 = [1,0,0]$ and x_2 corresponds to the tangent direction $t_0 = 0$. The homaloidal linear system consists of conics which pass through x_1, x_3 and have a common tangent $t_0 = 0$ at x_1. We can take a basis formed by the conics $V(t_0t_2), V(t_0t_1), V(t_1^2)$. The corresponding Cremona transformation is given by the formula

$$\tau_2 : [t_0, t_1, t_2] \mapsto [t_1^2, t_0t_1, t_0t_2]. \tag{7.28}$$

Any quadratic transformation with one infinitely near base point is equal to $\sigma \circ \tau_2 \circ \sigma'$ for some projective transformations σ, σ'.

In the affine coordinates as above, the transformation is given by

$$(x, y) \mapsto (x^{-1}, yx^{-2}). \tag{7.29}$$

Assume now that $x_3 \succ x_2 \succ x_1$. By a linear change of variables we may assume that $x_1 = [0,0,1]$, x_2 corresponds to the tangent direction $t_0 = 0$, and x_3 lies on the proper transform of the line $t_2 = 0$. The homaloidal linear system consists of conics which pass through x_1 and have a common tangent $t_0 = 0$, and after blowing up x_1 still intersect at one point. We can take a basis formed by the conics $V(t_0t_2 - t_1^2), V(t_0^2), V(t_0t_1)$. The corresponding Cremona transformation is given by the formula

$$\tau_3 : [t_0, t_1, t_2] \mapsto [t_0^2, t_0t_1, t_1^2 - t_0t_2]. \tag{7.30}$$

Any quadratic transformation with one infinitely near base point is equal to $\sigma \circ t_2 \circ \sigma'$ for some projective transformations σ, σ'.

In affine coordinates, the transformation is given by

$$(x, y) \mapsto (x, x^2 - y). \tag{7.31}$$

Note that the case $x_2 \succ_1 x_1, x_3 \succ_1 x_1$ is not realized since a general member of the linear system is singular at x_1.

It is easy to see that the quadratic transformations τ_i are involutorial.

Proposition 7.2.1. *Let ϕ be an involutorial quadratic Cremona transformation. Then there exists a projective transformation g such that $g \circ \phi \circ g^{-1} = \tau_i$ for some $i = 1, 2, 3$.*

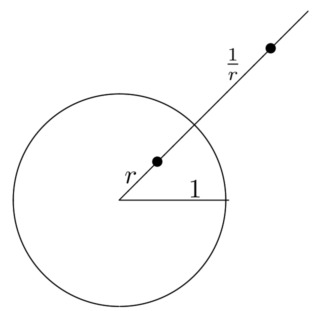
Proof. We assume that ϕ has no infinitely near base point and prove that $i = 1$ in this case. We leave other cases to the reader. Let p_1, p_2, p_3 be the base points. Choose a projective transformation g such that $g(p_1) = [1,0,0], g(p_2) = [0,1,0], g(p_3) = [0,0,1]$. Then $\phi' = g \circ \phi \circ g^{-1}$ is an involution and has the base points $[1,0,0], [0,1,0], [0,0,1]$. We can choose a basis of the homaloidal net of conics through these points in the form (t_1t_2, t_0t_2, t_0t_1). This shows that the transformation ϕ' is given by the formula

$$\phi'(x) = [a_1x_1x_2 + b_1x_0x_2 + c_1x_0x_1, a_2x_1x_2 + b_2x_0x_2 + c_2x_0x_1, a_3x_1x_2 + b_3x_0x_2 + c_3x_0x_1].$$

The image of the line t_0 is the point $[a_1, a_2, a_3]$. Since ϕ' is an involution, this point must be a base point of ϕ'. Similarly, we obtain that the points $[b_1, b_2, b_3]$ and $[c_1, c_2, c_3]$

are base points. Thus we may assume that the transformation $\phi' = \sigma \circ \tau_1$, where σ is the projective transformation which permutes the coordinates. It is directly verified by iteration that σ must be the identity. $\square$

Example 7.2.1. The first historical example of a Cremona transformation is the *inversion* map. Recall the inversion transformation from the plane geometry. Given a circle of radius R, a point $x \in \mathbb{R}^2$ with distance r from the center of the circle is mapped to the point on the same ray at the distance R/r. The following picture illustrates this in the case $R = 1$.



In the affine plane $\mathbb{C}^2$ the transformation is given by the formula

$$(x, y) \mapsto (\frac{Rx}{x^2 + y^2}, \frac{Ry}{x^2 + y^2}).$$

In projective coordinates, the transformation is given by the formula

$$(x_0, x_1, x_2) \mapsto (x_1^2 + x_2^2, Rx_1x_0, Rx_2x_0).$$

Note that the transformation has three base points $[1, 0, 0], [0, 1, i], [0, 1, -i]$. It is an involution and transforms lines not passing through the base points to conics (circles in the real affine plane). The lines passing though one of the base points are transformed to lines. The lines passing through the origin $(1, 0, 0)$ are invariant under the transformation. The conic $x_1^2 + x_2^2 - Rx_0^2 = 0$ is the closure of the set of fixed points.

Example 7.2.2. Let C_1 and C_2 be two conics intersecting at 4 distinct points. For each general point x in the plane let $\phi(x)$ be the intersection of the polar lines $P_x(C_1)$ and $P_x(C_2)$. Let us see that this defines an involutorial quadratic transformation with base points equal to the singular points of three reducible conics in the pencil generated by C_1 and C_2. It is clear that the transformation ϕ is given by three quadratic polynomials. Since $P_x(C_1) \cap P_x(C_2)$ is equal to $P_x(C) \cap P_x(C')$ for any two different members of the pencil, taking C to be a reducible conic and x to be its singular point, we obtain that ϕ is not defined at ϕ. Since the pencil contains three reducible members, we obtain that ϕ has three base points, hence ϕ is given by a homaloidal net and hence is a birational map. Obviously, $x \in P_{\phi(x)}(C_1) \cap P_{\phi(x)}(C_2)$, hence ϕ is an involution. Note that fixed points of the transformation are the base points of the pencil of conics.

7.2.2 Symmetric Cremona transformations

Assume that the bubble cycle defining the homaloidal net η consists of points taken with equal multiplicity m. The Cremona transformations defined by such a bubble cycle are called *symmetric*. Then the necessary conditions are

$$d^2 - Nm^2 = 1, \quad 3d - Nm = 3.$$

Multiplying the second equality by m and subtracting from the first one, we obtain $d^2 - 3dm = 1 - 3m$. This gives $(d-1)(d+1) = 3m(d-1)$. The case $d = 1$ corresponds to a projective transformation. Assume $d > 1$. Then we get $d = 3m - 1$ and hence $3(3m-1) - Nm = 3$. Finally, we obtain

$$(9 - N)m = 6, \quad d = 3m - 1.$$

This gives us 4 cases.

(i) $m = 1, N = 3, d = 2$;

(ii) $m = 2, N = 6, d = 5$;

(ii) $m = 3, N = 7, d = 8$;

(iii) $m = 6, N = 8, d = 17$.

The first case is obviously realized by a quadratic transformation with 3 fundamental points.

The second case is realized by the linear system of plane curves of degree 5 with 6 double points. Take a bubble cycle $\eta = 2x_1 + \cdots + 2x_6$, where the points x_i in the bubble space do not lie on a proper transforms of a conic and no three lie on the proper transforms of a line. I claim that the linear system $|V| = |\mathcal{O}_{\mathbb{P}^2}(2) - \eta|$ is homaloidal. The space of plane quintics is of dimension 20. The number of conditions for passing through a point with multiplicity ≥ 2 is equal to 3. Thus $\dim |\mathcal{O}_{\mathbb{P}^2}(2) - \eta| \geq 2$. It is easy to see that the linear system does not have fixed components. For example, if the fixed component is a line, it cannot pass through more than 2 points, hence the residual components are quartics with 4 double points, obviously reducible. If the fixed component is a conic, then it passes through at most 5 points, hence the residual components are cubics with at least one double points and passing through the remaining points. It is easy to see that the dimension of such linear system is at most 1. If the fixed component is a cubic, then by the previous analysis we may assume that it is irreducible. Since it has at most one singular point, the residual conics pass through at least 5 points and the dimension of the linear system is equal to zero (or it is empty). Finally, if the fixed component is a quartic, then the residual components are lines passing through 3 points, again a contradiction.

Applying Bezout's Theorem, we see that two general members of our linear system intersect at 1 point outside of the base locus, also their genus is equal to 0. Thus the linear system is a homaloidal.

Assume that all base points are proper points in the plane. Then the linear system blows down the six conics, each passing through 5 base points.

A homaloidal cycle of type (iii) is realized by a *Geiser involution*. We consider an irreducible net $\mathcal{N}$ of cubic curves through 7 general points $x_1, \ldots, x_7$ in the plane. For any general point x the subpencil of the net which consists of cubics passing through x has the base points $x_1, \ldots, x_7, x, y$. The Geiser transformation assigns to x the point y. It is clear that this transformation is involutorial and its base points are $x_1, \ldots, x_7$. To determine its degree, consider the rational map $f : \mathbb{P}^2 - \to \mathcal{N}^\vee$ given by the net. We have used this map already in Proposition 6.3.5 from Chapter 6. A pencil in $\mathcal{N}$ is a point in $\mathcal{N}^\vee$ and its preimage consists of the base points of the net outside of the base locus of $\mathcal{N}$. Thus f is of degree 2. The restriction of f to a general line ℓ is given by the linear series of degree 3 and dimension 2, hence $C = f(\ell)$ is a rational cubic curve. The preimage of a general line in $\mathcal{N}^\vee$ is a member of $\mathcal{N}$, i.e. a cubic through the base points. The preimage of C is a curve of degree 9 passing through the base points with multiplicity 3. It consists of the union of ℓ and the curve $\phi(\ell)$. Thus $\deg \phi(\ell)$ is equal to 8. Since ϕ is an involution, $\phi(\ell)$ is a general member of the homaloidal net defining the Geiser involution.

A homaloidal cycle of type (iv) is realized by a *Bertini involution*. We consider a pencil of cubic curves through a general set of 8 points $x_1, \ldots, x_8$. Let q be its ninth base point. For any general point x in the plane let $E(p)$ be the member of the pencil containing x. Let $\phi(x)$ be the residual point in the intersection of $E(p)$ with the line $\overline{x, q}$. The transformation $x \to \phi(x)$ is the Bertini involution. If we take q as the origin in the group law on a nonsingular $E(p)$, then $\phi(x) = -x$.

Consider the web $\mathcal{N}$ of curves of degree 6 whose general member passes through each point p_i with multiplicity 2. The restriction of $\mathcal{N}$ to any $E(p)$ is a pencil with fixed part $2p_1 + \cdots + 2p_8$ and the moving part g_2^1. One of the members of this g_2^1 is the divisor $2q$ cut out by $2E(p'), p \neq p'$. As we have seen in section 6.3.3 of Chapter 6, the members of this pencil are cut out by lines through the coresidual point on $E(p)$. This point must coincide with the base point of the pencil. Thus members of the pencil are divisors $x + \phi(x)$. Now we use that $\mathcal{N}$ defines a degree 2 rational map $f : \mathbb{P}^2 - \to Q \subset \mathbb{P}^3$, where Q is a singular irreducible quadric in $\mathbb{P}^3$. The image of the point q is equal to the singular point of Q. The restriction of f to a general line ℓ is given by the linear system of dimension 3 and degree 3. Its image of a line in Q is a rational curve R of degree 6 intersecting each line on Q at 3 points and not passing through the singular point of Q. It is easy to see that it is a singular curve of arithmetic genus 4 cut out by a cubic hypersurface. Since the preimage of a hyperplane section under f is a curve from $\mathcal{N}$, the preimage of R is a curve of degree 18 passing through the base points of $\mathcal{N}$ with multiplicities 6. As in the previous case, we see that the preimage of R is equal to the union of ℓ and $\phi(\ell)$. Since $\phi(\ell)$ is a member of the homaloidal linear system defining ϕ, we obtain that the characteristic of ϕ is equal to $(17, 6, 6, 6, 6, 6, 6, 6, 6)$.

7.2.3 De Jonquières transformations

Assume that there exists a point q in the support of η with multiplicity $d - 1$. We have

$$d^2 - (d-1)^2 - \sum_{i=2}^{N} = 1, \quad 3d - (d-1) - \sum_{i=2}^{N} = 3.$$

This easily implies $\sum_{i=2}^{N} m_i(m_i - 1) = 0$, hence

$$m_2 = \ldots = m_N = 1, \quad N = 2d - 1.$$

For simplicity of the exposition we assume that the base scheme is reduced, i.e. all base points are proper. The homaloidal system must consist of curves of degree d with singular point q of multiplicity $d-1$ (*monoidal curves*) passing simply through $2d-2$ points $x_1, \ldots, x_{2d-2}$. The corresponding Cremona transformation is called *De Jonquières transformation*.

Changing the projective coordinates in the source plane, we may assume that $q = [0,0,1]$. Then the equation of a curve from the homaloidal linear system must look like

$$f(t_0, t_1, t_2) = t_2 f_{d-1}(t_0, t_1) + f_d(t_0, t_1) = 0. \tag{7.32}$$

Since a general curve from the homaloidal linear system intersects a line through the points $q, x_i \neq x_j$ with degree $d+1 > d$, no such line can exist. The base points satisfy the condition

- no three points $q, x_i, x_j, 1 < i < j$ are collinear.

Let us see that this condition is sufficient for the existence of the homaloidal net. Counting constants, we see that the linear system curves of degree $d-1$ passing through the points $x_1, \ldots, x_{2d-2}$ with point of multiplicity $d-2$ at q is non-empty and its expected dimension is equal to zero. If it contains an irreducible curve, Bezout's Theorem implies that the linear system consists of this curve. Suppose there is an reducible curve C in the linear system. A general line through the point q intersects C at one point $p \neq q$. This implies that $C = C_1 + \ell_1 + \cdots + \ell_k$, where C_1 is an irreducible curve of degree $d-1-k$ and ℓ_i are lines passing through q. It follows from the assumption that C_1 passes through at least $2d-2-k$ points x_i's. Let us assume that it does not happen, i.e.

- no proper subset of $2d-2-k$ points x_i's lie on an irreducible curve of degree $d-1-k$ with singular point q of multiplicity $d-2-k$.

One can show that this condition is equivalent to the condition that the inverse of the transformation has no infinitely near base points.

Let Γ be the unique irreducible curve of degree $d-1$ with singular point of multiplicity $d-2$ at q and passing through the points $x_1, \ldots, x_{2d-2}$. Its equation must be of the form

$$g(t_0, t_1, t_2) = t_2 g_{d-2}(t_0, t_1) + g_{d-1}(t_0, t_1) = 0, \tag{7.33}$$

where the subscript indicates the degree of the binary form. The union of this curve and the pencil of lines through q is a pencil contained in the homaloidal net. Let $V(f)$, where f as in (7.32), be a curve from the net which is not contained in the pencil. The Cremona transformation $\phi : \mathbb{P}^2 - \to \mathbb{P}^2$ defined by the homaloidal net is obtained by a choice of a basis in the homaloidal net. Consider the Cremona transformation

$$\phi : [t_0, t_1, t_2] \mapsto [t_0 g(t_0, t_1, t_2), t_1 g(t_0, t_1, t_2), f(t_0, t_1, t_2)] \tag{7.34}$$

defined by the choice of a special basis. Any other transformation with the same homaloidal net is equal to the composition $s \circ \phi$, where s is a projective transformation.

It is easy to see that $s \circ \phi$ transforms the pencil of lines through q to the pencil of lines through the point $s(q)$. A De Jonquières transformation is called *special* if $s(q) = q$. A special transformation is given by the formula

$$[t_0, t_1, t_2] \mapsto [(a_1 t_0 + a_2 t_1)g, (a_3 t_0 + a_4 t_1)g, (a_5 t_0 + a_6 t_1)g + a_7 f]. \tag{7.35}$$

In affine coordinates $x = t_1/t_0, y = t_2/t_0$ it is given by the formula

$$(x, y) \mapsto (\frac{ax + b}{cx + d}, \frac{r_1(x)y + r_2(x)}{r_3(x)y + r_4(x)}, \tag{7.36}$$

where $r_i(x)$ are certain rational functions in x. All such transformations form a subgroup of the $\mathrm{Cr}(2)$. It is called a *De Jonquirères subgroup*. Conversely, if a Cremona transformation leaves invariant a pencil of lines, then it can be considered as an automorphism of the field $\mathbb{C}(x, y)$ leaving invariant the subfield $\mathbb{C}(x)$. It is easy to see that it can be given by a formula (7.36). After homogenizing, we get a formula of type (7.35).

It is easy to invert the transformation ϕ defined by formula (7.34). We find that ϕ^{-1} is a De Jonquières transformation given by the formula

$$[t_0, t_1, t_2] \mapsto [t_0 g'(t_0, t_1, t_2), t_1 g'(t_0, t_1, t_2), f'_d(t_0, t_1, t_2)], \tag{7.37}$$

where

$$\begin{aligned} g'(t_0, t_1, t_2) &= t_2 g_{d-2}(t_0, t_1) - f_{d-1}(t_0, t_1) \\ f'(t_0, t_1, t_2) &= -t_2 g_{d-1}(t_0, t_1) + f_d(t_0, t_1). \end{aligned}$$

Observe that ϕ^{-1} is also an De Jonquiéres transformation. Note that, if $f_{d-1} = -g_{d-1}$, the transformation is an involution.

Note the following properties of a De Jonquirères transformation. The lines $\overline{p_1, p_i}$ are blown down to $2d-2$ points $q_1, \ldots, q_{2d-2}$. The curve Γ is blown down to a point q'. If we resolve the map by $\pi : X \to \mathbb{P}^2$, then the exceptional curve $\pi^{-1}(q)$ is mapped to the curve Γ' of order $d-1$ with $(d-2)$-multiple point y_1. The exceptional curves $\pi^{-1}(p_i)$ are mapped to lines $\overline{q', y_i}$.

It is easy to see that the net of isologues of a De Jonquir̀es transformation is defined unless it is a special De Jonquir̀es transformation. Thus it has $d+2$ fixed points. Let us find the locus of fixed points of the special transformation T given by (7.34).

$$[t_0, t_1, t_2] \mapsto [(a_1 t_0 + a_2 t_1)g + b_1 f, (a_3 t_0 + a_4 t_1)g + b_2 f, (a_5 t_0 + a_6 t_1)g + a_7 f)]. \tag{7.38}$$

They satisfy

$$\operatorname{rank} \begin{pmatrix} t_0 & t_1 & t_2 \\ t_0 g & t_1 g & f \end{pmatrix} = 1. \tag{7.39}$$

Since we are excluding the point $[0, 0, 1]$, this condition is equivalent to the equation

$$t_2 g - f = t_2^2 g_{d-2}(t_0, t_1) + t_2 g_{d-1}(t_0, t_1) = t_2 f_{d-1}(t_0, t_1) + f_d(t_0, t_1). \tag{7.40}$$

The closure of this set is a plane curve X of degree d with a $(d-2)$-multiple point at q. It is birationally isomorphic to a hyperelliptic curve of genus $g = d-2$. The corresponding double cover $f : X \to \mathbb{P}^1$ is defined by the projection $[t_0, t_1, t_2] \mapsto [t_0, t_1]$. Its branch points are given by the discriminant of the quadratic equation (in the variable t_2):

$$D = (g_{d-1} - f_{d-1})^2 + 4f_d g_{d-2}.$$

We have $2d - 2 = 2g + 2$ points as expected.

A space construction of a De Jonquières transformation due to Cremona [105]. Consider a rational curve R of bidegree $(1, d-2)$ on a nonsingular quadric Q in $\mathbb{P}^3$. Let L be a line on Q which intersects R at $d-2$ distinct points. For each point x in the space there exists a unique line joining a point on L and on R. In fact, the plane spanned by x and L intersects R at a unique point r outside $R \cap L$ and the line $\overline{x, r}$ intersects L at a unique point s. Take two general planes Π and Π' and consider the following birational transformation $\phi : \Pi - \to \Pi'$. Take a general point $p \in \Pi$, find the unique line joining a point $r \in R$ and a point $s \in L$. It intersects Π' at the point $\phi(p)$. For a general line ℓ in Π the union of lines $\overline{r, s}, r \in R, s \in L$, which intersect ℓ is a ruled surface of degree d. Its intersection with Π' is a curve of degree d. This shows that the transformation ϕ is of degree d. It has $2d-2$ simple base points. They are $d-1$ points in $\Pi' \cap R$ and $d-1$ points which are common to the line $\Pi \cap \Pi'$ and the $d-1$ lines joining the point $L \cap \Pi$ with the points in the intersection $\Pi \cap R$. Finally the point $L \cap \Pi'$ is a base point of multiplicity $n-1$. Identifying Π and Π' by means of an isomorphism, we obtain a De Jonquières transformation.

7.2.4 De Jonquières involutions and hyperelliptic curves

Let C be a hyperelliptic curve of genus g and g^1_2 be its linear system defining a degree 2 map to $\mathbb{P}^1$. Consider the linear system $|D| = |g^1_2 + a_1 + \cdots + a_g|$, where $a_1, \ldots, a_g \in C$. We assume that the divisor $D_1 = a_1 + \cdots + a_g$ is not contained in the linear system $|(g-2)g^1_2|$ or, equivalently, $|K_C - D| = \emptyset$. By Riemann-Roch, $\dim |D| = 2$, hence the linear system $|D|$ defines a map $\varphi : C \to \mathbb{P}^2$. The image of φ is a plane curve H_{g+2} of degree $g+2$ with a g-multiple point q, the image of the divisor D_1.

By choosing projective coordinates such that $q = [0, 0, 1]$, we can write H_{g+2} by an equation

$$t_2^2 f_g(t_0, t_1) + 2t_2 f_{g+1}(t_0, t_1) + f_{g+2}(t_0, t_1) = 0. \tag{7.41}$$

Let ℓ be a general line through q. It intersects H_{g+2} at two points a, b not equal to q. For any point $x \in l$ let y be the fourth point such that the pairs (a, b) and (x, y) are harmonically conjugate.

We would like to define a birational map $T : \mathbb{P}^2 - \to \mathbb{P}^2$ whose restriction to a general line through q takes a point to its harmonically conjugate. Notice that such map is not defined at the points where a line ℓ through q is tangent to H_{g+2}. it is also undefined at the point q. Let $x_1, \ldots, x_{2g+2}$ be the tangency points. They correspond to the ramification points of the double map $C \to \mathbb{P}^1$. It is a fair guess that the transformation T must be a De Jonquières transformation defined by the linear system

$$|(g+2)\ell - (g+1)q - x_1 - \ldots - x_{2g+2}| \tag{7.42}$$

and the curve H_{g+2} must be the curve of fixed points. Let us see this.

Consider the first polar of H_{g+2} with respect to the point q. Its equation is

$$t_2 f_g(t_0,t_1) + f_{g+1}(t_0,t_1) = 0.$$

We know that it passes through the tangency points $x_1,\ldots,x_{2g+2}$. Also it follows from the equation that it has a g-multiple point at q. It suggests that the first polar is the curve Γ which was used to define a De Jonquières transformation. Thus we take $d = g+2$ and $g_i = f_i$ in equation (7.33). To show that we get an involutorial transformation, we need to check that the curve $V(t_2 f_{g+1}(t_0,t_1) + f_{g+2}(t_0,t_1))$ belongs to the linear system (7.42). The points $x_i = [1,a_i,b_i]$ belong to the intersection of curves Γ and H_{g+2}. In appropriate coordinate system, we may assume that $b_i \neq 0$. Plugging $f_g(1,a_i) = -f_{g+1}(1,a_i)/b_i$ in the equation of H_{g+2}, we obtain

$$\begin{aligned} &b_i^2(\frac{-f_{g+1}(1,a_i)}{b_i}) + 2b_i f_{g+1}(1,a_i) + f_{g+2}(1,a_i) \\ &= b_i f_{g+1}(1,a_i) + f_{g+2}(1,a_i) = 0. \end{aligned}$$

Thus the curve given by the equation

$$t_2 f_{g+1}(t_0,t_1) + f_{g+2}(t_0,t_1) = 0$$

belongs to the linear system (7.42). So, we can define the De Jonquières transformation by the formula

$$\begin{aligned} t_0' &= t_0\big(t_2 f_g(t_0,t_1) + f_{g+1}(t_0,t_1)\big), \\ t_1' &= t_1\big(t_2 f_g(t_0,t_1) + f_{g+1}(t_0,t_1)\big), \\ t_2' &= -t_2 f_{g+1}(t_0,t_1) - f_{g+2}(t_0,t_1). \end{aligned} \tag{7.43}$$

This transformation is an involution. It follows from (7.40) that the curve of fixed points is the curve H_{g+2}. Its restriction to a line $l = V(t_1 - tt_0)$ is given by the formula

$$\begin{aligned} t_0' &= t_2 f_g(1,t) + t_0 f_{g+1}(1,t), \\ t_1' &= t\big(t_2 f_g(1,t) + t_0 f_{g+1}(1,t)\big), \\ t_2' &= -t_2 f_{g+1}(1,t) - t_0 f_{g+2}(1,t). \end{aligned}$$

In affine coordinates t_2/t_0 on the line $t_1 - tt_0 = 0$, the transformation is

$$x \mapsto y = \frac{-x f_{g+1}(1,t) - f_{g+2}(1,t)}{x f_g(1,t) + f_{g+1}(1,t)}.$$

This gives

$$xy f_g(1,t) + (x+y) f_{g+1}(1,t) + f_{g+2}(1,t) = 0. \tag{7.44}$$

The pair (x,y) satisfies the quadratic equation $z^2 - z(x+y) + xy = 0$ and the pair (a,b), where a,b are the points of intersection of the line ℓ with H_{g+2} satisfies the quadratic equation $z^2 f_g(1,t) + 2z f_{g+1}(1,t) + f_{g+2}(1,t) = 0$. It follows from the definition (2.1) of harmonical conjugates that equation (7.44) expresses the condition that the pairs (x,y) and (a,b) are harmonic.

Definition 7.3. *The Cremona transformation defined by the formula* (7.43) *is called the* De Jonquières involution *defined by the hyperelliptic curve H_{g+2}* (7.41). *It is denoted by IH_{g+2}.*

Remark 7.2.1. By conjugating the De Jonquières involution with a Cremona transformation given by the formula

$$(x_0', x_1', x_2') = (x_0^{g+1}, x_0^g x_1, f_g(x_0, x_1)x_2 + f_{g+1}(t_0, t_1),$$

we may assume that the hyperelliptic curve (7.41) is given by the equation

$$t_2^2 t_0^{2g} + f_{g+2}(t_0, t_1)f_g(t_0, t_1) - f_{g+1}(t_0, t_1)^2 = 0.$$

Formula (7.43) simplifies. In affine coordinates it is given by

$$y' = -\frac{f(x)}{y}, \quad x' = x, \tag{7.45}$$

where $f(x)$ is the dehomogenized polynomial $f_{g+2}(t_0, t_1)f_g(t_0, t_1) - f_{g+1}(t_0, t_1)^2$. For any polynomial f this defines an involutary Cremona transformation which is conjugate to IH_{g+2} for some g.

7.3 Elementary transformations

7.3.1 Segre-Hirzebruch minimal ruled surfaces

First let us recall the definition of a minimal rational ruled surface $\mathbf{F}_n$. If $n = 0$ this is the surface $\mathbb{P}^1 \times \mathbb{P}^1$. If $n = 1$ it is isomorphic to the blow-up of one point in $\mathbb{P}^2$ with the ruling $\pi : \mathbf{F}_1 \to \mathbb{P}^1$ defined by the pencil of lines through the point. If $n > 1$, we consider the cone in $\mathbb{P}^{n+1}$ over a Veronese curve $v_n(\mathbb{P}^1) \subset \mathbb{P}^n$, i.e., we identify $\mathbb{P}^{n-1}$ with a hyperplane in $\mathbb{P}^n$ and consider the union of lines joining a point not on the hyperplane with all points in $v_n(\mathbb{P}^1)$. The surface $\mathbf{F}_n$ is a minimal resolution of its vertex. The exceptional curve of the resolution is a smooth rational curve E_n with $E_n^2 = -n$. The projection from the vertex of the cone extends to a morphism $p : \mathbf{F}_n \to \mathbb{P}^1$ which defines a ruling (a $\mathbb{P}^1$-bundle). The curve E_n is its section, called the *exceptional section*. In the case $n = 1$, the exceptional curve E_1 of the blow-up $\mathbf{F}_1 \to \mathbb{P}^2$ is also a section of the corresponding ruling $p : \mathbf{F}_1 \to \mathbb{P}^1$. It is also called the exceptional section.

We will see a little later that the ruling $p : \mathbf{F}_n \to \mathbb{P}^1$ is a projective vector bundle isomorphic to the projectivization of the vector bundle $\mathbb{V}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n))$. Recall that for any locally free sheaf $\mathcal{E}$ of rank $r+1$ over a scheme S one defines the *vector bundle* $\mathbb{V}(\mathcal{E})$ as the scheme Spec(Sym($\mathcal{E}$)) (see [206]). A local section $U \to \mathbb{V}(\mathcal{E})$ is defined by a homomorphism Sym($\mathcal{E}$) $\to \mathcal{O}(U)$ of $\mathcal{O}(U)$-algebras, and hence by a linear map $\mathcal{E}|U \to \mathcal{O}(U)$. Thus the sheaf of local sections of the vector bundle $\mathbb{V}(\mathcal{E})$ is isomorphic to the sheaf $\mathcal{E}^\vee$. The fibre $\mathbb{V}(\mathcal{E})_x$ over a point $x \in X$ is equal to Spec $S^\bullet(\mathcal{E}(x)) = \mathcal{E}(x)^\vee$, where $\mathcal{E}(x) = \mathcal{E} \otimes_{\mathcal{O}_{X,x}} \kappa(x)$ is the fibre of $\mathcal{E}$ at x considered as a vector space over the residue field $\kappa(x)$ of the point x.

The *projective bundle* associated with a vector bundle $\mathbb{V}(\mathcal{E})$ (or a locally free sheaf $\mathcal{E}$) is the scheme $\mathbb{P}(\mathcal{E}) = \text{Proj}(\text{Sym}(\mathcal{E}))$. It comes with the natural morphism $p : \mathbb{P}(\mathcal{E}) \to S$. In the same notation as above,

$$\mathbb{P}(\mathcal{E})|U \cong \text{Proj}(\text{Sym}(\mathcal{O}_U^{r+1})) \cong \text{Proj}(\mathcal{O}(U)[t_0, \ldots, t_r]) \cong \mathbb{P}_U^r.$$

For any point $x \in X$, the fibre $\mathbb{P}(\mathcal{E})_x$ over x is equal to $\mathbb{P}(\mathcal{E}(x)) = |\mathcal{E}(x)^\vee|$.

By definition of the projective spectrum, we have an invertible sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. Its sections over $p^{-1}(U)$ are homogeneous elements of degree 1 in $\text{Sym}(\mathcal{O}_U^{r+1})$. This gives for any $k \geq 0$,

$$p_*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(k) \cong \text{Sym}^k(\mathcal{E}).$$

Note that for any invertible sheaf $\mathcal{L}$ over S, we have $\mathbb{P}(\mathcal{E} \otimes \mathcal{L}) \cong \mathbb{P}(\mathcal{E})$ as schemes, however the sheaves $\mathcal{O}(1)$ are different.

For any scheme $\pi : X \to S$ over S a morphism of S-schemes $f : X \to \mathbb{P}(\mathcal{E})$ is defined by an invertible sheaf $\mathcal{L}$ over X and a surjection $\phi : \pi^*\mathcal{E} \to \mathcal{L}$. Then we trivialize $\mathbb{P}(\mathcal{E})$ over U, the surjection ϕ defines $r+1$ sections of $\mathcal{L}|\pi^{-1}(U)$. This gives a local map $x \mapsto [s_0(x), \ldots, s_r(x)]$ from $\pi^{-1}(U)$ to $p^{-1}(U) = \mathbb{P}_U^r$. These maps are glued together to define a global map. We have $\mathcal{L} = f^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$.

In particular, taking $X = \mathbb{P}(\mathcal{E})$ and f the identity morphism, we obtain a surjection $p^*\mathcal{E} \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. When we push it down, we get the identity map $p_*p^*\mathcal{E} = \mathcal{E} \to p_*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$.

Example 7.3.1. Let us take $X = S$. Then an S-morphism $S \to \mathbb{P}(\mathcal{E})$ is a section $s : S \to \mathbb{P}(\mathcal{E})$. It is defined by an invertible sheaf $\mathcal{L}$ on S and a surjection $\phi : \mathcal{E} \to \mathcal{L}$. We have $\mathcal{L} = s^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. Let $\mathcal{N} = \text{Ker}(\phi)$. This is a locally free sheaf of rank r.

Example 7.3.2. Take $x = \text{Spec}(\kappa(x))$ to be a point in S, and $i : x \to S$ be its inclusion in S. Then an invertible sheaf on a point is the constant sheaf κ_x and $i^*\mathcal{E} = \mathcal{E}_x = \mathcal{E}/\mathfrak{m}_x\mathcal{E} = \mathcal{E}(x)$ is the fibre of the sheaf $\mathcal{E}$. The inclusion of x in S is defined by a surjection $\mathcal{E}(x) \to \kappa_x$, i.e. by a point in the projective space $\mathbb{P}(\mathcal{E}(x)) = |\mathcal{E}(x)^\vee|$. This agrees with the description of fibres of a projective bundle from above.

Lemma 7.3.1. *Let $s : S \to \mathbb{P}(\mathcal{E})$ be a section, $\mathcal{L} = s^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ and $\mathcal{N} = \text{Ker}(\mathcal{E} \to \mathcal{L})$. Let us identify S with $s(S)$. Then $\mathcal{N} \otimes \mathcal{L}^{-1}$ is isomorphic to the conormal sheaf of $s(S)$ in $\mathbb{P}(\mathcal{E})$.*

Proof. Recall (see [206], Proposition 8.12) that for any closed embedding $i : Y \hookrightarrow X$ of a S-scheme defined by the ideal sheaf $\mathcal{I}$ we have an exact sequence

$$\mathcal{I}/\mathcal{I}^2 \to i^*\Omega^1_{X/S} \to \Omega^1_{Y/S} \to 0, \tag{7.46}$$

where the first homomorphism is injective if i is a regular embedding (e.g. X, Y are regular schemes). The sheaf $\mathcal{I}/\mathcal{I}^2$ is called the *conormal sheaf* of Y in X and is denoted by $\mathcal{N}_{Y/X}^\vee$. Its dual sheaf is called the normal sheaf of Y in X and is denoted by $\mathcal{N}_{Y/X}$.

Also recall that the sheaf $\Omega^1_{\mathbb{P}^n}$ of regular 1-forms on projective space can be defined by the exact sequence (the *dual Euler sequence*)

$$0 \to \Omega^1_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(-1) \to \mathcal{O}_{\mathbb{P}^n} \to 0. \tag{7.47}$$

It is generalized to any projective bundle

$$0 \to \Omega^1_{\mathbb{P}(\mathcal{E})/S} \to p^*\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1) \to \mathcal{O}_{\mathbb{P}(\mathcal{E})} \to 0. \tag{7.48}$$

Here the homomorphism $p^*\mathcal{E} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1) \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}$ is equal to the homomorphism $p^*\mathcal{E} \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ after twisting by -1. Thus

$$\Omega^1_{\mathbb{P}(\mathcal{E})/S}(1) \cong \mathrm{Ker}(p^*\mathcal{E} \to \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)). \tag{7.49}$$

Applying s^* to both sides we get

$$s^*\Omega^1_{\mathbb{P}(\mathcal{E})/S}(1) \cong \mathcal{N}. \tag{7.50}$$

Since $\Omega^1_{s(S)/S} = \{0\}$, we get from (7.46)

$$s^*(\mathcal{N}_{s(S)/\mathbb{P}(\mathcal{E})}) \cong s^*\Omega^1_{\mathbb{P}(\mathcal{E})/S} \cong \mathcal{N} \otimes \mathcal{L}^{-1}.$$

□

Let us apply this to minimal ruled surfaces $\mathbf{F}_n$. It is known that any locally free sheaf over $\mathbb{P}^1$ is isomorphic to the direct sum of invertible sheaves. Suppose $\mathcal{E}$ is of rank 2. Then $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$ for some integers a, b. Since the projective bundle $\mathbb{P}(\mathcal{E})$ does not change if we tensor $\mathcal{E}$ with an invertible sheaf, we may assume that $a = 0$ and $b = n \geq 0$.

Proposition 7.3.2. *Let $p : S \to \mathbb{P}^1$ be a morphism of a nonsingular surface such that all fibres are isomorphic to $\mathbb{P}^1$. Suppose S has a section E with $E^2 = -n$ for some $n \geq 0$, then $S \cong \mathbf{F}_n$.*

Proof. Let f be the divisor class of a fibre of p and s be the divisor class of the section E. For any divisor class d on S such that $d \cdot f = a$, we obtain $(d - as) \cdot f = 0$. If d represents an irreducible curve C, this implies that $p(C)$ is a point, and hence C is a fibre. Writing every divisor as a linear combination of irreducible curves, we obtain that any divisor class is equal to $af + bs$ for some integers a, b. Let us write $K_{\mathbb{P}(\mathcal{E})} = af + bs$. By adjunction formula, applied to a fibre and the section s, we get

$$-2 = (af + bs) \cdot f, \quad -2 + n = (af + bs) \cdot s = a - 2nb.$$

This gives

$$K_S = (-2 - n)f - 2s. \tag{7.51}$$

Assume $n \neq 0$. Consider the linear system $|nf + s|$. We have

$$(nf + s)^2 = n, \ (nf + s) \cdot ((-2 - n)f - 2s) = -2 - n.$$

By Riemann-Roch, $\dim |nf + s| \geq n + 1$. The linear system $|nf + s|$ has no base points because it contains the linear system $|nf|$ with no base points. Thus it defines a regular map $\mathbb{P}(\mathcal{E}) \to \mathbb{P}^n$. Since $(nf + s) \cdot s = 0$, it blows down the section s to a point p. Since$(nf + s) \cdot f = a$, it maps fibres to lines passing through p. The degree of the image is $(nf + s)^2 = n$. Thus the image of the map is a surface of degree n equal to the union of lines through a point. It must be a cone over the Veronese curve $v_n(\mathbb{P}^1)$ if $n > 1$ and $\mathbb{P}^2$ if $n = 1$. The map is its minimal resolution of singularities. This proves the assertion in this case.

Assume $n = 0$. We leave to the reader to check that the linear system $|f + s|$ maps S isomorphically to a quadric surface in $\mathbb{P}^3$. □

Corollary 7.3.3.

$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n)) \cong \mathbf{F}_n.$$

Proof. The assertion is obvious if $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$. Assume $n > 0$. Consider the section of $\mathbb{P}(\mathcal{E})$ defined by the surjection

$$\phi : \mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n) \to \mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(-n), \tag{7.52}$$

corresponding to the projection to the second factor. Obviously, $\mathcal{N} = \mathrm{Ker}(\phi) \cong \mathcal{O}_{\mathbb{P}^1}$. Applying Lemma 7.3.1, we get

$$\mathcal{N}_{s(\mathbb{P}^1)/\mathbb{P}(\mathcal{E})} \cong \mathcal{O}_{\mathbb{P}^1}(-n).$$

Now, if C is any curve on a surface X, its ideal sheaf is isomorphic to $\mathcal{O}_X(-C)$ and hence the conormal sheaf is isomorphic to $\mathcal{O}_X(-C)/\mathcal{O}_X(-2C)$. This easily implies that

$$\mathcal{N}_{C/X} = \mathcal{O}_X(C) \otimes \mathcal{O}_C. \tag{7.53}$$

In particular, we see that the degree of the invertible sheaf $\mathcal{N}_{C/X}$ on the curve C is equal to the self-intersection C^2.

Thus we obtain that the self-intersection of the section s defined by the surjection (7.52) is equal to $-n$. It remains to apply the previous Proposition. □

7.3.2 Elementary transformations

Let $p : \mathbf{F}_n \to \mathbb{P}^1$ be a ruling of $\mathbf{F}_n$ (the unique one if $n \neq 0$). Let $x \in \mathbf{F}_n$ and F_x be the fibre of the ruling containing x. If we blow up x, the proper inverse transform of F_x is an exceptional curve of the first kind. We can blow it down to obtain a nonsingular surface S'. The projection p induces a morphism $p' : S' \to \mathbb{P}^1$ with any fibre isomorphic to $\mathbb{P}^1$. Let E_n be the exceptional section or any section with the self-intersection 0 if $n = 0$ (such a section is of course equal to a fibre of the second ruling of $\mathbf{F}_0$). Assume that $x \notin E_n$. The proper inverse transform of E_n on the blow-up has the self-intersection equal to $-n$, and its image in S' has the self-intersection equal to $-n+1$. Applying Proposition 7.3.2, we obtain that $S' \cong \mathbf{F}_{n-1}$. This defines a birational map

$$\mathrm{elm}_x : \mathbf{F}_n - \to \mathbf{F}_{n-1}.$$

Assume that $x \in E_n$. Then the proper inverse transform of E_n on the blow-up has self-intersection $-n-1$ and its image in S' has the self-intersection equal to $-n-1$. Applying Proposition 7.3.2, we obtain that $S' \cong \mathbf{F}_{n+1}$. This defines a birational map

$$\mathrm{elm}_x : \mathbf{F}_n - \to \mathbf{F}_{n+1}.$$

A birational map elm_x is called an *elementary transformation*.

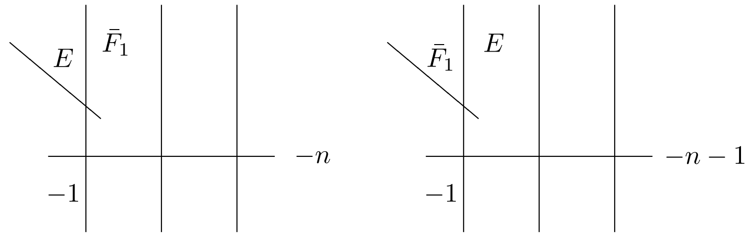


Figure 7.1: Elementary transformation

Remark 7.3.1. Let $\mathcal{E}$ be a locally free sheaf over a nonsingular curve B. As we explained in Example 7.3.2, a point $x \in \mathbb{P}(\mathcal{E})$ is defined by a surjection $\mathcal{E}(x) \to \kappa(x)$, where $\kappa(x)$ is considered as the structure sheaf of the closed point x. Composing this surjection with the natural surjection $\mathcal{E} \to \mathcal{E}(x)$, we get a surjective morphism of sheaves $\phi_x : \mathcal{E} \to \kappa(x)$. Its kernel $\mathrm{Ker}(\phi_x)$ is a subsheaf of $\mathcal{E}$ which has no torsion. Since the base is a regular one-dimensional scheme, the sheaf $\mathcal{E}' = \mathrm{Ker}(\phi_x)$ is locally free. Thus we have defined an operation on locally free sheaves. It is also called an elementary transformation.

Consider the special case when $B = \mathbb{P}^1$ and $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n)$. We have an exact sequence

$$0 \to \mathcal{E}' \to \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n) \xrightarrow{\phi_x} \kappa_x \to 0.$$

The point x belongs to the exceptional section E_n if and only if ϕ_x factors through $\mathcal{O}_{\mathbb{P}^1}(-n) \to \kappa_x$. Then $\mathcal{E}' \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n-1)$ and $\mathbb{P}(\mathcal{E}') \cong \mathbf{F}_{n+1}$. The inclusion of sheaves $\mathcal{E}' \subset \mathcal{E}$ gives rise to a rational map $\mathbb{P}(\mathcal{E}) - \to \mathcal{P}(\mathcal{E}')$ which coincides with elm_x. If $x \notin E_n$, then ϕ_x factors through $\mathcal{O}_{\mathbb{P}^1}$, and we obtain $\mathcal{E}' \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-n)$. In this case $\mathbb{P}(\mathcal{E}') \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n+1)) \cong \mathbf{F}_{n-1}$ and again, the inclusion $\mathcal{E}' \subset \mathcal{E}$ defines a rational map $\mathbb{P}(\mathcal{E}) - \to \mathbb{P}(\mathcal{E}')$ which coincides with elm_x. We refer for this sheaf-theoretical interpretation of elementary transformation to [205]. A more general definition applied to projective bundles over any algebraic variety can be found in [408].

Let $x, y \in \mathbf{F}_n$. Assume that $x \in E_n$, $y \notin E_n$ and $p(x) \neq p(y)$. Then the composition

$$t_{x,y} = \mathrm{elm}_y \circ \mathrm{elm}_x : \mathbf{F}_n - \to \mathbf{F}_n$$

is a birational automorphism of $\mathbf{F}_n$. Here we identify the point y with its image in $\mathrm{elm}_x(\mathbf{F}_n)$. Similarly, we get a birational automorphism $t_{y,x} = \mathrm{elm}_y \circ \mathrm{elm}_x$ of $\mathbf{F}_n$. We can also extend this definition to the case when $y \succ_1 x$, where y does not correspond to the tangent direction defined by the fibre passing through x or the exceptional section (or any section with self-intersection 0). We blow up x, then y, and then blow down the proper transform of the fibre through x and the proper inverse transform of the exceptional curve blown up from x.

7.3.3 Birational automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$

We will often identify $\mathbf{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ with a nonsingular quadric Q in $\mathbb{P}^3$. Let us fix a point $x_0 \in Q$. The linear projection $p_{x_0} : Q \setminus \{x_0\} \to \mathbb{P}^2$ defines a birational map. Let l_1, l_2 be two lines on Q passing through x_0 and q_1, q_2 be their projections. The inverse map $p_{x_0}^{-1}$ blows up the points q_1, q_2 and blows down the proper transform of the line $\overline{q_1, q_2}$. For any birational automorphism T of $\mathbb{P}^2$ the composition $p_{x_0}^{-1} \circ T \circ p_{x_0}$ is a birational transformation of Q. This defines an isomorphism of groups

$$\Phi_{x_0} : \mathrm{Bir}(\mathbb{P}^2) \cong \mathrm{Bir}(Q),\ T \mapsto p_{x_0}^{-1} \circ T \circ p_{x_0}.$$

Explicitly, choose coordinates in $\mathbb{P}^3$ such that $Q = V(z_0z_3 - z_1z_2)$ and $x_0 = [0, 0, 0, 1]$. The inverse map $p_{x_0}^{-1}$ can be given by the formulas

$$[t_0, t_1, t_2] \mapsto [t_0^2, t_0t_1, t_0t_2, t_1t_2].$$

If T is given by the polynomials f_0, f_1, f_2, then $\Phi_{x_0}(T)$ is given by the formula

$$[z_0, z_1, z_2, z_3] \mapsto [f_0(z')^2, f_0(z')f_1(z'), f_0(z')f_2(z'), f_1(z')f_2(z')], \tag{7.54}$$

where $f_i(z') = f_i(z_0, z_1, z_2)$.

Remark 7.3.2. Let $z_1, \ldots, z_n \in Q$ be F-points of T different from x_0. Let $T^{-1}(x_0)$ be a point if T^{-1} is defined at x_0 or the principal curve of T corresponding to x_0 with x_0 deleted if it contains it. The Cremona transformation $\Phi_{x_0}(T)$ is defined outside the set $q_1, q_2, p_{x_0}(z_1), \ldots, p_{x_0}(z_n), p_{x_0}(T^{-1}(x_0))$. Here, we also include the case of infinitely near fundamental points of T. If some of z_i's lie on a line l_i or infinitely near to points on l_i, their image under p_{x_0} is considered to be an infinitely near point to q_i.

Let $\mathrm{Aut}(Q) \subset \mathrm{Bir}(Q)$ be the subgroup of biregular automorphisms of Q. It acts naturally on $\mathrm{Pic}(Q) = \mathbb{Z}f + \mathbb{Z}g$, where $f = [l_1], g = [l_2]$. The kernel $\mathrm{Aut}(Q)^o$ of this action is isomorphic to $\mathrm{Aut}(\mathbb{P}^1) \times \mathrm{Aut}(\mathbb{P}^1) \cong \mathrm{PGL}(2) \times \mathrm{PGL}(2)$. The quotient group is of order 2, and its nontrivial coset can be represented by the automorphism τ of $\mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1)$ defined by $(a, b) \mapsto (b, a)$.

Proposition 7.3.4. *Let $\sigma \in \mathrm{Aut}(Q)^o$. If $\sigma(x_0) \neq x_0$, then $\Phi_{x_0}(\sigma)$ is a quadratic transformation with fundamental points $q_1, q_2, p_{x_0}(\sigma^{-1}(x_0))$. If $\sigma(x_0) = x_0$, then $\Phi_{x_0}(\sigma)$ is a projective transformation.*

Proof. It follows from Remark 7.3.2 that $\Phi_{x_0}(\sigma)$ has at most 3 fundamental points if $\sigma(x_0) \neq x_0$ and at most 2 fundamental points if $\sigma(x_0) = x_0$. Since any birational map with less than 3 fundamental points (including infinitely near) is regular, we see that in the second case $\Phi_{x_0}(\sigma)$ is a projective automorphism. In the first case, the image of the line $\overline{q_1, q_2}$ is equal to the point $p_{x_0}(\sigma(x_0))$. Thus $\Phi_{x_0}(\sigma)$ is not projective. Since it has at most 3 fundamental points, it must be a quadratic transformation. □

Remark 7.3.3. In general, the product of quadratic transformations is not a quadratic transformation. However, in our case all quadratic transformations from $\mathrm{Aut}(Q)$ have a common pair of fundamental points and hence their product is a quadratic transformation. The subgroup $\Phi_{x_0}(\mathrm{Aut}(Q))$ of $\mathrm{Cr}(2) = \mathrm{Bir}(\mathbb{P}^2)$ is an example of a subgroup of the Cremona group $\mathrm{Cr}(2)$ which is isomorphic to an algebraic linear group. According to a theorem of Enriques-Fano, any subgroup of $\mathrm{Cr}(2)$ which is isomorphic to a linear algebraic group, is contained in a subgroup isomorphic to $\mathrm{Aut}(\mathbf{F}_n)$ for some n. There is even a generalization of this result to the group $\mathrm{Cr}(n) = \mathrm{Bir}(\mathbb{P}^n)$ (see [122]). Instead of minimal ruled surfaces one considers smooth toric varieties of dimension n.

Take two points x, y which do not lie on the same fibre of each projection $p_1 : \mathbf{F}_0 \to \mathbb{P}^1, p_2 : \mathbf{F}_0 \to \mathbb{P}^1$. Let $x = F_1 \cap F_2$, $y = F_1' \cap F_2'$, where F_1, F_1' are two fibres of p_1 and F_2, F_2' are two fibres of p_2. Then $t_{x,y}$ is a birational automorphism of $\mathbf{F}_0$.

Proposition 7.3.5. *$\Phi_{x_0}(t_{x,y})$ is a product of quadratic transformations. If $x_0 \in \{x, y\}$, then $\Phi_{x_0}(t_{x,y})$ is a quadratic transformation. Otherwise, $\Phi_{x_0}(t_{x,y})$ is the product of two quadratic transformation.*

Proof. Assume first that y is not infinitely near to x. Suppose x_0 coincides with one of the points x, y, say $x_0 = x$. It follows from Remark 7.3.2 that $\Phi_{x_0}(T)$ is defined outside $q_1, q_2, p_{x_0}(y)$. On the other hand, the image of the line $\overline{q_1, p_{x_0}(y)}$ is a point. Here we assume that the projection $\mathbf{F}_0 \to \mathbb{P}^1$ is chosen in such a way that its fibres are the proper transforms of lines through q_1 under $p_{x_0}^{-1}$. Thus $\Phi_{x_0}(T)$ is not regular with at most three F-points, hence is a quadratic transformation.

If $x_0 \neq x, y$, we compose $t_{x,y}$ with an automorphism σ of Q such that $\sigma(x_0) = x$. Then

$$\Phi_{x_0}(t_{x,y} \circ \sigma) = \Phi_{x_0}(t_{x_0,\sigma^{-1}(y)}) = \Phi_{x_0}(t_{x,y}) \circ \Phi_{x_0}(\sigma).$$

By the previous lemma, $\Phi_{x_0}(\sigma)$ is a quadratic transformation. By the previous argument, $\Phi_{x_0}(t_{x_0,\sigma^{-1}(y)})$ is a quadratic transformation. Also the inverse of a quadratic transformation is a quadratic transformation. Thus $\Phi_{x_0}(t_{x,y})$ is a product of two quadratic transformations.

Now assume that $y \succ x$. Take any point $z \neq x$. Then one can easily checks that $t_{x,y} = t_{z,y} \circ t_{x,z}$. Here we view y as an ordinary point on $t_{x,z}(\mathbf{F}_0)$. $\square$

Proposition 7.3.6. *Let $T : \mathbf{F}_n - \to \mathbf{F}_m$ be a birational map. Assume that T commutes with the projections of the minimal ruled surfaces to $\mathbb{P}^1$. Then T is a composition of biregular maps and elementary transformations.*

Proof. Let (X, π, σ) be a resolution of T. Let $p_1 : \mathbf{F}_n \to \mathbb{P}^1$ and $p_2 : \mathbf{F}_m \to \mathbb{P}^1$ be the projections. We have

$$\phi = p_1 \circ \pi = p_2 \circ \sigma : X \to \mathbb{P}^1.$$

Let $a_1, \ldots, a_k$ be points in $\mathbb{P}^1$ such that $C_i = \phi^{-1}(a_i) = \pi^*(p_1^{-1}(a_i))$ is a reducible curve. We have $\pi_*(C_i) = p_1^{-1}(a_i)$ and $\sigma_*(C_i) = p_2^{-1}(a_i)$. Let E_i be the unique component of C_i which is mapped surjectively to $p_1^{-1}(a_i)$ and E_i' be the unique component of C_i which is mapped surjectively to $p_2^{-1}(a_i)$. Let π be a composition of blow-ups of points $x_1, \ldots, x_N$ and let f be a composition of blow-ups of points $y_1, \ldots, y_N$. The preimages in X of the maximal points (with respect to the partial order on the set of infinitely near points) are irreducible curves with self-intersection -1. Let E be a component of C_i with $E^2 = -1$ which is different from E_i, E_i'. We can reorder the order of the blow-ups to assume that $\pi(E) = x_N$ and $f(E) = y_N$. Let $\pi_N : X \to X_{N-1}$ be the blow-up x_N and $f_N : X \to Y_{N-1}$ be the blow-up y_N. Since π_N and f_N are given by the same linear system, there exists an isomorphism $t : X_{N-1} \cong Y_{N-1}$. Thus, we can replace the resolution (X, π, f) with

$$(X_{N-1}, \pi_1 \circ \ldots \circ \pi_{N-1}, f_1 \circ \ldots \circ f_{N-1} \circ t).$$

Continuing in this way, we may assume that x_N and y_N are the only maximal points of π and σ such that $p_1(x_N) = p_2(y_N) = a_i$. Let $E = \pi^{-1}(x_N)$ and $E' = f^{-1}(y_N)$. Let $R \neq E'$ be a component of $\phi^{-1}(a_i)$ which intersects E. Let $x = \pi(R)$. Since $x_N \succ x$, and no other points is infinitely near to x, we get $R^2 = -2$. Blowing down E, we get that the image of R has self-intersection -1. Continuing in this way we get two possibilities

$$C_i = E_i + E_i', \quad E_i^2 = E_i'^2 = -1, E_i \cdot E_i' = 1,$$

$$C_i = E_i + R_1 + \cdots + R_k + E_i', \quad E_i^2 = E_i'^2 = -1,$$
$$R_i^2 = -1, E_i \cdot R_1 = \ldots = R_i \cdot R_{i+1} = R_k \cdot E_i' = 1$$

and all other intersections are equal to zero.

In the first case, $T = \mathrm{elm}_{x_N}$. In the second case, let $g : X \to X'$ be the blow-down E_i, let $x = \pi(R_1 \cap E_i)$. Then $T = T' \circ \mathrm{elm}_x$, where T' satisfies the assumption of the proposition. Continuing in this way we write T as the composition of elementary transformations. $\square$

7.3.4 De Jonquières transformations again

Let T be a De Jonquières transformation of degree d with fundamental points $p_1, \ldots, p_{2d-1}$. Consider the pencil of lines through p_1. The restriction of the linear system $|d\ell - \eta|$ to a general line from this pencil is of degree 1, and hence maps this line to a line. Since each such line ℓ intersects X at 2 points different from p_1, the image of ℓ is equal to ℓ. Thus T leaves any line from the pencil invariant or blows down it to a point. Let us blow up p_1 to get a birational map $\pi_1 : S_1 \to \mathbb{P}^2$. The surface S_1 is isomorphic to $\mathbf{F}_1$. Its exceptional section is $E_1 = \pi_1^{-1}(p_1)$. The proper transform of the curve Γ is a nonsingular curve $\bar{\Gamma}$. It intersects E_1 at $d-1$ points $z_1, \ldots, z_{2d-2}$ corresponding to the branches of Γ at p_1. Let $l_1, \ldots, l_{d-1}$ be the fibres of the projection $\phi : S_1 \to \mathbb{P}^1$ corresponding to the lines $\overline{p_1, p_i}$, where $i = 2, \ldots, 2d-1$. The curve $\bar{C}$ passes through the points $\bar{p}_i = \pi_1^{-1}(p_i) \in l_i$. Let $\pi : X \to \mathbb{P}^2, f : X \to \mathbb{P}^2$ be the resolution of T obtained by blowing up the cycle η. The map factors through $\pi' : X \to S$ which is the blow-up with center at the points $\bar{p}_i$. The proper transform of $\Gamma' = \bar{\Gamma}$ in X is an exceptional curve of the first kind. The map f blows down the proper inverse transforms of the fibres l_i and the curve Γ'. If we stop before blowing down Γ' we get a surface isomorphic to S_1. Thus T can be viewed also as a birational automorphism of $\mathbf{F}_1$ which is the composition of $2d-2$ elementary transformations

$$\mathbf{F}_1 \overset{\mathrm{elm}_{\bar{p}_2}}{- \to} \mathbf{F}_0 - \to \mathbf{F}_1 - \to \ldots - \to \mathbf{F}_0 - \to \mathbf{F}_1.$$

If we take x_0 to be the image of l_1 under $\mathrm{elm}_{\bar{p}_2}$, to define an isomorphism Φ_{x_0} : $\mathrm{Bir}(\mathbf{F}_0) \to \mathrm{Bir}(\mathbb{P}^2)$, then we obtain that $T = \Phi_{x_0}(T')$, where T' is the composition of transformations $t_{\bar{p}_i,\bar{p}_{i+1}} \in \mathrm{Bir}(\mathbf{F}_0)$, where $i = 3, 5, \ldots, 2d-3$. Applying Proposition 7.3.5, we obtain the following.

Theorem 7.3.7. *A De Jonquières transformation is equal to a composition of quadratic transformations.*

7.4 Characteristic matrices

Consider a resolution (7.6) of a Cremona transformation ϕ

$$\begin{tikzcd} & X \arrow[dl, "\pi"'] \arrow[dr, "\sigma"] & \\ \mathbb{P}^2 \arrow[rr, dashed, "\phi"] & & \mathbb{P}^2. \end{tikzcd}$$

Obviously, it gives a resolution of the inverse transformation ϕ^{-1}. The roles of π and σ are interchanged. Let

$$\sigma : X = X_M \xrightarrow{\sigma_M} X_{M-1} \xrightarrow{\sigma_{M-1}} \ldots \xrightarrow{\sigma_2} X_1 \xrightarrow{\sigma_1} X_0 = \mathbb{P}^2 \tag{7.55}$$

be the factorization into a sequence of blow-ups similar to the one we had for π. It defines a bubble cycle ξ and the homaloidal net $|d'h - \xi|$ defining ϕ^{-1}. Let $\mathcal{E}'_1, \ldots, \mathcal{E}'_M$ be the corresponding exceptional configurations. We will always take for X a minimal resolution. It must be isomorphic to the minimal resolution of the graph of ϕ.

Lemma 7.4.1. *Let $\mathcal{E}_1, \ldots, \mathcal{E}_N$ be the exceptional configurations for π and $\mathcal{E}'_1, \ldots, \mathcal{E}'_M$ be the exceptional configurations for σ. Then*

$$N = M.$$

Proof. Let S be a nonsingular projective surface and $\pi : S' \to S$ be a blow-up map. Then the Picard group $\mathrm{Pic}(S')$ is generated by the preimage $\pi^*(\mathrm{Pic}(S))$ and the divisor class $[E]$ of the exceptional curve. Also we know that $[E]$ is orthogonal to any divisor class from $\pi^*(\mathrm{Pic}(S))$ and this implies that

$$\mathrm{Pic}(S') = \mathbb{Z}[E] \oplus \pi^*(\mathrm{Pic}(S)).$$

In particular, taking $S = \mathbb{P}^2$, we obtain, by induction that

$$\mathrm{Pic}(X) = \pi^*(\mathrm{Pic}(\mathbb{P}^2)) \bigoplus_{i=1}^{N} [\mathcal{E}_i].$$

This implies that $\mathrm{Pic}(X)$ is a free abelian group of rank $N+1$. Replacing π with σ, we obtain that the rank is equal to $1 + M$. Thus $N = M$.

$\square$

Remark 7.4.1. It could happen that all exceptional configurations of π are irreducible (i.e. no infinitely points are used to define π) but some of the exceptional configurations of σ are reducible. This happens in the case of the transformation given in Exercise 7.2.

Definition 7.4. *An* ordered resolution *of a Cremona transformation is the diagram* (7.6) *together with an order of a sequence of the exceptional curves for* σ *and* π.

Any ordered resolution of T defines two bases in Pic(X). The first basis is

$$\underline{e}' : e_0' = \sigma^*(\ell),\ e_1' = [\mathcal{E}_1'], \ldots, e_N' = [\mathcal{E}_N'].$$

The second basis is

$$\underline{e} : e_0 = \pi^*(\ell),\ e_1 = [\mathcal{E}_1], \ldots, e_N = [\mathcal{E}_N].$$

Write

$$e_0' = de_0 - \sum_{i=1}^N m_i e_i, \quad e_j' = d_j e_0 - \sum_{i=1}^N m_{ij} e_i,\ j > 0.$$

The matrix

$$A = \begin{pmatrix} d & d_1 & \ldots & d_N \\ -m_1 & -m_{11} & \ldots & -m_{1N} \\ \vdots & \vdots & \vdots & \vdots \\ -m_N & -m_{N1} & \ldots & -m_{NN} \end{pmatrix} \tag{7.56}$$

is called the *characteristic matrix* of T with respect to an ordered resolution. It is the matrix of change of basis from $\underline{e}$ to $\underline{e}'$.

The first column of A is the vector $(d, -m_1, \ldots, -m_N)$, where $(d; m_1, \ldots, m_N)$ is the characteristic of ϕ. We write other columns in the form $(d_j, -m_{1j}, \ldots, -m_{Nj})$. They describe the exceptional configurations $\mathcal{E}_j'$ of σ. If $d_j > 0$, then images of $\mathcal{E}_j'$ in $\mathbb{P}^2$ under the map π is called a *total principal curves* or *total P-curves* of ϕ. Its degree is equal to d_j. It passes through the base points x_k of ϕ with multiplicities $\geq m_{jk}$. The equality takes place if and only if no irreducible component of $\mathcal{E}_j$ is mapped to x_k under the map $X \to X_{k-1}$. The total P-curve could be reducible, its irreducible components are *principal curves* or P*-curves*.

Note that each $\mathcal{E}_i'$ contains an irreducible component E_i with self-intersection -1. Under the map π it cannot be mapped to a point. In fact, assume that it is blown down to a point x_i, a base point of height 1 of ϕ. Since the self-intersection increases under blowing-down, $E_i = \mathcal{E}_j$ for some j. Let $\alpha : X \to X'$ be the blowing-down of E_i. Then $\pi \circ \alpha^{-1} : X' \to \mathbb{P}^2$ is a regular map, and $\pi \circ \alpha^{-1} : X' \to \mathbb{P}^2$ is a regular map. Thus X' is nonsingular and resolves the indeterminacy points of ϕ. This contradiction proves the claim and shows that the image of E_i is a principal curve.

The characteristic matrix defines a homomorphism of free abelian groups

$$\phi_A : \mathbb{Z}^{1+N} \to \mathbb{Z}^{1+N}.$$

We equip $\mathbb{Z}^{1+N}$ with the standard hyperbolic inner product where the norm of a vector $v = (a_0, a_1, \ldots, a_N)$ defined by

$$v^2 = a_0^2 - a_1^2 - \ldots - a_N^2.$$

The group $\mathbb{Z}^{1+N}$ equipped with this integral quadratic form is customary denoted by $I^{1,N}$. It is an example of a *quadratic lattice*, a free abelian group equipped with an integral valued quadratic form. We will discuss quadratic lattices in Chapter 9. Since both bases $\underline{e}$ and $\underline{e'}$ are orthonormal with respect to the inner product, we obtain that the characteristic matrix is orthogonal, i.e. belongs to the group $\mathrm{O}(I^{1,N}) \subset \mathrm{O}(1,N)$.

Recall that the orthogonal group $\mathrm{O}(1,N)$ consists of $N+1 \times N+1$ matrices M such that

$$M^{-1} = J_{N+1} \cdot {}^tM \cdot J_{N+1}, \tag{7.57}$$

where J_{N+1} is the diagonal matrix $\mathrm{diag}[1, -1, \ldots, -1]$.

In particular, the characteristic matrix A^{-1} of ϕ^{-1} satisfies

$$A^{-1} = J \cdot A^t \cdot J = \begin{pmatrix} d & m_1 & \ldots & m_N \\ -d_1 & -m_{11} & \ldots & -m_{N1} \\ \vdots & \vdots & \vdots & \vdots \\ -d_N & -m_{1N} & \ldots & -m_{NN} \end{pmatrix} \tag{7.58}$$

Remark 7.4.2. A Cremona map given by polynomials (f_0, f_1, f_2) can be considered as a regular map $\mathbb{C}^3 \to \mathbb{C}^3$ of degree 1. Its divisor of critical points is equal to the jacobian determinant

$$\mathrm{jac}(f_0, f_1, f_2) := \det \begin{pmatrix} \frac{\partial f_0}{\partial t_0} & \frac{\partial f_0}{\partial t_1} & \frac{\partial f_0}{\partial t_2} \\ \frac{\partial f_1}{\partial t_0} & \frac{\partial f_1}{\partial t_1} & \frac{\partial f_1}{\partial t_2} \\ \frac{\partial f_2}{\partial t_0} & \frac{\partial f_2}{\partial t_1} & \frac{\partial f_2}{\partial t_2} \end{pmatrix} = 0.$$

Since the degree of the jacobian is equal to $3d-3$ we expect that the degree of the union of principal curves is equal to $3d-3$. Using (7.58), we find that $(d, d_1, \ldots, d_N)$ is the characteristic vector of the transformation ϕ^{-1}. Hence it satisfies $3d - \sum_{i=1}^N d_i = 3$. So, we confirm that the sum $\sum_{i=1}^N d_i$ of the degrees of principal curves is equal to $3d-3$.

Recall that, for any rational map $\phi : X' - \to X$ of irreducible algebraic varieties, one can define the image $\phi(Z')$ of an irreducible subvariety of X' and the preimage $\phi^{-1}(Z)$ of an irreducible subvariety of X. We choose an open subset U' where ϕ is defined, and define $\phi(Z)$ to be the closure of $\phi(U' \cap Z')$ in X. Similarly, we choose an open subset U of X, where ϕ^{-1} is defined and define $\phi^{-1}(Z)$ to be equal to the closure of $\phi^{-1}(U \cap Z)$ in X.

The image of a total principal curve $\pi(\mathcal{E}_j)$ under the Cremona map is equal to the image of the base point y_j of ϕ^{-1} in the plane. It is the unique base point y_i of ϕ^{-1} of height 0 such that $y_j \succ y_i$. Conversely, any irreducible curve blown down to a point under ϕ coincides with a P-curve.

Proposition 7.4.2. *Let $\phi : \mathbb{P}^2 - \to \mathbb{P}^2$ be a Cremona transformation with F-points $x_1, \ldots, x_N$ and F-points $y_1, \ldots, y_N$ of ϕ^{-1}. Let A be the characteristic matrix A. Let C be an irreducible curve on $\mathbb{P}^2$ of degree n which passes through the points y_i with multiplicities n_i. Let n' be the degree of $\phi(C)$ and let n'_i be the multiplicity of $\phi(C)$ at x_i. Then the vector $v = (n', -n'_1, \ldots, -n'_N)$ is equal to $A^{-1} \cdot v$, where $v = (n, -n_1, \ldots, -n_N)$.*

Proof. Let (X, π, σ) be a minimal resolution of ϕ. The divisor class of the proper inverse transform $\pi^{-1}(C)$ in X is equal to $v = ne_0 - \sum n_i e_i$. If we rewrite it in terms of the basis $(e'_0, e'_1, \ldots, e'_N)$ we obtain that it is equal to $v' = n'e_0 - \sum n'_i e_i$, where $v' = Av$. Now the image of $\pi^{-1}(C)$ under σ coincides with $\phi(C)$. By definition of the curves $\mathcal{E}_i$, the curve $\phi^{-1}(C)$ is a curve of degree n' passing through the fundamental points y_i of ϕ^{-1} with multiplicities n'_i. □

Let C be a principal curve of ϕ and $ce_0 - \sum_{i=1}^{N} c_i$ be the class of $\pi^{-1}(C)$. Let $v = (c, -c_1, \ldots, -c_N)$. Since $\phi(C)$ is a point, $A \cdot v = -e'_j$ for some j.

Example 7.4.1. The following matrix is a characteristic matrix of the standard quadratic transformation τ_1 or its degenerations τ_2, τ_3.

$$A = \begin{pmatrix} 2 & 1 & 1 & 1 \\ -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{pmatrix}. \tag{7.59}$$

Consider the case $\phi = \tau_1$. Since $\phi = \phi^{-1}$, the fundamental points p_1, p_2, p_3 of ϕ and ϕ^{-1} are the same and we choose the same order on them. Let E_1, E_2, E_3 be the exceptional curves of π and E'_1, E'_2, E'_3 be the exceptional curves of σ. We know that T blows down the line $\ell_{ij} = \overline{p_i, p_j}$ to the point p_k, where $\{i, j, k\} = \{1, 2, 3\}$. The linear system $|\sigma^*(e'_0)|$ is $|2e_0 - e_1 - e_2 - e_3|$, the proper inverse transform of ℓ_{ij} in X has the divisor class $e_0 - e_i - e_j$. Thus $e'_k = e_0 - e_i - e_j$. This gives us the matrix (7.59).

Now assume that $\phi = \tau_2$. The resolution $\pi : X \to \mathbb{P}^2$ is the composition of the blow-up of the point $p_1 = [0, 0, 1]$, followed by blowing up an infinitely near point p_2 corresponding to the tangent direction $t_0 = 0$, and followed by the blowing up the point $p_3 = [1, 0, 0]$. Let $\mathcal{E}_1 = E_1 + E_2$, $\mathcal{E}_2 = E_2$, $\mathcal{E}_3 = E_3$. Here $E_1^2 = -2, E_2^2 = E_3^2 = -1$. One sees easily that under the map σ, the proper transform of the line $t_0 = 0$ is blown down to the point p_3, the proper transform of the line x_1 together with the curve E_1 is blown down to the point p_1. Thus $e'_1 = (e_0 - e_1 - e_3) + (e_1 - e_2) = e_0 - e_2 - e_3$, $e'_2 = e_0 - e_1 - e_3, e'_3 = e_0 - e_1 - e_2$. We get the same matrix. Note that the second column describes the P-curve as a curve from the linear system $|\ell - p_2 - p_3|$. Here p_2 is infinitely near point to p_1. By definition, $p_2 + p_3$ is not a bubble cycle since p_1 is absent. So, $|\ell - p_2 - p_3|$ is not representing a curve on $\mathbb{P}^2$. In fact, $\mathcal{E}'_1$ is reducible and contains a component which is blown down to a point under π.

Now assume $\phi = \tau_3$. The resolution π is the composition of the blow-up of $p = [0, 0, 1]$, followed by the blow-up the infinitely near point corresponding to the direction $t_0 = 0$, and then followed by the blow-up the intersection point of the proper transform of the line $l = V(t_0)$ with the exceptional curve of the first blow-up. We have $\mathcal{E}_1 = E_1+E_2+E_3, \mathcal{E}_2 = E_2+E_3, \mathcal{E}_3 = E_3$. Here $E_1^2 = E_2^2 = -2, E_3^2 = -1$. The blowing down $\sigma : X \to \mathbb{P}^2$ consists of blowing down the proper inverse transform of the line ℓ equal to $e_0 - e_1 - e_2$, followed by the blowing down the image of E_2 and then blowing down the image of E_1. We have $e_1' = (e_0-e_1-e_2)+(e_2-e_3)+(e_1-e_2) = e_0-e_2-e_3$, $e_2' = (e_0 - e_1 - e_2) + (e_2 - e_3) = e_0 - e_1 - e_3, e_3' = e_0 - e_1 - e_2$. Again we get the same matrix.

Observe that the canonical class K_X is an element of Pic(X) which can be written in both bases as

$$K_X = -3e_0 + \sum_{i=1}^{N} e_i = -3e_0' + \sum_{i=1}^{n} e_i'.$$

This shows that the matrix A considered as an orthogonal transformation of $I^{1,N}$ leaves the vector

$$k_N = -3\mathbf{e}_0 + \mathbf{e}_1 + \cdots + \mathbf{e}_N = (-3, 1, \ldots, 1)$$

invariant. Here, $\mathbf{e}_i$ denotes the unit vector in $\mathbb{Z}^{1+N}$ with $(i+1)$-th coordinate equal to 1 and other coordinates equal to zero.

The matrix A defines an orthogonal transformation of $(\mathbb{Z}k_N)^\perp$.

Lemma 7.4.3. *The following vectors form a basis of* $(\mathbb{Z}k_N)^\perp$.

$$\begin{aligned} N \geq 3 : \alpha_0 &= \mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3, \quad \alpha_i = \mathbf{e}_i - \mathbf{e}_{i+1},\ i = 1, \ldots, N, \\ N = 2 : \alpha_0 &= \mathbf{e}_0 - 3\mathbf{e}_1, \quad \alpha_1 = \mathbf{e}_1 - \mathbf{e}_2 \\ N = 1 : \alpha_0 &= \mathbf{e}_0 - 3\mathbf{e}_1. \end{aligned}$$

Proof. Obviously, the vectors α_i are orthogonal to the vector k_N. Suppose a vector $v = (a_0, a_1, \ldots, a_N) \in (\mathbb{Z}k_N)^\perp$. Thus $3a_0+\sum_{i=1}^N a_i = 0$, hence $-a_N = 3a_0+\sum_{i=1}^{N-1} a_i$. Assume $N \geq 3$. We can write

$$v = a_0(\mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3) + (a_0 + a_1)(\mathbf{e}_1 - \mathbf{e}_2) + (2a_0 + a_1 + a_2)(\mathbf{e}_2 - \mathbf{e}_3) + \sum_{i=3}^{N-1} (3a_0 + a_1 + \cdots + a_i)(\mathbf{e}_i - \mathbf{e}_{i+1}).$$

If $N = 2$, we write $v = a_0(\mathbf{e}_0 - 3\mathbf{e}_1) + (3a_0 + a_1)(\mathbf{e}_1 - \mathbf{e}_2)$. If $N = 1$, $v = a_0(\mathbf{e}_0 - 3\mathbf{e}_1)$. $\square$

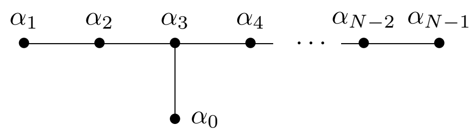
It is easy to compute the matrix $Q_N = (a_{ij})$ of the restriction of the inner product to $(\mathbb{Z}k_N)^\perp$ with respect to the basis (α_0, α_{N-1}). We have

$$(-8), \quad \text{if } N = 1, \quad \begin{pmatrix} -8 & 3 \\ 3 & -2 \end{pmatrix}, \quad \text{if } N = 2.$$

If $N \geq 3$, we have

$$(a_{ij}) = \begin{cases} -2 & \text{if } i = j, \\ 1 & \text{if } |i - j| = 1 \text{ and } i, j \geq 1, \\ 1 & \text{if } i = 0, j = 3, \\ 0 & \text{otherwise.} \end{cases}$$

For $N \geq 3$ the matrix $A + 2I_N$ is the incidence matrix of the following graph (the *Coxeter-Dynkin diagram* of type $T_{2,3,N-3}$).



For $3 \leq N \leq 8$ this is the Coxeter-Dynkin diagram of the root system of the semi-simple Lie algebra $\mathfrak{sl}_3 \oplus \mathfrak{sl}_2$ of type $A_2 + A_1$ if $N = 3$, of $\mathfrak{sl}_5$ of type $\mathfrak{A}_4$ if $N = 4$, of $\mathfrak{so}_{10}$ of type D_5 if $N = 5$ and of the exceptional simple Lie algebra of type E_N if $N = 6, 7, 8$.

We have

$$k_N^2 = 9 - N.$$

This shows that the matrix Q_N is negative definite if $N < 9$, semi-negative definite with one-dimensional null-space for $N = 9$, and of signature $(1, N-1)$ for $N \geq 10$. By a direct computation one checks that its determinant is equal to $N - 9$.

Proposition 7.4.4. *Assume $N \leq 8$. There are only finitely many posssible characteristic matrices. In particular, there are only finitely many possible characteristics of a homaloidal net with ≤ 8 base points.*

Proof. Let G be the group of real matrices $M \in \mathrm{GL}(N)$ such that ${}^tMQ_NM = Q_N$. Since Q_N is negative definite for $N \leq 8$, the group G is isomorphic to the orthogonal group $\mathrm{O}(N)$. The latter group is a compact Lie group. A characteristic matrix belongs to the subgroup $\mathrm{O}(Q_N) = G \cap \mathrm{GL}(N, \mathbb{Z})$. Since the latter is discrete, it must be finite. $\square$

There are further properties of characteristic matrices for which we refer to [1] for the modern proofs. The most important of these is the following *Clebsch Theorem*.

Theorem 7.4.5. *Let A be the characteristic matrix. There exists a bijection $\beta : \mathbb{N} \to \mathbb{N}$ such that for any set I of columns with $d_i = n, i \in I$, there exists a set of rows J with $\#I = \#J$ such that $\mu_j = \beta(a), j \in J$.*

Note that subtracting two columns (or rows) with the same first entry, and taking the inner product square, we easily get that they differ only at two entries by ± 1. This implies a certain symmetry of the matrix if reorder the columns and rows according to the Clebsch Theorem. We refer for the details to [1].

7.4.1 Composition of characteristic matrices

Suppose we have two birational maps $\phi : \mathbb{P}^2 - \to \mathbb{P}^2, \phi' : \mathbb{P}^2 - \to \mathbb{P}^2$. We would like to compute the characteristic matrix of the composition $\phi' \circ \phi$. Let

$$\begin{tikzcd} & X \arrow[dl, "\pi"'] \arrow[dr, "\sigma"] & \\ \mathbb{P}^2 \arrow[rr, dashed] & & \mathbb{P}^2 \end{tikzcd} \quad , \quad \begin{tikzcd} & X' \arrow[dl, "\pi'"'] \arrow[dr, "\sigma'"] & \\ \mathbb{P}^2 \arrow[rr, dashed] & & \mathbb{P}^2 \end{tikzcd} \tag{7.60}$$

be resolutions of ϕ and ϕ'. We want to construct a resolution of $\phi' \circ \phi$. Let

$$\sigma : X = X_N \xrightarrow{\sigma_N} X_{N-1} \xrightarrow{\sigma_{N-1}} \dots \xrightarrow{\sigma_2} X_1 \xrightarrow{\sigma_1} X_0 = \mathbb{P}^2$$

be a composition of blow-ups of σ and

$$\pi' : Y = Y_M \xrightarrow{\pi'_M} Y_{M-1} \xrightarrow{\pi'_{M-1}} \dots \xrightarrow{\pi'_2} Y_1 \xrightarrow{\pi'_1} Y_0 = \mathbb{P}^2$$

be a composition of blow-ups of π'. Let $x_1, \dots, x_N$ be the fundamental points of T and $y_1, \dots, y_N$ be the fundamental points of ϕ^{-1}. Let $x'_1, \dots, x'_M$ be the fundamental points of T' and $y'_1, \dots, y'_M$ be the fundamental points of ϕ'^{-1}. For simplicity we will assume that no infinitely near points occur as fundamental points of $\phi, \phi', \phi^{-1}, \phi'^{-1}$. We refer to the general case to [1].

Some of the fundamental points of ϕ^{-1} may coincide with fundamental points of ϕ'. This happens when a P-curve of ϕ contains a fundamental point of ϕ'. Let us assume that

$$y_i = x'_i, \quad i = 1, \dots, r.$$

In this case the fibred product of $X \xrightarrow{\sigma} \mathbb{P}^2$ and $Y \xrightarrow{\pi'} \mathbb{P}^2$ contains $\mathcal{E}_i'^{(1)} \times \mathcal{E}_i^{(2)},\ i = 1, \dots, r$, as irreducible components. When we throw them away, we obtain an ordered resolution $(Z, \pi \circ g, \sigma' \circ h)$ of $\phi' \circ \phi$, where $g : Z \to X$ is a composition of blow-ups $x'_{r+1}, \dots, x'_M$ and $h : Z \to Y$ is the composition of the blow-ups of $y_{r+1}, \dots, y_N$. Consider the following bases of $\mathrm{Pic}(Z)$.

$$\underline{e}_1 = \big(g^*(e_0^{(1)}), g^*(e_1^{(1)}), \dots, g^*(e_N^{(1)}), h^*(e_{r+1}^{(2)}), \dots, h^*(e_M^{(2)})\big),$$

$$\underline{e}_2 = \big(g^*(e_0'^{(1)}), g^*(e_1'^{(1)}), \dots, g^*(e_N'^{(1)}), h^*(e_{r+1}^{(2)}), \dots, h^*(e_M^{(2)})\big),$$

$$\underline{e}'_2 = \big(h^*(e_0'^{(2)}), h^*(e_1'^{(2)}), \dots, h^*(e_M'^{(2)}), g^*(e_{r+1}'^{(1)}), \dots, g^*(e_N'^{(1)})\big),$$

$$\underline{e}_3 = \big(h^*(e_0^{(2)}), h^*(e_1^{(2)}), \dots, h^*(e_M^{(2)}), g^*(e_{r+1}'^{(1)}), \dots, g^*(e_N'^{(1)})\big),$$

Note that

$$g^*(e_0^{(1)}) = h^*(e_0'^{(2)}).$$

The transition matrix from basis $\underline{e}_1$ to basis $\underline{e}_2$ is

$$\tilde{A}_1 = \begin{pmatrix} A_1 & 0_{N,M-r} \\ 0_{M-r,N} & I_{M-r} \end{pmatrix},$$

where A_1 is the characteristic matrix of ϕ. The transition matrix from basis $\underline{e}_2$ to basis $\underline{e}'_2$ is

$$P = \begin{pmatrix} I_{r+1} & 0_{r+1,N-r} & 0_{r+1,N-r} \\ 0_{N-r,r+1} & 0_{N-r,N-r} & I_{N-r} \\ 0_{M-r,r+1} & I_{M-r} & 0_{M-r,M-r} \end{pmatrix}.$$

The transition matrix from basis $\underline{e}'_2$ to basis $\underline{e}_3$ is

$$\tilde{A}_2 = \begin{pmatrix} A_2 & 0_{M,N-r} \\ 0_{N-r,M} & I_{N-r} \end{pmatrix},$$

where A_2 is the characteristic matrix of τ_2. The characteristic matrix of $t_2 \circ t_1$ is equal to the product

$$A = \tilde{A}_1 \circ P \circ \tilde{A}_2.$$

In the special case, when $r = N$, i.e., all fundamental points of ϕ^{-1} are fundamental points of ϕ', we obtain that the characteristic matrix of $\phi' \circ \phi$ is equal to

$$\begin{pmatrix} A_1 & 0_{N,M-N} \\ 0_{M-N,N} & I_{M-N} \end{pmatrix} \cdot A_2. \tag{7.61}$$

Example 7.4.2. Assume that $r = 0$, i.e. no F-point of ϕ^{-1} coincide with a F-point of ϕ'. Then the characteristic matrix of $\phi' \circ \phi$ is equal to

$$\begin{pmatrix} dd' & dd'_1 & \dots & dd'_M & d_1 & \dots & d_N \\ -d'm_1 & -d'_1m_1 & \dots & -d'_Mm_1 & -m_{11} & \dots & -m_{N1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -d'm_N & -d'_1m_N & \dots & -d'_Mm_N & -m_{1N} & \dots & -m_{NN} \\ -m'_1 & m'_{11} & \dots & m'_{1M} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -m'_M & m'_{1M} & \dots & m'_{MM} & 0 & \dots & 0 \end{pmatrix}$$

with the obvious meanings of $d, m_i, m_{ij}, d', m'_j, m'_{ij}$. In particular, we see that the degree of the composition is equal to the product of the degrees of the factors.

Example 7.4.3. Consider the standard quadratic transformation τ_1 with base points x_1, x_2, x_3. Let y_1, y_2, y_3 be the base points of τ_1^{-1}. Let τ be a Cremona transformation with base points $x_1, \dots, x'_M$ and base points $y'_1, \dots, y'_M$ of τ^{-1}. Assume that $y_i = x'_i$ for $i \le r$. Let A be the characteristic matrix of the composition $\tau \circ \tau_1$. If $r = 3$, we obtain from (6.30)

$$A = \begin{pmatrix} 2 & 1 & 1 & 1 & 0 & 0 & \dots & 0 \\ -1 & 0 & -1 & -1 & 0 & 0 & \dots & 0 \\ -1 & -1 & 0 & -1 & 0 & 0 & \dots & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & \dots & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & 0 & 0 & 0 & \dots & 1 \end{pmatrix} \cdot \begin{pmatrix} d & d_1 & \dots & d_M \\ -m_1 & -m_{11} & \dots & -m_{1M} \\ \vdots & \vdots & \vdots & \vdots \\ -m_M & -m_{M1} & \dots & -m_{MM} \end{pmatrix}.$$

Here we choose some order on the points y_1, y_2, y_3 which affects the matrix A_1.

For example, we obtain that the characteristic of the composition map is equal to

$$(2d - m_1 - m_2 - m_3; d - m_2 - m_3, d - m_1 - m_3, d - m_1 - m_2, m_4, \dots, m_M) \tag{7.62}$$

Assume $r < 3$. We leave to the reader to check that the characteristic of the composition map is equal to

$$(2d - m_1 - m_2; d - m_2, d - m_1, d - m_1 - m_2, m_3, \dots, m_M), \quad r = 2, \tag{7.63}$$

$$(2d - m_1; d, d - m_1, d - m_1, m_2, \dots, m_M), \quad r = 1. \tag{7.64}$$

It is not difficult to see that the same formulae are true in the case when some of the points $y_i = x_i$ are infinitely near.

7.4.2 The Weyl groups

Let $\mathbf{E}_N = (\mathbb{Z}k_N)^\perp \cong \mathbb{Z}^N$ equipped with the quadratic form obtained by the restriction of the inner product in $I^{1,N}$. Assume $N \geq 3$. For any vector $\alpha \in \mathbf{E}_N$ with $\alpha^2 = -2$, we define the following element in $\mathrm{O}(E_N)$:

$$r_\alpha : v \mapsto v + (v, \alpha)\alpha.$$

It is called a *reflection* with respect to α. It leaves the orthogonal complement to α pointwisely fixed, and maps α to $-\alpha$.

Definition 7.5. *The subgroup $W(\mathbf{E}_N)$ of $\mathrm{O}(\mathbf{E}_N)$ generated by reflections r_{α_i} is called the* Weyl group *of $\mathbf{E}_N$.*

The following proposition is stated without proof. It follows from the theory of groups generated by reflections.

Proposition 7.4.6. *The Weyl group $W(\mathbf{E}_N)$ is of infinite index in $\mathrm{O}(\mathbf{E}_N)$ for $N > 10$. For $N \leq 10$,*

$$\mathrm{O}(\mathbf{E}_N) = W(\mathbf{E}_N) \rtimes (\tau),$$

where $\tau^2 = 1$ and $\tau = 1$ if $N = 7, 8$, $\tau = -1$ if $N = 9, 10$ and τ is induced by the symmetry of the Coxeter-Dynkin diagram for $N = 4, 5, 6$.

Note that any reflection can be extended to an orthogonal transformation of the lattice $I^{1,N}$ (use the same formula). The subgroup generated by reflections $r_{\alpha_i}, i \neq 0$, acts as the permutation group $\mathfrak{S}_N$ of the vectors $\mathbf{e}_1, \dots, \mathbf{e}_N$.

Lemma 7.4.7. *(Noether's inequality) Let $v = (d, m_1, \dots, m_N)$. Assume $d > 0, m_1 \geq \dots \geq m_N \geq 0, m_3 \neq 0$, and*

(i) $\sum_{i=1}^n m_i^2 = d^2 + a$;

(ii) $\sum_{i=1}^N m_i = 3d - 2 + a$,

where $a \in \{-1, 0, 1\}$. Then

$$m_1 + m_2 + m_3 \geq d + 1.$$

Proof. We have

$$m_1^2 + \dots + m_N^2 = d^2 - 1, \quad m_1 + \dots + m_N = 3d - 3.$$

Multiplying equality (ii) by m_3 and subtracting it from the first one, we get

$$m_1(m_1 - m_3) + m_2(m_2 - m_3) - \sum_{i \geq 4} m_i(m_3 - m_i) = d^2 + a - 3m_3(d - \tfrac{2-a}{3}).$$

We can rewrite the previous equality in the form

$$(d-\tfrac{2-a}{3})(m_1+m_2+m_3-d-\tfrac{2-a}{3}) = (m_1-m_3)(d-\tfrac{2-a}{3}-m_1)+$$

$$(m_2-m_3)(d-\tfrac{2-a}{3}-m_2)+\sum_{i\geq 4} m_i(m_3-m_i)+a+(\tfrac{2-a}{3})^2.$$

Since $a \geq 0$ and equality (i) implies $m_i < d$, we obtain that the right-hand side positive. Since $m_3 \neq 0$ we get $d > 1$ if $a = -1$ and $d \geq 1$ if $a = 0, 1$. In any case we have $d - \frac{2-a}{3} > 0$. This implies that $m_1 + m_2 + m_3 > d + \frac{2-a}{3} > d$. □

Corollary 7.4.8.

$$m_1 > d/3.$$

We can apply Noether's Lemma to the case when $v = (d, m_1, \ldots, m_N)$ is the characteristic vector of a homaloidal net or when $d\mathbf{e}_0 - \sum m_i\mathbf{e}_i$ is the class of an exceptional configuration.

Definition 7.6. *Let $v = d\mathbf{e}_0 - \sum_{i=1}^N m_i\mathbf{e}_i \in I^{1,N}$. We say that v is of* homaloidal type *(resp.* exceptional type*) if it satisfies conditions (i) and (ii) from above with $a = -1$ (resp. $a = 1$). We say that v is of* proper *homaloidal (exceptional type) if there exists a Cremona transformation whose characteristic matrix has v as the first (resp. second column).*

Lemma 7.4.9. *Let $v = d\mathbf{e}_0 - \sum_{i=1}^n m_i\mathbf{e}_i$ belong to the W_N-orbit of $\mathbf{e}_1$. Then $d \geq 0$. Let $\eta = \sum_{i=1}^N x_i$ be a bubble cycle and $\alpha_\eta : I^{1,N} \to \mathrm{Pic}(Y_\eta)$ be an isomorphism of lattices defined by choosing some admissible order of η. Then $\alpha_\eta(v)$ is an effective divisor.*

Proof. The assertion is true for $v = \mathbf{e}_1$. In fact, $\alpha_\eta(v)$ is the divisor class of the first exceptional configuration $\mathcal{E}_1$. Let $w = s_1 \circ \cdots \circ s_1 \in W_N$ be written as the product of simple reflections and $v = w(\mathbf{e}_1) = (d', m'_1, \ldots, m'_N)$. Let us prove the assertion by using induction on the length of w as the minimal product of simple reflections that $d' \geq 0$. The assertion is obvious if $k = 1$ since $v' = \mathbf{e}_0 - \mathbf{e}_i - \mathbf{e}_j$ or differs from v by a permutation of the m_i's. Suppose the assertion is true for $t = k$. Without loss of generality, we may assume that s_{k+1} is the reflection with respect to some root $\mathbf{e}_0-\mathbf{e}_1-\mathbf{e}_2-\mathbf{e}_3$. Then $d' = 2d-m_1-m_2-m_3 < 0$ implies $4d^2 < (m_1+m_2+m_3)^2 \leq 3(m_1^2+m_2^2+m_3^2)$, hence $d^2 - m_1^2 - m_2^2 - m_3^2 < -\frac{d^2}{3}$. If $d \geq 2$, this contradicts condition (i) of the exceptional type. If $d = 1$, we check the assertion directly by listing all exceptional types.

To prove the second assertion, we use Riemann-Roch Theorem applied to the divisor class $D = \alpha_\eta(v)$. We have $D^2 = -1, D \cdot K_{Y_\eta} = -1$, hence $h^0(D) + h^0(K_{Y_\eta} - D) \geq 1$. Assume $h^0(K_{Y_\eta} - D) > 0$. Intersecting $K_Y - D$ with $e_0 = \alpha_\eta(\mathbf{e}_0)$, we obtain a negative number. However, the divisor class e_0 is nef on Y_η. This shows that $h^0(D) > 0$ and we are done. □

Lemma 7.4.10. *Let v be a proper homaloidal type. Then it belongs to the W_N-orbit of the vector $\mathbf{e}_0$.*

Proof. Let $\mathbf{v} = d\mathbf{e}_0 - \sum_{i=1}^N m_i\mathbf{e}_i$ be a proper homaloidal type and η be the corresponding homaloidal bubble cycle. Let $w \in W_N$ and $v' = w(v) = d'\mathbf{e}_0 - \sum_{i=1}^N m_i'\mathbf{e}_i$. We have $m_i' = \mathbf{e}_i \cdot v' = w^{-1}(\mathbf{e}_i) \cdot v$. Since $w^{-1}(\mathbf{e}_i)$ represents an effective divisor on Y_η and v is the characteristic vector of the corresponding homaloidal net, we obtain $w^{-1}(\mathbf{e}_i) \cdot v \geq 0$, hence $m_i \geq 0$.

Obviously, $m_i \geq 0$. We may assume that $v \neq \mathbf{e}_0$, i.e. the homaloidal net has at least 3 base points. Applying the Noether inequality, we find m_i, m_j, m_k such that $m_i + m_j + m_k > d$. We choose maximal possible such m_i, m_j, m_k. After reordering, we may assume that $m_1 \geq m_2 \geq m_3 \geq \ldots \geq m_N$. Note that this preserves the properness of the homaloidal type since the new order on η is still admissible. Applying the reflection s with respect to the vector $\mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3$, we obtain a new homaloidal type $v' = d'\mathbf{e}_0 - \sum_{i=1}^N m_i'\mathbf{e}_i$ with $d' = 2d - m_1 - m_2 - m_3 < d$. By above, each $m_i \geq 0$. So, we can apply Noether's inequality again until we get $w \in W_N$ such that the number of nonzero coefficients m_i' of $v' = w(v) =$ is at most 2 (i.e. we cannot apply Noether's inequality anymore). A direct computation shows that such vector must be equal to $\mathbf{e}_0$. $\square$

Remark 7.4.3. It follows from Proposition 7.4.2 that the composition of a quadratic transformation with base points x_i, x_j, x_k and a Cremona transformation with characteristic vector v has characteristic vector equal to $v' = s(w')$ where s is the reflection with respect to the vector $\mathbf{e}_0 - \mathbf{e}_i - \mathbf{e}_j - \mathbf{e}_k$. It is important to understand that the proof does not show that v' is obtained in this way, and, in particular, is a proper homaloidal type. If this were true we obtain a proof that any Cremona transformation is the product of quadratic transformations. This is the content of the Noether Theorem below whose proof is different. The original proof of Noether was along these lines, where he wrongly presumed that one can always perform a standard quadratic transformation with base points equal to the highest multiplicities, say m_1, m_2, m_3. The problem here is that the three points x_1, x_2, x_3 may not represent the base points of a standard Cremona transformation when one of the following three cases happen for the three base points x_1, x_2, x_3 of highest multiplicities

(i) $x_2 \succ x_1, x_3 \succ x_1$);

(ii) the base ideal in an affine neighborhood of x_1 is equal to (u^2, v^3).

Theorem 7.4.11. *Let A be a characteristic matrix of a homaloidal net. Then A belongs to the Weyl group $W(\mathbf{E}_N)$.*

Proof. Let $A_1 = (d, -m_1, \ldots, -m_N)$ be the first column of A. Applying the previous lemma, we obtain $w \in W_N$, identified with a $(N+1) \times (N+1)$-matrix such that the $w \cdot A_1 = \mathbf{e}_0$. Thus the matrix $A' = w \cdot A$ has the first column equal to the vector $(1, 0, \ldots, 0)$. Since A' is an orthogonal matrix (with respect to the hyperbolic inner product), it must be the block matrix of the unit matrix I_1 of size 1 and an orthogonal matrix O of size $n-1$. Since O has integer entries it is equal to the product of a permutation matrix P and the diagonal matrix with ± 1 at the diagonal. Since $A \cdot k_N = k_N$ and $w \cdot k_N = k_N$, this easily implies that O is the identity matrix I_N. Thus $w \cdot A = I_{N+1}$ and $A \in W_N$. $\square$

Proposition 7.4.12. *Every vector v in the W_N-orbit of $\mathbf{e}_0$ is a proper homaloidal type.*

Proof. Let $v = w(\mathbf{e}_0)$ for some $w \in W_N$. Write w as the composition of simple reflections $s_k \circ \cdots \circ s_1$. Choose an open subset U of $(\mathbb{P}^2)^N$ such that an ordered set of points $(x_1, \ldots, x_N) \in U$ satisfies the following conditions:

(i) $x_i \neq x_j$ for $i \neq j$;

(ii) if $s_1 = s_{\mathbf{e}_0 - \mathbf{e}_i - \mathbf{e}_j - \mathbf{e}_k}$, then x_i, x_j, x_k are not collinear;

(iii) let α_1 be the involutive quadratic transformation with base points x_i, x_j, x_k and $(y_1, \ldots, y_N)$ be the set of points with $y_i = x_i, y_j = x_j, y_k = x_k$ and $y_h = \alpha_1(x_h)$ for $h \neq i, j, k$. Then $(y_1, \ldots, y_N)$ satisfies conditions (i) and (ii) for s_1 is replaced with s_2. Next do it again by taking s_3 and so on. It is easy to see that in this way U is a non-empty Zariski open subset of $(\mathbb{P}^2)^N$ such that $w(\mathbf{e}_0)$ represents the characteristic vector of a homaloidal net. $\square$

Corollary 7.4.13. *Every vector v in the W_N-orbit of $\mathbf{e}_1$ can be realized as a proper exceptional type.*

Proof. Let $v = w(\mathbf{e}_1)$ for some $w \in W_N$. Then η be a bubble cycle realizing the homaloidal type $w(\mathbf{e}_0)$ and ϕ be the corresponding Cremona transformation with characteristic matrix A. Then v is its second column, and hence corresponds to the first exceptional configuration $\mathcal{E}'_1$ for ϕ^{-1}. $\square$

Remark 7.4.4. It follows from Proposition 7.4.12 (resp. Corollary 7.4.13) that any vector $v = d\mathbf{e}_0 - \sum_{i=1}^N m_i \mathbf{e}_i \in W_N \cdot \mathbf{e}_0$ (resp. $v \in W_N \cdot \mathbf{e}_1) \setminus \{\mathbf{e}_1, \ldots, \mathbf{e}_N\}$) satisfies $d > 0, m_i \geq 0$. We do not know a purely group theoretical proof of this fact.

7.4.3 Noether-Fano inequality

Let T be a Cremona transformation of $\mathbb{P}^2$ defined by a linear system $|d\ell - \sum m_i x_i|$. We order the multiplicities $m_1 \geq \ldots \geq m_N$. Obviously, we may assume that $x_1 \in \mathbb{P}^2$. Assume that one of the points x_2 and x_3 is not infinitely near to x_1 of the first order. Then replacing T with $T \circ Q$, where Q is a quadratic transformation such that the fundamental points of Q^{-1} are equal to x_1, x_2, x_3, we obtain that $T \circ Q$ is given by a linear system of degree $2d - m_1 - m_2 - m_3 < d$ (see (7.62)). Continuing in this way we obtain that $Q_k \circ \cdots Q_1 \circ T$ is given by a linear system of degree 1, i.e. a projective transformation. Unfortunately, this proof is wrong (as was the original proof of M. Noether). The reason is that at a certain step, maybe even at the first one, a quadratic transformation cannot be applied because of infinitely near points $x_2 \succ x_1, x_3 \succ x_1$. We will give a modified version of this proof due to V. Iskovskikh.

First we generalize Corollary 7.4.8 to birational maps of any rational surfaces. The same idea works even for higher-dimensional varieties. Let $T : S - \to S'$ be a birational map of surfaces. Let $\pi : X \to S, \sigma : X \to S'$ be its resolution. Let $|H'|$ be a linear system on X' without base points. Let

$$\sigma^*(H') \sim \pi^*(H) - \sum_i m_i \mathcal{E}_i$$

for some divisor H on S and exceptional configurations $\mathcal{E}_i$ of the map π. Since $|H'|$ has no base points, $|f^*(H')|$ has no base points. Thus $\sigma^*(H')$ intersects nonnegatively any curve on X. In particular,

$$\sigma^*(H') \cdot \mathcal{E}_i = -m_i \mathcal{E}_i^2 = m_i \geq 0. \tag{7.65}$$

This can be interpreted by saying that $T^{-1}(H')$ belongs to the linear system $|H - \eta|$, where $\eta = \sum m_i x_i$ is a bubble cycle on S.

Theorem 7.4.14. *(Noether-Fano inequality) Assume that there exists some integer $m_0 \geq 0$ such that $|H' + mK_{S'}| = \emptyset$ for $m \geq m_0$. For any $m \geq m_0$ such that $|H + mK_S| \neq \emptyset$ there exists i such that*

$$m_i > m.$$

Moreover, we may assume that $x_i \in S$, i.e. $\mathrm{ht}(x_i) = 0$.

Proof. We know that $K_X = \pi^*(K_S) + \sum_i \mathcal{E}_i$. Thus we have the equality in $\mathrm{Pic}(X)$

$$\sigma^*(H') + mK_X = (\pi^*(H + mK_S)) + \sum (m - m_i)\mathcal{E}_i.$$

Applying σ_* to the left-hand side we get the divisor class $H' + mK_{S'}$ which, by assumption cannot be effective. Since $|\pi^*(H+mK_S)| \neq \emptyset$, applying σ_* to the right-hand side, we get the sum of an effective divisor and the image of the divisor $\sum_i (m-m_i)\mathcal{E}_i$. If all $m - m_i$ are nonnegative, it is also an effective divisor, and we get a contradiction. Thus there exists i such that $m - m_i < 0$.

The last assertion follows from the fact that $m_i \geq m_j$ if $x_j \succ x_i$. □

Note that

Example 7.4.4. Assume $S = S' = \mathbb{P}^2$, $H = d\ell$ and $H' = \ell$. We have $|H + K_{S'}| = |-2\ell| = \emptyset$. Thus we can take $m_0 = 1$. If $d \geq 3$, we have for any $1 \leq a \leq d/3$, $|H' + aK_S| = |(d - 3a)\ell| \neq \emptyset$. This gives $m_i > d/3$ for some i. This is Corollary 7.4.8.

Example 7.4.5. Let $S = \mathbf{F}_n$ and $S' = \mathbf{F}_r$ be the minimal Segre-Hirzebruch ruled surfaces. Let $|H'| = |f'|$ be the linear system defined by the ruling on S'. It has no base points, so we can write $[\sigma^*(H')] = \pi^*(af + bs) - \sum m_i e_i$, where f, s the divisor classes of a fibre and the exceptional section on S, and $m_i \geq 0$. Here (X, π, σ) is a resolution of T. Thus $H = af + bs$.

Recall that $K_S = -2s - (2+n)f, K_{S'} = -2s' - (2+r)f'$. Thus $|H' + K_{S'}| = |(-1-n)f - 2s| = \emptyset$. We take $m_0 = 1$. We have

$$|af + bs + mK_S| = |(a - m(2+n))f + (b - 2m)s|.$$

Assume that

$$1 < b \leq \frac{2a}{2+n}.$$

If $m = [b/2]$, then $m \geq m_0$ and both coefficients $a - m(2+n)$ and $b - 2m$ are nonnegative. Thus we can apply Theorem 7.4.14 to find an index i such that $m_i > m \geq b/2$.

In the special case, when $n = 0$, i.e. $S = \mathbb{P}^1 \times \mathbb{P}^1$, the inequality $b \le a$ implies that there exists i such that $m_i > b/2$.

Similar argument can be also applied to the case $S = \mathbb{P}^2, S' = \mathbb{F}_r$. In this case, $|H| = |a\ell|$ and $|aH + mK_S| = |(a-3m)\ell|$. Thus, we can take $m = [a/3]$ and find i such that $m_i > a/3$.

7.4.4 Noether's Reduction Theorem

We shall prove the following.

Theorem 7.4.15. *The group* $\mathrm{Bir}(\mathbf{F}_0)$ *is generated by biregular automorphisms and a birational automorphism* $t_{x,y}$ *for some pair of points* x, y*.*

Applying Proposition 7.3.5, we obtain the following Noether's Reduction Theorem.

Corollary 7.4.16. $\mathrm{Bir}(\mathbb{P}^2)$ *is generated by projective automorphisms and quadratic transformations.*

Now let us prove Theorem 7.4.15.

Let $T : \mathbf{F}_n - \to \mathbf{F}_m$ be a birational map. Let

$$\mathrm{Pic}(\mathbf{F}_n) = \mathbb{Z}f + \mathbb{Z}s, \quad \mathrm{Pic}(\mathbf{F}_m) = \mathbb{Z}f' + \mathbb{Z}s',$$

where f, f' are the divisor classes of fibres, and s, s' are the divisor classes of exceptional sections. Similar to the case of birational maps of projective plane, we can define an ordered resolution (X, π, σ) of T and its characteristic matrix A. We have two bases in $\mathrm{Pic}(X)$

$$\underline{e} : \pi^*(f),\ s = \pi^*(s),\ e_i = [\mathcal{E}_i],\ i = 1, \ldots, N,$$

$$\underline{e'} : \pi^*(f'),\ s' = \pi^*(s'),\ e_i' = [\mathcal{E}_i'],\ i = 1, \ldots, N.$$

For simplicity of notation, let us identify f, s, f', s' with their inverse transforms in $\mathrm{Pic}(X)$. As in the case of Cremona transformations, one can define the characteristic matrix of T. For example, its first column $(a, b; m_1, \ldots, m_N)$ expresses that the preimage of the linear system $|f'|$ on $\mathbf{F}_m$ is the linear system $|af + bs - \eta|$, where $\eta = \sum m_i x_i$ is a bubble cycle over $\mathbf{F}_n$. The first column of the inverse matrix defines preimage of $|f|$ under T^{-1} (the same as the image under T).

Example 7.4.6. Let $T = \mathrm{elm}_x : \mathbf{F}_n - \to \mathbf{F}_{n\pm 1}$. Let f, s, e be the classes of a fibre, the exceptional section, and the exceptional curve E on the blow-up $\pi : X \to \mathbf{F}_n$ of x. Suppose $|s - x| = \emptyset$, i.e., x does not lie on the exceptional divisor. Let $f : X \to \mathbf{F}_{n-1}$ be the blow-down the proper transform $\bar{F}$ of F. Then

$$f' = f, \quad s' = s + f - e, \quad e' = f - e.$$

If $|s - x| \neq \emptyset$, we have

$$f' = f, \quad s' = s - e, \quad e' = f - e.$$

It is easy to see that these transformations are inverse to each other, as it should be. Thus we get

$$f = f', \quad s = s' - e', \quad e = f' - e', \quad \text{if } |s - x| \neq \emptyset,$$
$$f = f', \quad s = s' + f' - e', \quad e = f' - e', \quad \text{otherwise.}$$

Let $T : \mathbf{F}_n - \to \mathbf{F}_m$. Composing T with elm_x, we get a map $\mathrm{elm}_x \circ T : \mathbf{F}_n - \to \mathbf{F}_{m\pm 1}$. The image of $|f|$ on $\mathbf{F}_{m\pm 1}$ is equal to

$$|(a - m_i)f + bs - (b - m_x)x' - \sum_{y \neq x} m_y y|, \quad \text{if } |s - x| = \emptyset, \tag{7.66}$$
$$|(a + b - m_i)f + bs - (b - m_x)x' - \sum_{y \neq x} m_y y|, \quad \text{if } |s - x| \neq \emptyset,$$

where x' is the image of the proper transform of the fibre passing through x.

Lemma 7.4.17. *Let $T : \mathbf{F}_0 - \to \mathbf{F}_0$ be a birational automorphism equal to a composition of elementary transformations. Then T is equal to a composition of biregular automorphisms of $\mathbf{F}_0$ and a transformation $t_{x,y}$ for a fixed pair of points x, y, where y is not infinitely near to x.*

Proof. It follows from Proposition 7.3.5 that $t_{x,y}$, where $y \succ_1 x$ can be written as a composition of two transformations of type $t_{x',y'}$ with no infinitely near points. Now notice that the transformations $t_{x,y}$ and $t_{x',y'}$ for different pairs of points differ by an automorphism of $\mathbf{F}_0$ which sends x to x' and y to y'. Suppose we have a composition T of elementary transformations.

$$\mathbf{F}_0 \overset{\mathrm{elm}_{x_1}}{-- \to} \mathbf{F}_1 \overset{\mathrm{elm}_{x_2}}{-- \to} \ldots \overset{\mathrm{elm}_{x_{k-1}}}{-- \to} \mathbf{F}_1 \overset{\mathrm{elm}_{x_k}}{-- \to} \mathbf{F}_0.$$

If no $\mathbf{F}_0$ occurs among the surfaces $\mathbf{F}_n$ here, then T is a composition of even number k of elementary transformations preserving the projections to $\mathbb{P}^1$. It is clear that not all points x_i are images of points in $\mathbf{F}_0$ lying on the same exceptional section as x_1. Let x_i be such a point (maybe infinitely near to x_1). Then we compose T with t_{x_i,x_1} to obtain a birational map $T' : \mathbf{F}_0 - \to \mathbf{F}_0$ which is a composition of $k - 2$ elementary transformations. Continuing in this way we write T as a composition of transformations $t_{x',y'}$.

If $\mathbf{F}_1 \overset{\mathrm{elm}_{x_{i-1}}}{-- \to} \mathbf{F}_0 \overset{\mathrm{elm}_{x_i}}{-- \to} \mathbf{F}_1$ occurs, then elm_{x_i} may be defined with respect to another projection to $\mathbb{P}^1$. Then we write as a composition of the switch automorphism τ and the elementary transformation with respect to the first projection. Then we repeat this if such $(\mathbf{F}_0, \mathrm{elm}_{x_j})$ occurs again. □

Let $T : \mathbf{F}_0 - \to \mathbf{F}_0$ be a birational transformation. Assume the image of $|f|$ is equal to $|af + bs - \sum m_x x|$. Applying the automorphism τ, if needed, we may assume that $b \leq a$. Thus, using Example 7.4.5, we can find a point x with $m_x > b/2$. Composing T with elm_x, we obtain that the image of $|f|$ in $\mathbf{F}_1$ is the linear system $|a'f' + bs' - m_{x'}x' - \sum_{y \neq x'} m_y y|$, where $m_{x'} = b - m_x < m_x$. Continuing in this way using formula (7.66), we get a map $T' : \mathbf{F}_0 - \to \mathbf{F}_q$ such that the image of $|f|$ is the linear system $|a'f' + bs' - \sum m_x x|$, where all $m_x \le b/2$. If $b = 1$, we get all $m_i = 0$. Thus T' is everywhere defined and hence $q = 0$. The assertion of the Theorem is verified.

Assume $b \ge 2$. Since all $m_i \le b/2$, we must have, by Example 7.4.5,

$$b > \frac{2a'}{2+q}.$$

Since the linear system $|a'f' + bs'|$ has no fixed components, we get

$$(a'f' + bs') \cdot s' = a' - bq \ge 0.$$

Thus $q \le a'/b < (2+q)/2$, and hence $q \le 1$. If $q = 0$, we get $b > a'$. Applying τ, we will decrease b and will start our algorithm again until we either arrive at the case $b = 1$, and we are done, or arrive at the case $q = 1$, and $b > 2a'/3$ and all $m_{x'} \le b/2$.

Let $\pi : \mathbf{F}_1 \to \mathbb{P}^2$ be the blowing down the exceptional section s' to a point q. Then the image of a fibre $|f|$ on $\mathbf{F}_1$ under π is equal to $|\ell - q|$. Hence the image of our linear system in $\mathbb{P}^2$ is equal to $|a'\ell - (a'-b)q - \sum_{p \ne q} m'_p p|$. Obviously, we may assume that $a' \ge b$, hence the coefficient at q is non-negative. Since $b > 2a'/3$, we get $a' - b < a'/3$. By Example 7.4.5, there exists a point $p \ne q$ such that $m'_p > a'/3$. Let $\pi(x) = p$ and $\mathcal{E}_1$ be the exceptional curve corresponding to x and S be the exceptional section in $\mathbf{F}_1$. If $x \in S$, the divisor class $s - e_1$ is effective and is represented by the proper inverse transform of S in the blow-up of x. Then

$$(a'f + bs - m'_x e_1 - \sum_{i>1} m'_i e_i) \cdot (s - e_1) \le a' - b - m'_x < 0.$$

This is impossible because the linear system $|a'f + bs - m_x x - \sum_{y \ne x} y|$ on $\mathbf{F}_1$has no fixed part. Thus x does not lie on S. If we apply elm_x, we arrive at $\mathbf{F}_0$ and may assume that the new coefficient at f' is equal to $a' - m'_x$. Since $m'_x > a'/3$ and $a' < 3b/2$, we see that $a' - m'_x < b$. Now we apply τ to decrease b. Continuing in this way we obtain that T is equal to a product of elementary transformations and automorphisms of $\mathbf{F}_0$. We finish the proof of Theorem 7.4.15 by applying Lemma 7.4.17.

Corollary 7.4.18. *The group* $\mathrm{Cr}(2)$ *of Cremona transformations of* $\mathbb{P}^2$ *is generated by projective automorphisms and the standard Cremona transformation* t_0.

Proof. It is enough to show that the standard quadratic transformations τ_2 and τ_3 are generated by t_0 and projective transformations. Let τ_2 has fundamental points at p_1, p_2 and an infinitely near point $p_3 \succ_1 p_1$. Choose a point q different from p_1, p_2, p_3 and not lying on the line $\overline{p_1, p_2}$. Let T be a quadratic transformation with F-points at p_1, p_2, q. It is easy to check that $T \circ \tau_2$ is a quadratic transformation with F-points $(p_1, p_2, \tau_2(q))$. Composing it with projective automorphisms we get the standard quadratic transformation t_1.

Now let us consider the standard quadratic transformation τ_3 with F-points $p_3 \succ p_2 \succ p_1$. Take a point q which is not on the line in the linear system $|\ell - p_1 - p_2|$. Consider a quadratic transformation T with F-points p_1, p_2, q. It is easy to see that

$T \circ \tau_3$ is a quadratic transformation with F-points $p_1, p_2, \tau_3(q)$. Composing it with projective transformations we get the standard quadratic transformation τ_2. Then we write τ_2 as a composition of τ_1 and projective transformations. $\square$

Exercises

7.1 Consider a minimal resolution X of the standard quadratic transformation τ_1. Show that τ_1 lifts to an automorphism σ of X. Show that σ has 4 fixed points and the orbit space $X/(\sigma)$ is isomorphic to the cubic surface with 4 nodes given by the equation $t_0t_1t_2 + t_0t_1t_3 + t_1t_2t_3 + t_0t_2t_3 = 0$.

7.2 Consider the rational map defined by

$$[t_0, t_1, t_2] \mapsto [t_1t_2(t_0 - t_2)(t_0 - 2t_1), t_0t_2(t_1 - t_2)(t_0 - 2t_1), t_0t_1(t_1 - t_2)(t_0 - t_2)].$$

Show that it is a Cremona transformation and find the Enriques diagram of the corresponding bubble cycle.

7.3 Let C be a plane curve of degree d with a singular point p. Let $\pi : X \to \mathbb{P}^2$ be a sequence of blow-ups which resolves the singularity. Define the bubble cycle $\eta(C, p) = \sum m_i x_i$ as follows: $x_1 = p$ and $m_1 = \mathrm{mult}_p C$, $x_2, \ldots, x_k$ are infinitely near points to p of order 1 such that the proper transform C' of C under the blow-up at p contains these points, $m_i = \mathrm{mult}_{x_i} C', i = 2, \ldots, k$, and so on.

(i) Show that the arithmetic genus of the proper transform of C in X is equal to $\frac{1}{2}(d-1)(d-2) - \frac{1}{2}\sum_i m_i(m_i - 1)$.

(ii) Describe the Enriques diagram of $\eta(C, p)$, where $C = V(t_0^{b-a}t_1^a + t_2^b), p = [1, 0, 0]$, and $a \leq b$ are positive integers.

7.4 Show that two hyperelliptic plane curves H_{g+2} and H'_{g+2} are birationally isomorphic if and only if there exists a De Jonquières transformation which transforms one curve to another.

7.5 Let H_{g+2} be a hyperelliptic curve given by the equation (7.41). Consider the linear system of hyperelliptic curves $H_{q+2} = V(t_2^2 g_q(t_0, t_1) + 2t_2 g_{q+1}(t_0, t_1) + g_{q+2}(t_0, t_1))$ such that $f_g g_{q+2} - 2f_{g+1} g_{q+1} + f_{g+2} g_q = 0$. Show that

(i) the curves H_{q+2} exist if $q \geq (g-2)/2$;

(ii) the branch points of H_{g+2} belong to H_{q+2} and vice versa;

(iii) the curve H_{q+2} is invariant with respect to the De Jonquières involution IH_{g+2} defined by the curve H_{g+2} and the curve H_{g+2} is invariant with respect to the De Jonquières involution IH_{q+2} defined by the curve H_{q+2};

(iv) the involutions IH_{g+2} and IH_{q+2} commute with each other;

(v) the fixed locus of the composition $H_{g+2} \circ H_{q+2}$ is given by the equation $V(f_{g+q+3})$, where

$$f_{g+q+3} = \det \begin{pmatrix} f_g & f_{g+1} & f_{g+2} \\ g_q & g_{q+1} & g_{q+2} \\ 1 & -t_2 & t_2^2. \end{pmatrix};$$

(vi) the De Jonquières transformations which leave the curve H_{g+2} invariant form a group. It contains an abelian subgroup of index 2 which consist of transformations which leave H_{g+2} pointwisely fixed.

7.6 Show that any De Jonquières transformation of finite order leaves a pencil of lines invariant.

7.7 Consider the linear system $L_{a,b} = |af + bs|$ on $\mathbf{F}_n$, where s is the divisor class of the exceptional section, and f is the divisor class of a fibre. Assume $a, b \geq 0$. Show that

(i) $L_{a,b}$ has no fixed part if and only if $a \geq nb$;

(ii) $L_{a,b}$ has no base points if and only if $a \geq nb$;

(iii) Assume $b = 1$ and $a \geq n$. Show that the linear system $L_{a,1}$ maps $\mathbf{F}_n$ in $\mathbb{P}^{2a-n+1}$ onto a surface $X_{a,n}$ of degree $2a - n$;

(iv) show that the surface $X_{a,n}$ is isomorphic to the union of lines $\overline{v_a(x), v_{a-n}(x)}$, where $v_a : \mathbb{P}^1 \to \mathbb{P}^a, v_{2a-n} : \mathbb{P}^1 \to \mathbb{P}^{a-n}$ are the Veronese maps, and $\mathbb{P}^a$ and $\mathbb{P}^{a-n}$ are identified with two disjoint projective subspaces of $\mathbb{P}^{2a-n+1}$.

7.8 Show that the surface $X_{a,n} \subset \mathbb{P}^{2a-n+1}$ contains a nonsingular curve C of genus $g = 2a - n + 2$ which is embedded in $\mathbb{P}^{2a-n+1}$ by the canonical linear system $|K_C|$.

7.9 Find the automorphism group of the surface $\mathbf{F}_n$.

7.10 Show that a projective automorphism T of $\mathbb{P}^2$ which fixes two points is equal to $\Phi_{x_0}(g)$ for some automorphism of $\mathbf{F}_0$ and a point $x_0 \in \mathbf{F}_0$.

7.11 Compute a characteristic matrix of a De Jonquières transformation.

7.12 Compute a characteristic matrix of symmetric Cremona transformation.

7.13 Let C be an irreducible plane curve of degree $d > 1$ passing through the points $x_1, \ldots, x_n$ with multiplicities $m_1 \geq \ldots \geq m_n$. Assume that its proper inverse transform under the blowing up the points $x_1, \ldots, x_n$ is a smooth rational curve $\bar{C}$ with $\bar{C}^2 = -1$. Show that $m_1 + m_2 + m_3 > d$.

7.14 Let $(m, m_1, \ldots, m_n)$ be the characteristic vector of a Cremona transformation. Show that the number of base points with $m_i > m/3$ is less than 9.

7.15 Compute the characteristic matrix of the composition $T \circ T'$ of a De Jonquières transformation T with F-points $p_1, p_2, \ldots, p_{2d-1}$ and characteristic vector $(d, d - 1, 1, \ldots, 1)$ and a quadratic transformation T' with F-points p_1, p_2, p_3.

7.16 Let $\sigma : \mathbb{A}^2 \to \mathbb{A}^2$ be an automorphism of the affine plane given by a formula $(x, y) \to (x + P(y), y)$, where P is a polynomial of degree d in one variable. Consider σ as a Cremona transformation. Compute its characteristic matrix. In the case $d = 3$ write as a composition of projective transformations and quadratic transformations.

7.17 Show that every Cremona transformation is a composition of the following maps ("links"):

(i) the switch involution $\tau : \mathbf{F}_0 \to \mathbf{F}_0$;

(ii) the blow-up $\sigma : \mathbf{F}_1 \to \mathbb{P}^2$;

(iii) the inverse $\sigma^{-1} : \mathbb{P}^2 - \to \mathbf{F}_1$;

(iv) an elementary transformation $\mathrm{elm}_x : \mathbb{F}_q - \to \mathbf{F}_{q\pm 1}$.

7.18 Show that any Cremona transformation is a composition of De Jonquières transformations and projective automorphisms.

7.19 Let $x_0 = [0,1] \times [1,0] \in \mathbb{P}^1 \times \mathbb{P}^1$, $y_0 = \tau(x_0)$, where $\tau : \mathbb{P}^1 \to \mathbb{P}^1$ is the switch of the factors. Show that t_{y_0,x_0} is given by the formula $[u_0, u_1] \times [v_0, v_1] \mapsto [u_0, u_1] \times [u_0 v_1, u_2 v_0]$. Check that the composition $T = \tau \circ t_{y_0,x_0}$ satisfies $T^3 = \mathrm{id}$.

7.20 Let C_1 and C_2 be two plane conics that span an irreducible pencil of conics. For any point x in the plane let $T(x)$ be the intersection of the polar lines $P_x(C_1)$ and $P_x(C_2)$. Show that T is a quadratic Cremona transformation.

Historical Notes

A comprehensive history of the theory of Cremona transformations can be found in several sources [224], [389], [94]. Here we give only a brief sketch.

The general study of plane Cremona transformations was first initiated by L. Cremona in his two papers [104] and [105] published in 1863 and 1864. However, examples of birational transformations were known since the antiquity, for example, the inversion transformation. The example of a quadratic transformation which we presented in Example 7.2.2 goes back to Poncelet [324], although the first idea of a general quadratic transformation must be credited to C. MacLaurin [275]. It was generally believed that all birational transformations must be quadratic and much work was done in developing the general theory of quadratic transformations. The first transformation of arbitrary degree was constructed in 1859 by E. De Jonquières in [119], the De Jonquières transformations. His memoir remained unpublished until 1885 although an abstract was published in 1864 [118]. In his first memoir [104] Cremona gives a construction of a general De Jonquières transformation without reference to De Jonquières. We reproduced his construction in section 7.2.3. Cremona gives the credit to De Jonquières in his second paper. Symmetric transformations of order 5 were first studied by M. Sturm [399], of order 8 by C. Geiser [182], and of order 17 much later by E. Bertini [27].

In his second paper Cremona lays foundation of the general theory of plane birational transformations. He introduces the notion of fundamental points and principal curves, establishes the equalities (7.1.7), proves that the numbers of base points of the transformation and its inverse coincide, proves that principal curves are rational and computes all possible characteristic vectors up to degree 10. The notion of a homaloidal linear system was introduced by Cremona later, first for space transformations in [108] and then for plane transformations in [109]. The word homaloid means flat and was used by J. Sylvester to mean a linear subspace of a projective space. More generally it was applied by A. Cayley to rational curves and surfaces. Cremona also introduces the net of isologues and proves that the number of fixed points of a general transformation of degree d is equal to $d+2$. In the special case of De Jonquière transformations this was also done by De Jonquière in [119]. The notion of isologue curves belongs to him as well as the formula for the number of fixed points.

The first major result in the theory of plane Cremona transformations after Cremona's work was Noether's Theorem. The statement of the theorem was guessed by

W. Clifford in 1869 [84]. The original proof of M. Noether in [301] based on Noether's inequality contained a gap which we explained in Remark 7.4.3. Independently, J. Rosanes found the same proof and made the same mistake [340] . In [302] Noether tried to correct his mistake, taking into account the presence of infinitely near base points of highest multiplicities where one cannot apply a quadratic transformation. He took into account the case of infinitely near points with different tangent direction but overlooked the cuspidal case. The theorem was accepted for thirty years until in 1901 C. Segre pointed out that the cuspidal case was overlooked [374]. In the same year G. Castelnuovo [45] gave a complete proof along the same lines as used in this chapter. In 1916 J. Alexander [3] raised objections to Castelnuovo's proof and gives a proof without using De Jonquières transformations [3]. This seems to be a still accepted proof. It is reproduced, for example, in [2].

The characteristic matrices of Cremona transformation were used by S. Kantor [241] and later by P. Du Val [144]. The latter clearly understood the connection to reflection groups. The description of proper homaliodal and exceptional types as orbits of the Weyl groups were essentially known to H. Hudson. There are numerous modern treatment of this started from M. Nagata [296] and culminated in the monography of M. Alberich-Carramiana [1]. A modern account of Clebsch's Theorem and its history can be also found there.

We intentionally omitted the discussion of finite subgroups of the Cremona group $\mathrm{Cr}(2)$, the modern account of this classification and the history can be found in [141].

Chapter 8

Del Pezzo surfaces

8.1 First properties

8.1.1 Varieties of minimal degree

Recall that a subvariety $X \subset \mathbb{P}^n$ is called *nondegenerate* if it is not contained in a proper linear subspace. Let $d = \deg(X)$. We have the following well-known (i.e., can be found in modern text-books, e.g. [197], [203]) result.

Theorem 8.1.1. *Let X be an irreducible nondegenerate subvariety of $\mathbb{P}^n$ of dimension k and degree d. Then $d \geq n-k+1$, and the equality holds only in one of the following cases:*

(i) X is an irreducible quadric hypersurface;

(ii) a Veronese surface $v_2(\mathbb{P}^2)$ in $\mathbb{P}^5$;

(iii) a cone over a Veronese surface $v_2(\mathbb{P}^2)$ in $\mathbb{P}^5$;

(iv) a rational normal scroll.

Recall that a *rational normal scroll* is defined as follows. Choose k disjoint linear subspaces $L_1, \ldots, L_k$ in $\mathbb{P}^n$ which span the space. Let $a_i = \dim L_i$. We have $\sum_{i=1}^k a_i = n-k+1$. Consider Veronese maps $v_{a_i} : \mathbb{P}^1 \to L_i$ and define $S_{a_1,\ldots,a_k;n}$ to be the union of linear subspaces spanned by the points $v_{a_1}(x), \ldots, v_{a_k}(x)$, where $x \in \mathbb{P}^1$. It is clear that $\dim S_{a_1,\ldots,a_k;n} = k$ and it is easy to see that $\deg S_{a_1,\ldots,a_k;n} = a_1 + \cdots + a_k$ and $\dim S_{a_1,\ldots,a_k;n} = k$. In this notation, it is assumed that $a_1 \leq a_2 \leq \ldots \leq a_k$.

A rational normal scroll of dimension 2 with $a_1 = a, a_2 = n-1-a$ will be denoted by $S_{a,n}$. Its degree is $n-1$ and it lies in $\mathbb{P}^n$. For example, $S_{1,3}$ is a nonsingular quadric in $\mathbb{P}^3$ and $S_{0,3}$ is an irreducible quadric cone.

Corollary 8.1.2. *Let S be an irreducible nondegenerate surface in $\mathbb{P}^n$ of degree d. Then $d \geq n-1$ and the equality holds only in one of the following cases:*

(i) X is a nonsingular quadric in $\mathbb{P}^3$;

(ii) X is an irreducible quadric cone in $\mathbb{P}^3$;

(iii) X is a Veronese surface $v_2(\mathbb{P}^2)$ in $\mathbb{P}^5$;

(iv) X is a rational normal scroll $S_{a,n} \subset \mathbb{P}^n$.

Del Pezzo surfaces come next.

Definition 8.1. *A* Del Pezzo surface *is a nonsingular surface with ample $-K_S$. A* weak Del Pezzo surface *is a nonsingular surface with $-K_S$ nef and big.*

Recall that a divisor D is called *nef* if for any irreducible curve C the intersection number $C \cdot D$ is non-negative. It is called *big* if $D^2 > 0$. Note that, if we require $C \cdot D > 0$ instead of $C \cdot D \geq 0$, then D is an ample divisor. This follows from the Moishezon-Nakai criterion of ampleness .

8.1.2 A blow-up model

Lemma 8.1.3. *Let S be a weak Del Pezzo surface. Then, any irreducible curve C on S with negative self-intersection is a smooth rational curve with $C^2 = -1$ or -2.*

Proof. By adjunction

$$C^2 + C \cdot K_S = \deg \omega_C = 2 \dim H^1(C, \mathcal{O}_C) - 2.$$

By definition of a weak Del Pezzo surface, we have $C \cdot K_S \leq 0$. Thus $0 > C^2 > -2$ and $H^1(C, \mathcal{O}_C) = 0$. It is easy to show that the latter equality implies that $C \cong \mathbb{P}^1$ (the genus of the normalization of an irreducible curve is less or equal to the arithmetic genus defined as $\dim H^1(C, \mathcal{O}_C)$ and the difference is positive if the curve is singular). □

We will call a smooth rational curve with negative self-intersection $-n$ a $(-n)$-*curve*.

Lemma 8.1.4. *Let S be a weak Del Pezzo surface. Then*

$$H^i(S, \mathcal{O}_S) = 0, \; i \neq 0.$$

Proof. We write $0 = -K_S + K_S$ and apply the following Ramanujam's Vanishing Theorem ([265], vol. I, Theorem 4.3.1): for any nef and big divisor D on a nonsingular projective variety X

$$H^i(X, \mathcal{O}_X(K_X + D)) = 0, \; i > 0.$$

□

Theorem 8.1.5. *Let S be a weak Del Pezzo surface. Then, either $X \cong \mathbf{F}_n, n = 0, 2$, or X is obtained from $\mathbb{P}^2$ by blowing up $N \leq 8$ points in the bubble space.*

Proof. Since K_S is not nef, a minimal model for S is either a minimal ruled surface V (over some base curve B) or $\mathbb{P}^2$. Since $H^1(S, \mathcal{O}_S) = 0$, we must have $B \cong \mathbb{P}^1$ (use that the projection $p : V \to B$ satisfies $p_*\mathcal{O}_V \cong \mathcal{O}_B$ and this defines a canonical injective map $H^1(B, \mathcal{O}_B) \to H^1(V, \mathcal{O}_V)$). Thus $V = \mathbf{F}_n$ or $\mathbb{P}^2$. Assume $V = \mathbf{F}_n$. If $n > 2$, the exceptional section of V has self-intersection $r < -2$. Its proper inverse transform on S has self-intersection $\leq r$. This contradicts Lemma 8.1.3. Thus $n \leq 2$. If $n = 1$, then composing the map $S \to \mathbf{F}_1$ with p, we get a birational morphism $S \to \mathbb{P}^2$. Assume $X \not\cong \mathbf{F}_n$, where $n = 0, 2$. Then the birational morphism $f : X \to \mathbf{F}_n$ is equal to the composition of $\phi : S \to V'$ and a blow-up $b : V' \to \mathbf{F}_n$ of a point $p \in \mathbf{F}_n$. Assume $n = 0$, and let ℓ_1, ℓ_2 be two lines on $\mathbf{F}_0$ containing p. Let $V' \to \mathbb{P}^2$ be the blow-down of the proper transforms of the lines. Then the composition $S \to V' \to \mathbb{P}^2$ is a birational morphism to $\mathbb{P}^2$. Assume $n = 2$. The point p does not belong to the exceptional section since otherwise its proper inverse transform in S has self-intersection < -2. Let ℓ be the fibre of $p : \mathbf{F}_2 \to \mathbb{P}^1$ which passes through p. Then elm_p maps $\mathbf{F}_2$ to $\mathbf{F}_1$ and hence blowing down the proper inverse transform of ℓ defines a birational morphism $S \to V' \to \mathbf{F}_1$. Composing it with the birational morphism $\mathbf{F}_1 \to \mathbb{P}^2$, we get a birational morphism $\pi : S \to \mathbb{P}^2$.

The last assertion follows from the known behavior of the canonical class of S under a blow-up. If $\pi : S \to \mathbb{P}^2$ is a birational morphism which is a composition of N blow-ups, then

$$K_S^2 = K_{\mathbb{P}^2}^2 - N = 9 - N. \tag{8.1}$$

By definition, $K_S^2 > 0$, so $N < 9$.

□

Definition 8.2. *The number $d = K_S^2$ is called the* degree *of a weak Del Pezzo surface.*

Lemma 8.1.6. *Let X be a nonsingular projective surface with $H^1(X, \mathcal{O}_X) = 0$. Let C be an irreducible curve on X such that $|-K_X - C| \neq \emptyset$ and $C \notin |-K_X|$. Then $C \cong \mathbb{P}^1$.*

Proof. We have $-K_X \sim C + D$ for some nonzero effective divisor D, and hence $K_X + C \sim -D \not\sim 0$. This shows that $|K_X + C| = \emptyset$. By Riemann-Roch,

$$0 = \dim H^0(X, \mathcal{O}_X(K_X + C)) = \tfrac{1}{2}((K_X + C)^2 - (K_X + C) \cdot K_X) + 1$$

$$-\dim H^1(X, \mathcal{O}_X) + \dim H^2(X, \mathcal{O}_X) \geq 1 + \tfrac{1}{2}(C^2 + K_X \cdot C) = \dim H^1(C, \mathcal{O}_C).$$

Thus $H^1(C, \mathcal{O}_C) = 0$, and as we noted earlier, this implies that $C \cong \mathbb{P}^1$. □

Proposition 8.1.7. *Let S be a weak Del Pezzo surface.*

(i) Let $f : S \to \bar{S}$ be a blowing down a (-1)-curve E. Then $\bar{S}$ is a weak Del Pezzo surface.

(ii) Let $\pi : S' \to S$ be the blowing-up with center at a point x not lying on any (-2)-curve. Assume $K_S^2 > 1$. Then S' is a weak Del Pezzo surface.

Proof. (i) We have $K_S = f^*(K_{\bar{S}}) + E$, and hence, for any curve C on $\bar{S}$, we have

$$K_{\bar{S}} \cdot C = f^*(K_{\bar{S}}) \cdot f^*(C) = (K_S - E) \cdot f^*(C) = K_S \cdot f^*(C) \leq 0.$$

Also $K_{\bar{S}}^2 = K_S^2 + 1 > 0$. Thus $\bar{S}$ is a weak Del Pezzo surface.

(ii) Since $K_S^2 > 2$, we have $K_{S'}^2 = K_S^2 - 1 > 0$. By Riemann-Roch,

$$\dim|-K_{S'}| \geq \tfrac{1}{2}((-K_{S'})^2 - (-K_{S'} \cdot K_{S'})) = K_{S'}^2 \geq 0.$$

Thus $|-K_{S'}| \neq \emptyset$, and hence, any irreducible curve C with $-K_{S'} \cdot C < 0$ must be a proper component of some divisor from $|-K_{S'}|$ (it cannot be linearly equivalent to $-K_{S'}$ because $(-K_{S'})^2 > 0$). Let $E = \pi^{-1}(x)$. We have $-K_{S'} \cdot E = 1 > 0$. So we may assume that $C \neq E$. Let $\bar{C} = f(C)$. We have

$$-K_{S'} \cdot C = \pi^*(-K_S) \cdot C - E \cdot C = -K_S \cdot \bar{C} - \mathrm{mult}_x(\bar{C}).$$

Since $f_*(K_{S'}) = K_S$ and $C \neq E$, the curve $\bar{C}$ is a proper irreducible component of some divisor from $|-K_S|$. By Lemma 8.1.6, $\bar{C} \cong \mathbb{P}^1$. Thus $\mathrm{mult}_x\bar{C} \leq 1$ and hence $0 > -K_{S'} \cdot C \geq -K_S \cdot \bar{C} - 1$. This gives $-K_S \cdot \bar{C} = 0$ and $x \in \bar{C}$ and hence $\bar{C}$ is a (-2)-curve. Since x does not lie on any (-2)-curve we get a contradiction. □

Definition 8.3. *A* blowing down structure *on a weak Del Pezzo surface S is a composition of birational morphisms*

$$\pi : S = S_N \xrightarrow{\pi_N} S_{N-1} \xrightarrow{\pi_{N-1}} \ldots \xrightarrow{\pi_2} S_1 \xrightarrow{\pi_1} \mathbb{P}^2,$$

where each $\pi : S_i \to S_{i-1}$ is the blow-up a point x_i in the bubble space of $\mathbb{P}^2$.

Recall from section 7.4 that a blowing-down structure of a weak Del Pezzo surface defines a basis $(e_0, e_1, \ldots, e_N)$ in $\mathrm{Pic}(S)$, where e_0 is the class of the full preimage of a line and e_i is the class of the exceptional configurations $\mathcal{E}_i$ defined by the point x_i. We call it *geometric basis*. A blowing-down structure defines an isomorphism of free abelian groups

$$\phi : \mathbb{Z}^{N+1} \to \mathrm{Pic}(S) \quad \text{such that } \phi(k_N) = K_S,$$

where $k_N = -3\mathbf{e}_0 + \mathbf{e}_1 + \cdots + \mathbf{e}_N$. We call such an isomorphism a *geometric marking*.

Definition 8.4. *A pair (S, ϕ), where S is a weak Del Pezzo surface and ϕ is a marking (resp. geometric marking) $\mathbb{Z}^{N+1} \to \mathrm{Pic}(S)$ is called a* marked weak Del Pezzo surface *(resp.* geometrically marked weak Del Pezzo surface*).*

Corollary 8.1.8. *Let $\eta = \sum_{i=1}^r x_i$ be a bubble cycle on $\mathbb{P}^2$ and S_η be its blow-up. Then S_η is a weak Del Pezzo surface if and only if*

(i) $r \leq 8$*;*

(ii) the Enriques diagram of η is the disjoint union of chains;

(iii) $|\mathcal{O}_{\mathbb{P}^2}(1) - \eta'| = \emptyset$ for any $\eta' \subset \eta$ consisting of four points;

(iv) $|\mathcal{O}_{\mathbb{P}^2}(2) - \eta'| = \emptyset$ for any $\eta' \subset \eta$ consisting of 7 points.

Proof. The necessity of condition (i) is clear. We know that S does not contain curves with self-intersection < -2. In particular, any exceptional cycle $\mathcal{E}_i$ of the birational morphism $\pi : S \to \mathbb{P}^2$ contains only smooth rational curves E with $E^2 = -1$ or -2. This easily implies that the bubble points corresponding to each exceptional configuration $\mathcal{E}_i$ represent a totally ordered chain. This checks condition (ii).

Suppose (iii) does not hold. Let D be an effective divisor from the linear system $|\mathcal{O}_{\mathbb{P}^2}(1) - \eta'|$. We can change the admissible order on η to assume that $\eta' = x_1 + x_2 + x_3 + x_4$. Then the divisor class of the proper transform of D in Y_η is equal to $e_0 - e_1 - e_2 - e_3 - e_4 - \sum_{i\geq 4} m_i e_i$. Its self-intersection is obviously ≤ -3.

Suppose (iv) does not hold. Let $D \in |\mathcal{O}_{\mathbb{P}^2}(2) - \eta'|$. Arguing as above we find that the divisor class of the proper transform of D is equal to $2e_0 - \sum_{i=1}^7 e_i - \sum_{i\geq 7} m_i e_i$. Its self-intersection is again ≤ -3.

Let us prove the sufficiency. Let $\mathcal{E}_N = \pi_N^{-1}(x_N)$ be the last exceptional configuration of the blow-down $Y_\eta \to \mathbb{P}^2$. It is an irreducible (-1)-curve. Obviously, $\eta' = \eta - x_N$ satisfies conditions (i)-(iv). By induction, we may assume that $S' = S_{\eta'}$ is a weak Del Pezzo surface. Applying Proposition 8.1.7, we have to show that x_N does not lie on any (-2)-curve on S'. Condition (ii) implies that it does not lie on any irreducible component of the exceptional configurations $\mathcal{E}_i, i \neq N$. We will show in the next section that any (-2)-curve on a week Del Pezzo surface S' of degree ≤ 7 is either blown down to a point under the canonical map $S_{\eta'} \to \mathbb{P}^2$ or equal to the proper inverse transform of a line through 3 points, or a conic through 5 points. If x_N lies on the proper inverse transform of such a line (resp. a conic), then condition (iii) (resp. (iv)) is not satisfied. This proves the assertion. □

A set of bubble points satisfying conditions (i)-(iv) is called a set of points in *almost general position*.

We say that the points are in *general position* if the following holds:

(i) all points are proper points;

(ii) no three points are on a line;

(iii) no 6 points on a conic;

(iv) no cubic passes through the points with one of the point being a singular point.

Proposition 8.1.9. *The blow-up of* $N \leq 8$ *points in* $\mathbb{P}^2$ *is a Del Pezzo surface if and only if the points are in general position.*

8.2 The $\mathbf{E}_N$-lattice

8.2.1 Lattices

A (quadratic) *lattice* is a free abelian group $M \cong \mathbb{Z}^r$ equipped with a symmetric bilinear form $M \times M \to \mathbb{Z}$. A relevant example of a lattice is the second cohomology group modulo torsion of a compact 4-manifold (e.g. a nonsingular projective surface) with respect to the cup-product. Another relevant example is the Picard group modulo numerical equivalence of a nonsingular projective surface equipped with the intersection pairing.

The values of the symmetric bilinear form will be often denoted by (x, y) or $x \cdot y$. We write $x^2 = (x, x)$. The map $x \mapsto x^2$ is an integral valued quadratic form on M. Conversely, such a quadratic form $q : M \to \mathbb{Z}$ defines a symmetric bilinear form by the formula $(x, y) = q(x + y) - q(x) - q(y)$. Note that $x^2 = 2q(x)$.

Let $M^\vee = \mathrm{Hom}_Z(M, \mathbb{Z})$ and

$$\iota_M : M \to M^\vee, \quad \iota_M(x)(y) = x \cdot y.$$

We say that M is *nondegenerate* if the homomorphism ι_M is injective. In this case the group

$$\mathrm{Disc}(M) = M^\vee/\iota_M(M)$$

is a finite abelian group. It is called the *discriminant group* of M. If we choose a basis to represent the symmetric bilinear form by a matrix A, then the order of $\mathrm{Disc}(M)$ is equal to $|\det(A)|$. The number $\mathrm{disc}(M) = \det(A)$ is called the *discriminant* of M. A different choice of a basis changes A to ${}^tCAC$ for some $C \in \mathrm{GL}(n, \mathbb{Z})$, so it does not change $\det(A)$. A lattice is called *unimodular* if $|\mathrm{disc}(M)| = 1$.

Tensoring M with reals, we get a real symmetric bilinear form on $M_\mathbb{R} \cong \mathbb{R}^r$. We can identify M with an abelian subgroup of the inner product space $\mathbb{R}^r$ generated by a basis in $\mathbb{R}^r$. The Sylvester signature (t_+, t_-, t_0) of the inner product space $M_\mathbb{R}$ is called the *signature* of M. We write (t_+, t_-) if $t_0 = 0$. For example, the signature of $H^2(X, \mathbb{Z})/\mathrm{Torsion} \cong \mathbb{Z}^{b_2}$ for a nonsingular projective surface X is equal to $(2p_g + 1, b_2 - 2p_g - 1)$, where $p_g = \dim H^0(X, \mathcal{O}_X(K_X))$. The signature on the lattice of divisor classes modulo numerical equivalence $\mathrm{Num}(X) = \mathrm{Pic}(X)/_\equiv \cong \mathbb{Z}^\rho$ is equal to $(1, \rho - 1)$ (this is called the *Hodge Index Theorem*, see [206], Chap. V, Thm. 1.9).

Let $N \subset M$ be a subgroup of M. The restriction of the bilinear form to N defines a structure of a lattice on N. We say that N together with this form is a *sublattice* of M. We say that N is of *finite index* m if M/N is a finite group of order m. Let

$$N^\perp = \{x \in M : x \cdot y = 0, \forall y \in N\}.$$

Note that $N \subset (N^\perp)^\perp$ and the equality takes place if and only if N is a *primitive sublattice* (i.e. M/N is torsion-free).

We will need the following lemmas.

Lemma 8.2.1. *Let M be a nondegenerate lattice and N be its nondegenerate sublattice of finite index m. Then*

$$|\mathrm{disc}(N)| = m^2|\mathrm{disc}(M)|.$$

Proof. Since N is of finite index in M the restriction homomorphism $M^\vee \to N^\vee$ is injective. We will identify $M^\vee$ with its image in $N^\vee$. We will also identify M with its image $\iota_M(M)$ in $M^\vee$. Consider the chain of subgroups

$$N \subset M \subset M^\vee \subset N^\vee.$$

Choose a basis in M, a basis in N, and the dual bases in $M^\vee$ and $N^\vee$. The inclusion homomorphism $N \to M$ is given by a matrix A and the inclusion $N^\vee \to M^\vee$ is given by its transpose ${}^tA$. The order m of the quotient M/N is equal to $|\det(A)|$. The order of $N^\vee/M^\vee$ is equal to $|\det({}^tA)|$. They are equal. Now the chain from above has the first and the last quotient of order equal to m and the middle quotient is of order $|\text{disc}(M)|$. The total quotient $N^\vee/N$ is of order $|\text{disc}(N)|$. The assertion follows. $\square$

Lemma 8.2.2. *Let M be a unimodular lattice and N be its nondegenerate primitive sublattice. Then*

$$|\text{disc}(N^\perp)| = |\text{disc}(N)|.$$

Proof. Consider the restriction homomorphism $r : M \to N^\vee$, where we identify M with $M^\vee$ by means of ι_M. Its kernel is equal to $N^\perp$. Composing r with the projection $N^\vee/\iota_N(N)$ we obtain an injective homomorphism

$$M/(N + N^\perp) \to N^\vee/\iota_N(N).$$

Notice that $N^\perp \cap N = \{0\}$ because N is a nondegenerate sublattice. Thus $N^\perp + N = N^\perp \oplus N$ is of finite index i in M. Also the sum is orthogonal, so that the matrix representing the symmetric bilinear form on $N \oplus N^\perp$ can be chosen to be a block matrix. We denote the orthogonal direct sum of two lattices M_1 and M_2 by $M_1 \oplus M_2$. This shows that $\text{disc}(N \perp N^\perp) = \text{disc}(N)\text{disc}(N^\perp)$. Applying Lemma 8.2.1, we get

$$\#(M/N \perp N^\perp) = \sqrt{|\text{disc}(N^\perp)||\text{disc}(N)|} \le \#(N^\vee/N) = |\text{disc}(N)|.$$

This gives $|\text{disc}(N^\perp)| \le |\text{disc}(N)|$. Since $N = (N^\perp)^\perp$, exchanging the roles of N and $N^\perp$, we get the opposite inequality. $\square$

Lemma 8.2.3. *Let N be a nondegenerate sublattice of a unimodular lattice M. Then*

$$\iota_M(N^\perp) = \text{Ann}(N) := \text{Ker}(r : M^\vee \to N^\vee) \cong (M/N)^\vee.$$

Proof. Under the isomorphism $\iota_M : M \to M^\vee$ the image of $N^\perp$ is equal to $\text{Ann}(N)$. Since the functor $\text{Hom}_\mathbb{Z}(-, \mathbb{Z})$ is left exact, applying it to the exact sequence

$$0 \to N \to M \to M/N \to 0,$$

we obtain an isomorphism $\text{Ann}(N) \cong (M/N)^\vee$. $\square$

A morphism of lattices $\sigma : M \to N$ is a homomorphism of abelian groups preserving the bilinear forms. If M is a nondegenerate lattice, then σ is necessary injective. We say in this case that σ is an *embedding* of lattices. An embedding is called *primitive* if its image is a primitive sublattice. An invertible morphism of lattices is called an *isometry*. The group of isometries of a lattice M to itself is denoted by $\text{O}(M)$ and is called the *orthogonal group* of M.

Let $M_\mathbb{Q} := M \otimes \mathbb{Q} \cong \mathbb{Q}^n$ with the symmetric bilinear form of M extended to a symmetric $\mathbb{Q}$-valued bilinear form on $M_\mathbb{Q}$. The group $M^\vee$ can be identified with the subgroup of $M_\mathbb{Q}$ consisting of vectors v such that $(v, m) \in \mathbb{Z}$ for any $m \in M$.

Suppose that M is nondegenerate lattice. The finite group $\mathrm{Disc}(M)$ can be equipped with a quadratic form defined by

$$q(\bar{x}) = (x,x) \mod \mathbb{Z},$$

where $\bar{x}$ denotes a coset $x + \iota_M(M)$. If M is an *even lattice*, i.e. $m^2 \in 2\mathbb{Z}$ for all $m \in M$, then we take values modulo $2\mathbb{Z}$. The group of automorphisms of $\mathrm{Disc}(M)$ leaving the quadratic form invariant is denoted by $\mathrm{O}(\mathrm{Disc}(M))$.

The proof of the next Lemma can be found in [300].

Lemma 8.2.4. *Let $M \subset N$ be a sublattice of finite index. Then the inclusion $M \subset N \subset N^\vee \subset M^\vee$ defines the subgroup N/M in $\mathrm{Disc}(M) = M^\vee/M$ such that the restriction of the quadratic form of $\mathrm{Disc}(M)$ to it is equal to zero. Conversely, any such subgroup defines a lattice N containing M as a sublattice of finite index.*

The group $\mathrm{O}(M)$ acts naturally on the dual group $M^\vee$ preserving its bilinear form and leaving the subgroup $\iota_M(M)$ invariant. This defines a homomorphism of groups

$$\alpha_M : \mathrm{O}(M) \to \mathrm{O}(\mathrm{Disc}(M)).$$

Lemma 8.2.5. *Let N be a primitive sublattice in a nondegenerate lattice M. Then an isometry $\sigma \in \mathrm{O}(N)$ extends to an isometry of M acting identically on $N^\perp$ if and only if $\sigma \in \mathrm{Ker}(\alpha_N)$.*

8.2.2 The E_N-lattice

Let $I^{1,N} = \mathbb{Z}^{N+1}$ equipped with the symmetric bilinear form defined by the diagonal matrix $\mathrm{diag}(1,-1,\ldots,-1)$ with respect to the standard unit basis

$$\mathbf{e}_0 = (1,0,\ldots,0),\ \mathbf{e}_1 = (0,1,0,\ldots,0),\ldots,\mathbf{e}_N = (0,\ldots,0,1)$$

of $\mathbb{Z}^{N+1}$. Any basis defining the same matrix will be called an *orthonormal basis*. The lattice $I^{1,N}$ is a unimodular lattice of signature $(1,N)$.

Consider the special vector in $I^{1,N}$ defined by

$$\mathbf{k}_N = (-3,1,\ldots,1) = -3\mathbf{e}_0 + \sum_{i=1}^{N} \mathbf{e}_i. \tag{8.2}$$

We define the *E_N-lattice* as a sublattice of $I^{1,N}$ given by

$$\mathbf{E}_N = (\mathbb{Z}\mathbf{k}_N)^\perp.$$

Since $\mathbf{k}_N^2 = 9-N$, it follows from Lemma 8.2.2, that $\mathbf{E}_N$ is a negative definite lattice for $N \le 8$. Its discriminant group is a cyclic group of order $9-N$. Its quadratic form is given by the value on its generator equal to $-\frac{1}{9-N} \mod \mathbb{Z}$ (or $2\mathbb{Z}$ if N is odd).

Lemma 8.2.6. *Assume $N \ge 3$. The following vectors form a basis of $\mathbf{E}_N$*

$$\alpha_1 = \mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3,\ \alpha_i = \mathbf{e}_{i-1} - \mathbf{e}_i,\ i = 2,\ldots,N.$$

The matrix of the symmetric bilinear form of $\mathbf{E}_N$ with respect to this basis is equal to

$$C_N = \begin{pmatrix} -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \dots & 0 & 0 & 0 & -2 & 1 \end{pmatrix}.$$

Proof. By inspection, each α_i is orthogonal to $\mathbf{k}_N$. Suppose $(a_0, a_1, \dots, a_N)$ is orthogonal to $\mathbf{k}_N$. Then

$$3a_0 + a_1 + \cdots + a_N = 0. \tag{8.3}$$

We can write this vector as follows

$$(a_0, a_1, \dots, a_N) = a_0\alpha_1 + (a_0 + a_1)\alpha_2 + (2a_0 + a_1 + a_2)\alpha_3$$

$$+(3a_0 + a_1 + a_2 + a_3)\alpha_4 + \cdots + (3a_0 + a_1 + \cdots + a_{N-1})\alpha_N.$$

We use here that (8.3) implies that the last coefficient is equal to $-a_N$. We leave the computation of the matrix to the reader. □

One can express the matrix C_N by means of the incidence matrix A_N of the following graph with N vertices We have $C_N = -2I_N + A_N$.

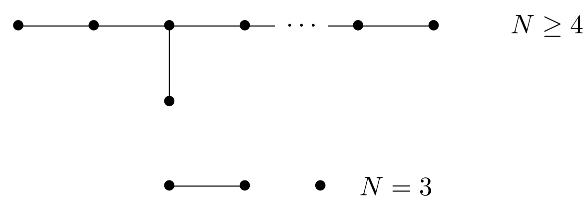


Table 8.1: Coxeter-Dynkin diagram of type E_N

8.2.3 Roots

A vector $\alpha \in \mathbf{E}_N$ is called a *root* if $\alpha^2 = -2$. A vector $(d, m_1, \dots, m_N) \in I^{1,N}$ is a root if and only if

$$d^2 - \sum_{i=1}^{N} m_i^2 = -2, \quad 3d - \sum_{i=1}^{N} m_i = 0. \tag{8.4}$$

Using the inequality $(\sum_{i=1}^N m_i)^2 \leq N \sum_{i=1}^N m_i^2$, it is easy to find all solutions.

Proposition 8.2.7. *Let $N \leq 8$ and*

$$\begin{aligned}\alpha_{ij} &= \mathbf{e}_i - \mathbf{e}_j, 1 \leq i < j \leq N,\\ \alpha_{ijk} &= \mathbf{e}_0 - \mathbf{e}_i - \mathbf{e}_j - \mathbf{e}_k,\ 1 \leq i < j < k \leq N.\end{aligned}$$

Any root in $\mathbf{E}_N$ is equal to one of the following:

N= 3 : $\pm\alpha_{ij}, \pm\alpha_{1,2,3}$. *Their number is* 8.

N= 4 :$\pm\alpha_{ij}, \pm\alpha_{i,j,k}$. *Their number is* 20.

N= 5 : $\pm\alpha_{ij}, \pm\alpha_{i,j,k}$. *Their number is* 40.

N= 6 :$\pm\alpha_{ij}, \pm\alpha_{i,j,k}, \pm(2\mathbf{e}_0 - \mathbf{e}_1 - \ldots - \mathbf{e}_6)$. *Their number is* 72.

N= 7 : $\pm\alpha_{ij}, \pm\alpha_{i,j,k}, \pm(2\mathbf{e}_0 - \mathbf{e}_1 - \ldots - \mathbf{e}_7 - e_i)$. *Their number is* 126.

N= 8 :$\pm\alpha_{ij}, \pm\alpha_{i,j,k}, \pm(2\mathbf{e}_0 - \mathbf{e}_1 - \ldots - \mathbf{e}_8 - e_i - e_j), \pm(3\mathbf{e}_0 - \mathbf{e}_1 - \ldots - \mathbf{e}_8 - e_i)$. *Their number is* 240.

For $N \geq 9$, the number of roots is infinite. From now on we assume

$$3 \leq N \leq 8.$$

An ordered set B of roots $\{\beta_1, \ldots, \beta_r\}$ is called a *root basis* if they are linearly independent over $\mathbb{Q}$ and

$$\beta_i \cdot \beta_j \geq 0.$$

A root basis is called irreducible if it is not equal to the union of non-empty subsets B_1 and B_2 such that $\beta_i \cdot \beta_j = 0$ if $\beta_i \in B_1$ and $\beta_j \in B_2$. The symmetric $r \times t$-matrix $C = (a_{ij})$, where $a_{ij} = \beta_i \cdot \beta_j$ is called the *Cartan matrix* of the root basis.

Recall that a *symmetric Cartan matrix* is a symmetric negative (positive) definite matrix $C = (a_{ij})$ of size n with $a_{ii} = -2(2)$ and $a_{ij} \geq 0(\leq 0)$ for $i \neq j$. All such matrices can be classified. Each Cartan matrix is a block-sum of irreducible Cartan matrices. There are two infinite series of irreducible matrices of types A_n and D_n and three exceptional irreducible matrices of type E_n, where $n = 6, 7, 8$. The matrix $C + 2I_n(C - 2I_n)$, where C is an irreducible Cartan matrix, is the incidence matrix of the Coxeter-Dynkin diagram of type A_n, D_n, E_n.

For $3 \leq n \leq 5$, we will use E_n to denote the Coxeter-Dynkin diagrams of types $A_2 + A_1(N = 3)$, $A_4(N = 4)$ and $D_5(N = 5)$.

Since $\mathbf{E}_N$ is a negative definite lattice for $N \leq 8$, any root basis generates a negative definite sublattice. Hence the matrix of the symmetric form satisfies the conditions of a Cartan matrix. This gives the following.

Proposition 8.2.8. *The Cartan matrix C of an irreducible root basis in $\mathbf{E}_N$ is equal to an irreducible Cartan matrix of type A_r, D_r, E_r with $r \leq N$.*

Definition 8.5. *A* canonical root basis *in $\mathbf{E}_N$ is a root basis with Cartan matrix of type E_N.*

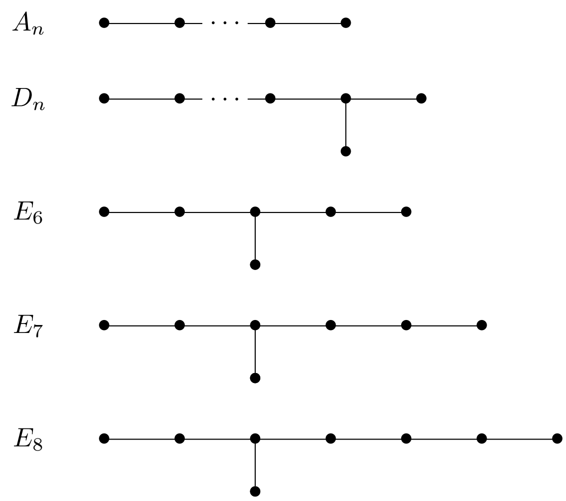


Table 8.2: Coxeter-Dynkin diagrams of types A,D, E

An example of a canonical root basis is the basis formed by the roots

$$\beta_1 = \alpha_{123},\ \beta_i = \alpha_{i-1,i}, \quad i = 2, \ldots, N. \tag{8.5}$$

Theorem 8.2.9. *Any canonical root basis is obtained from a unique orthonormal basis $(v_0, v_1, \ldots, v_n)$ in $I^{1,N}$ such that $\mathbf{k}_N = -3v_0 + v_1 + \cdots + v_N$ by the formula*

$$\beta_1 = v_0 - v_1 - v_2 - v_3,\ \beta_i = v_{i-1} - v_i,\ i = 2, \ldots, N. \tag{8.6}$$

Proof. Given a canonical root basis $(\beta_1, \ldots, \beta_N)$ we solve for v_i in the system of equations (8.6). We have

$$v_i = v_N + \sum_{i=2}^{N} \beta_i,\ i = 1, \ldots, N-1,$$

$$v_0 = \beta_1 + v_1 + v_2 + v_3 = \beta_1 + 3v_N + 3\sum_{i=4}^{N} \beta_i + 2\beta_3 + \beta_2,$$

$$-\mathbf{k}_N = 3v_0 - v_1 - \cdots - v_N = 9v_N + 9\sum_{i=4}^{N} \beta_i + 6\beta_3 + 3\beta_2$$

$$-(v_N + \sum_{i=2}^{N} \beta_i) - (v_N + \sum_{i=3}^{N} \beta_i) - \ldots - (v_N + \beta_N) - v_N.$$

This gives

$$v_N = -\frac{1}{9-N}(\mathbf{k}_N + 3\beta_1 + 2\beta_2 + 4\beta_3 + \sum_{i=3}^{N}(9-i)\beta_{i+1}).$$

Intersecting both sides with β_i we find $(v_N, \beta_i) = 0, i = 1, \ldots, N-1$, and $(v_N, \beta_N) = 1$. Thus all v_i belong to $(\mathbf{k}_N \perp \mathbf{E}_N)^\vee$. The discriminant group of this lattice is isomorphic to $(\mathbb{Z}/(9-N)\mathbb{Z})$ and the only isotropic subgroup of order $9-N$ is the diagonal subgroup. This shows that $\mathbf{E}_N^\vee$ is the only sublattice of $(\mathbf{k}_N \perp \mathbf{E}_N)^\vee$ of index $9-N$, hence $v_i \in \mathbf{E}_N^\vee$ for all i. It is immediately checked that $(v_0, v_1, \ldots, v_N)$ is an orthonormal basis and $\mathbf{k}_N = -3v_0 + v_1 + \cdots + v_N$. $\square$

Corollary 8.2.10. *Let $\mathrm{O}(I^{1,N})_{\mathbf{k}_N}$ be the stabilizer of the vector $\mathbf{k}_N$ in $\mathrm{O}(I^{1,N})$. Then $\mathrm{O}(I^{1,N})_{\mathbf{k}_N}$ acts simply transitively on the set of canonical root bases in $\mathbf{E}_N$.*

Let $\underline{\beta} = (\beta_1, \beta_2, \ldots, \beta_N)$ be a canonical root basis and α be a root. By applying a unique $\sigma \in \mathrm{O}(I^{1,N})_{\mathbf{k}_N}$ we may assume that $\sigma(\beta_1) = \mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3, \sigma(\beta_i) = \mathbf{e}_i - \mathbf{e}_{i+1}, i \geq 2$, and $\sigma(\alpha)$ is one of the vectors from Lemma 8.2.7. It is immediately checked that each such vector is equal to a linear combination of the roots $\sigma(\alpha_i)$ with either all non-negative or all non-positive integer coefficients. So each canonical root basis $\underline{\beta} = (\beta_1, \ldots, \beta_N)$ defines a partition of the set of roots $\mathcal{R}$

$$\mathcal{R} = \mathcal{R}_+ \coprod \mathcal{R}_-,$$

where $\mathcal{R}_+$ is the set of non-negative linear combinations of β_i. The roots from $\mathcal{R}_+$ ($\mathcal{R}_-$) are called *positive* (*negative*) roots with respect to the root basis $\underline{\beta}$. It is clear that $\mathcal{R}_- = \{-\alpha : \alpha \in \mathcal{R}_+\}$.

For any canonical root basis $\underline{\beta}$, the subset

$$C_{\underline{\beta}} = \{x \in I^{1,N}_{\mathbb{R}} : (x, \beta_i) \geq 0\}$$

is called a *Weyl chamber* with respect to $\underline{\beta}$. A subset of a Weyl chamber which consists of vectors such that $(v, \beta_i) = 0$ for some subset $I \subset \{1, \ldots, N\}$ is called a *face*. A face corresponding to the empty set is equal to the interior of the Weyl chamber. The face corresponding to the subset $\{1, \ldots, N\}$ is spanned by the vector $\mathbf{k}_N$.

For any root α and any $x \in I^{1,N}$, let

$$r_\alpha(v) = v + (v, \alpha)\alpha.$$

It is immediately checked that $r_\alpha \in \mathrm{O}(I^{1,N})_{\mathbf{k}_N}$, $r_\alpha(\alpha) = -\alpha$ and $r_\alpha(v) = v$ if $(v, \alpha) = 0$. The isometry r_α is called the *reflection* in the root α. By linearity r_α acts as an orthogonal transformation of the real inner product space $\mathbb{R}^{1,N} := I^{1,N}_{\mathbb{R}}$.

The following is a basis fact from the theory of finite reflection groups. We refer for the proof to numerous text-books on this subject (e.g. [37], [240]).

Theorem 8.2.11. *Let C be a Weyl chamber defined by a canonical root basis $\underline{\beta}$. Let $W(\mathbf{E}_N)$ be the subgroup of $\mathrm{O}(\mathbf{E}_N)$ generated by reflections r_{β_i}. For any $x \in \mathbb{R}^{1,N}$ there exists $w \in W(\mathbf{E}_N)$ such that $w(x) \in C$. If $x, w(x) \in C$, then $x = w(x)$ and x belongs to a face of C. The union of Weyl chambers is equal to $\mathbb{R}^{1,N}$. Two Weyl chambers intersect only along a common face.*

Corollary 8.2.12. *The group $W(\mathbf{E}_N)$ acts simply transitively on canonical root bases, and Weyl chambers. It coincides with the group* $\mathrm{O}(I^{1,N})_{\mathbf{k}_N}$.

The first assertion follows from the theorem. The second assertion follows from Corollary 8.2.10 since $W(\mathbf{E}_N)$ is a subgroup of $\mathrm{O}(I^{1,N})_{\mathbf{k}_N}$.

Corollary 8.2.13. *Then*

$$\mathrm{O}(\mathbf{E}_N) = W(\mathbf{E}_N) \times \langle \tau \rangle,$$

where τ is an isometry of $\mathbf{E}_N$ which is realized by a permutation of roots in a canonical basis leaving invariant the Coxeter-Dynkin diagram. We have $\tau = 1$ for $N = 7, 8$ and $\tau^2 = 1$ for $N \neq 7, 8$.

Proof. By Lemma 8.2.5, the image of the restriction homomorphism $\mathrm{O}(I^{1,N})_{\mathbf{k}_N} \to \mathrm{O}(\mathbf{E}_N)$ is equal to the kernel of the homomorphism $\alpha : \mathrm{O}(\mathbf{E}_N) \to \mathrm{O}(\mathrm{Disc}(\mathbf{E}_N))$. It is easy to compute $\mathrm{O}(\mathrm{Disc}(\mathbf{E}_N))$ and find that it is isomorphic to $\mathbb{Z}/\tau\mathbb{Z}$. Also it can be checked that α is surjective and the image of the symmetry of the Coxeter-Dynkin diagram is the generator of $\mathrm{O}(\mathrm{Disc}(\mathbf{E}_N))$. It remains to apply the previous corollary. $\square$

The definition of the group $W(\mathbf{E}_N)$ does not depend on the choice of a canonical basis and hence coincides with the definition of Weyl groups $W(\mathbf{E}_N)$ from Chapter 7. Note that Corollary 8.2.12 also implies that $W(\mathbf{E}_N)$ is generated by reflections r_α for all roots α in $\mathbf{E}_N$. This is true for $N \leq 10$ and is not true for $N \geq 11$.

Proposition 8.2.14. *If $N \geq 4$, the group $W(\mathbf{E}_N)$ acts transitively on the set of roots.*

Proof. Let $(\beta_1, \ldots, \beta_N)$ be a canonical basis from (8.5). Observe that the subgroup of $W(\mathbf{E}_N)$ generated by the reflections with respect to the roots $\beta_2, \ldots, \beta_N$ is isomorphic to the permutation group $\mathfrak{S}_N$. It acts on the set $\{\mathbf{e}_1, \ldots, \mathbf{e}_N\}$ by permuting its elements and leaves $\mathbf{e}_0$ invariant. This implies that $\mathfrak{S}_N$ acts on the roots $\alpha_{ij}, \alpha_{ijk}$, via its action on the set of subsets of $\{1, \ldots, N\}$ of cardinality 2 and 3. Thus it acts transitively on the set of roots α_{ij} and on the set of roots α_{ijk}. Similarly, we see that it acts transitively on the set of roots $2\mathbf{e}_0 - \mathbf{e}_{i_1} - \ldots - \mathbf{e}_{i_6}$ and $-\mathbf{k}_8 - \mathbf{e}_i$ if $N = 8$. Also applying r_α to α we get $-\alpha$. Now the assertion follows from the following computation

$$\begin{aligned} r_{\beta_1}(-\mathbf{k}_8 - \mathbf{e}_8) &= 2\mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_4 - \ldots - \mathbf{e}_8, \\ r_{\beta_1}(2\mathbf{e}_0 - \mathbf{e}_1 - \ldots - \mathbf{e}_6) &= \alpha_{456}, \\ r_{\beta_1}(\alpha_{124}) &= \alpha_{34}. \end{aligned}$$

$\square$

In a similar way one defines the Weyl group associated to any Cartan matrix and Coxeter-Dynkin diagram of some type T. We consider the negative lattice with the symmetric bilinear form defined by the Cartan matrix. We call it a *root lattice* of the corresponding type. A basis in this lattice in which the matrix of the symmetric bilinear form is equal to the Cartan matrix is called a *root basis*. The subgroup of the orthogonal group of a root lattice generated by reflection in basis vectors is the Weyl group $W(T)$

of type T. It acts transitively on the set of root bases and coincides with the group generated by reflections in all roots.

The types of root bases in the lattice $\mathbf{E}_N$ can be classified by the following procedure due to A. Borel and J. De Siebenthal [33] and, independently by E. Dynkin [148].

Let D be the Coxeter-Dynkin diagram. Consider the extended diagram by adding one more vertex which is connected to other edges as shown on the following *extended Coxeter-Dynkin diagrams*. Consider the following set of elementary operations over the diagrams D and their disconnected sums $D_1 + \cdots + D_k$. Extend one of the components D_i to get the extended diagram. Consider its subdiagram obtained by deleting subset of vertices. Now all possible root bases are obtained by applying recursively the elementary operations to the initial Coxeter-Dynkin diagram of type $\mathbf{E}_N$ and all its descendants.

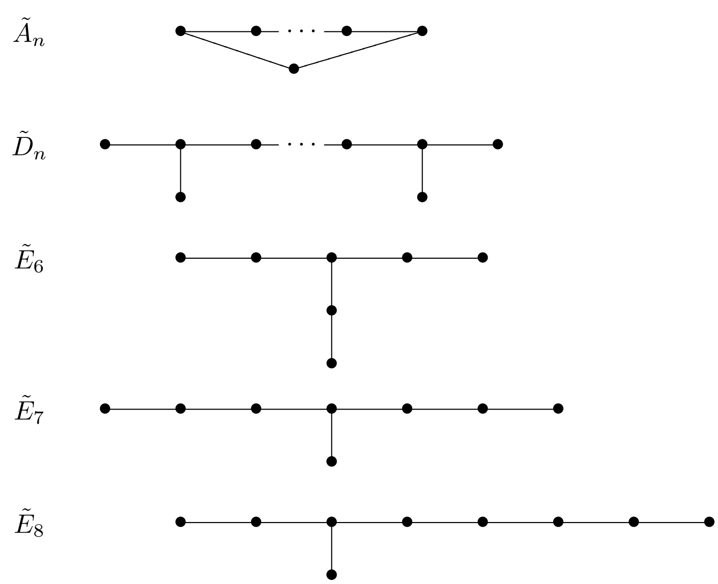


Table 8.3: Extended Coxeter-Dynkin diagrams of types $\tilde{A}, \tilde{D}, \tilde{E}$

8.2.4 Fundamental weights

Let $\underline{\beta} = (\beta_1, \beta_2, \ldots, \beta_N)$ be a canonical root basis in $\mathbf{E}_N$. Consider its dual basis $(\omega_1, \ldots, \omega_N)$ in $\mathbf{E}_N^* \otimes \mathbb{Q}$ defined by $\omega_i(\beta_j) = \delta_{ij}$. Its elements are called *fundamental weights*. We use the expressions for β_i from Theorem 8.2.9. Let us identify $\mathbf{E}_N^*$ with $(\mathbf{k}_N^\perp)^* = I^{1,N}/\mathbb{Z}\mathbf{k}_N$. Then we can take for representatives of ω_j the following vectors from $I^{1,N}$:

$$\begin{aligned}\omega_1 &= v_0,\\ \omega_2 &= v_0 - v_1,\\ \omega_3 &= 2v_0 - v_1 - v_2,\\ \omega_i &= v_i + \cdots + v_N,\ i = 4, \ldots, N.\end{aligned}$$

Definition 8.6. *A vector in $I^{1,N}$ is called an* exceptional vector *if it belongs to the $W(\mathbf{E}_N)$-orbit of ω_N.*

Proposition 8.2.15. *A vector $v \in I^{1,N}$ is exceptional if and only if $\mathbf{k}_N \cdot v = -1$ and $v^2 = -1$. The set of exceptional vectors is the following*

$$\begin{aligned}N = 3,4 &: \mathbf{e}_i, \mathbf{e}_0 - \mathbf{e}_i - \mathbf{e}_j;\\ N = 5 &: \mathbf{e}_i, \mathbf{e}_0 - \mathbf{e}_i - \mathbf{e}_j, 2\mathbf{e}_0 - \mathbf{e}_1 - \ldots - \mathbf{e}_5;\\ N = 6 &: \mathbf{e}_i, \mathbf{e}_0 - \mathbf{e}_i - \mathbf{e}_j, 2\mathbf{e}_0 - \mathbf{e}_1 - \ldots - \mathbf{e}_6 + e_i;\\ N = 7 &: \mathbf{e}_i, \mathbf{e}_0 - \mathbf{e}_i - \mathbf{e}_j, 2\mathbf{e}_0 - \mathbf{e}_1 - \ldots - \mathbf{e}_7 + e_i + e_j; -k_7 - \mathbf{e}_i;\\ N = 8 &: \mathbf{e}_i, \mathbf{e}_0 - \mathbf{e}_i - \mathbf{e}_j, 2\mathbf{e}_0 - \mathbf{e}_1 - \ldots - \mathbf{e}_7 + e_i + e_j; -k_8 + \mathbf{e}_i - \mathbf{e}_j;\\ & \quad -k_8 + \mathbf{e}_0 - \mathbf{e}_i - \mathbf{e}_j - \mathbf{e}_k, -k_8 + 2\mathbf{e}_{i_1} - \ldots - \mathbf{e}_{i_6}.\end{aligned}$$

The number of exceptional vectors is given by the following table:

N	3	4	5	6	7	8
#	6	10	16	27	56	240

Proof. Similarly to the case of roots, we solve the equations

$$d^2 - \sum_{i=1}^{N} m_i^2 = -1,\ 3d - \sum_{i=1}^{N} m_i = 1.$$

First we immediately get the inequality $(3d-1)^2 \le N(d^2+1)$ which gives $0 \le d \le 4$. If $d = 0$, the condition $\sum m_i^2 = d^2 + 1$ and $k_N \cdot v = -1$ gives the vectors $\mathbf{e}_i$. If $d = 1$, this gives the vectors $\mathbf{e}_0 - \mathbf{e}_i - \mathbf{e}_j$, and so on. Now we use the idea of Noether's inequality from Chapter 7 to show that all these vectors $(d, m_1, \ldots, m_N)$ belong to the same orbit of $W(\mathbf{E}_N)$. We apply permutations from $\mathfrak{S}_N$ to assume $m_1 \ge m_2 \ge m_3$, then use the reflection $r_{\alpha_{123}}$ to decrease d. $\square$

Corollary 8.2.16. *The orders of the Weyl groups $W(\mathbf{E}_N)$ are given by the following table:*

N	3	4	5	6	7	8
$\#W(\mathbf{E}_N)$	12	5!	$2^4 \cdot 5!$	$2^3 \cdot 3^2 \cdot 6!$	$2^6 \cdot 3^2 \cdot 7!$	$2^7 \cdot 3^3 \cdot 5 \cdot 8!$

Proof. Observe that the orthogonal complement of $\mathbf{e}_N$ in $I^{1,N}$ is isomorphic to I^{N-1}. Since $\mathbf{e}_N^2 = -1$, by Lemma 8.2.5, the stabilizer subgroup of $\mathbf{e}_N$ in $\mathrm{O}(I^{1,N})$ is equal to $\mathrm{O}(I^{1,N-1})$. This implies that the stabilizer subgroup of $\mathbf{e}_N$ in $W(\mathbf{E}_N)$ is equal to $W(\mathbf{E}_{N-1})$. Obviously, $W(\mathbf{E}_3) \cong \mathfrak{S}_3 \times \mathfrak{S}_2$ and $W(\mathbf{E}_3) \cong \mathfrak{S}_5$. Thus $\#W(\mathbf{E}_5) = 16 \cdot \#W(\mathbf{E}_4) = 2^4 \cdot 5!, \#W(\mathbf{E}_6) = 27 \cdot \#W(\mathbf{E}_5) = 2^3 \cdot 3^2 \cdot 6!,\ \#W(\mathbf{E}_7) = 56 \cdot \#W(\mathbf{E}_6) = 2^6 \cdot 3^2 \cdot 7!, \#W(\mathbf{E}_8) = 240 \cdot \#W(\mathbf{E}_7) = 2^7 \cdot 3^3 \cdot 5 \cdot 8!$. □

Proposition 8.2.17. *For any two different exceptional vectors v, w*

$$(v, w) \in \{0, 1, 2\}.$$

Proof. This can be seen directly from the list, however we prefer to give a proof independent of the classification. It is immediately seen that all vectors $\mathbf{e}_i$ are exceptional. Thus the orbit $W(\mathbf{E}_N) \cdot \omega_N = W(\mathbf{E}_N) \cdot \mathbf{e}_1$ is contained in the orbit of $W(\mathbf{E}_8) \cdot \mathbf{e}_1$. Thus exceptional vectors in $\mathbf{E}_N$ are exceptional vectors in $\mathbf{E}_8$. So it suffices to consider the case $N = 8$. Since $(v, \mathbf{k}_8) = (w, \mathbf{k}_8)$, we have $v - w \in \mathbf{E}_8$. Since $\mathbf{E}_8$ is a negative definite even lattice we have $(v - w, v - w) = -2 - 2(v, w) \leq -2$. This gives $(v, w) \geq 0$. NowAssume $(v, w) > 2$. Let $h = 2\mathbf{k}_8 + v + w$. We have $(v + w)^2 = -2 + 2(v, w) \geq 4$ and $h^2 = 4 - 8 + (v + w)^2 \geq 0,\ h \cdot \mathbf{k}_8 = 0$. Thus $I^{1,8}$ contains two non-proportional orthogonal vectors h and $\mathbf{k}_8$ with non-negative norm square. Since the signature of $I^{1,N}$ is equal to $(1, N)$, we get a contradiction.

□

8.2.5 Gosset polytopes

Consider the real vector space $I^{1,N} \otimes \mathbb{R}$ with the inner product $\langle , \rangle$ defined by the quadratic form on $I^{1,N}$ multiplied by -1. All exceptional vectors lie in the affine space $V_N = \{x \in V_N : (\mathbf{k}_N, x) = 1\}$ and belong to the unit sphere S_N. Let Σ_N be the convex hull of the exceptional vectors. It follows from Proposition 8.2.17 (or from the list of exceptional vectors) that all exceptional vectors are the vertices of Σ_N. For any two vectors $w, w' \in S_N$, the vector $w - w'$ belongs to the even quadratic lattice $\mathbf{E}_N$, hence $2 \leq \langle w - w', w - w' \rangle = 2 - 2\langle w, w' \rangle$. This shows that the minimal distance $\langle w - w', w - w' \rangle^{1/2}$ between two vertices is equal to $\sqrt{2}$ and occurs only when the vectors w and w' are orthogonal. This implies that the edges of Σ_N correspond to pairs of orthogonal exceptional vectors. The difference of such vectors is a root $\alpha = w - w'$ such that $\langle \alpha, w \rangle = 1$. The reflections $s_\alpha : x \mapsto x - \langle x, \alpha \rangle \alpha$ sends w to w'. Thus the reflection hyperplane $H_\alpha = \{x \in V_N : \langle x, \alpha \rangle = 0\}$ intersects the edge at the mid-point. It permutes two adjacent vertices. The Weyl group $W(\mathbf{E}_N)$ acts on Σ_N with the set of vertices forming one orbit. The edges coming out of a fixed vertex correspond to exceptional vectors orthogonal to the vertex. For example, if we take the vertex corresponding to e_N, then the edges correspond to exceptional vectors for the root system $\mathbf{E}_{N-1}$. Thus the vertex figure at each vertex (i.e. the convex hull of mid-points of edges coming from the vertex) is isomorphic to Σ_{N-1}. A convex polytope with isomorphic vertex figures is called a semi-regular polytope (a regular polytope satisfies the additional property that all facets are isomorphic).

The polytopes Σ_N are Gosset polytopes discovered by T. Gosset in 1900 [190]. Following Gosset they are denoted by $(N - 4)_{21}$. We refer to [98], p. 202, for their following facts about their combinatorics. Each polytope Σ_N has two $W(\mathbf{E}_N)$-orbits on the set of facets. One of them is represented by the convex hull of exceptional vectors $e_1, \ldots, e_N$ orthogonal to the vector e_0. It is a $(N-1)$-simplex α_{N-1}. The second one is represented by the convex hull of exceptional vectors orthogonal to $e_0 - e_1$. It is a cross-polytope β_{N-1} (a cross-polytope β_i is the bi-pyramide over β_{i-1} with β_2 being a square). The number of facets is equal to the index of the stabilizer group of e_0 or $e_0 - e_1$ in the Weyl group. The rest of faces are obtained by induction on N. Let N_k be the number of k-faces of Σ_N.

Their number is given in the following table (see [98], 11.8).

k/N	3	4	5	6	7	8
0	6	10	16	27	56	240
1	$3\alpha+6\alpha$	30	80	216	756	6720
2	$2\alpha+3\beta$	$10\alpha+20\alpha$	160	720	4032	60480
3		$5\alpha+5\beta$	$40\alpha+80\alpha$	1080	10080	241920
4			$16\alpha+10\beta$	$432\alpha+$ 216α	12096	483840
5				$72\alpha_5+$ 27β	$2016\alpha+$ 4032α	483840
6					$576\alpha+$ 126β	$69120\alpha+$ 138240α
7						$17280\alpha+$ $+2160\beta$

Table 8.4: Gosset polytopes

The Weyl group $W(\mathbf{E}_N)$ acts transitively on the set of k-faces when $k \leq N-2$. Othwerwise there are two orbits, their cardinality can be found in the table. The dual (reciprocal) polytopes are not semi-regular anymore since the group of symmetries has two orbits on the set of vertices. One is represented by the vector e_0 and another by $e_0 - e_1$.

8.2.6 (-1)-curves on Del Pezzo surfaces

Let $\phi : I^{1,N} \to \text{Pic}(S)$ be a geometric marking of a weak Del Pezzo surface S. The intersection form on Pic(S) equips it with a structure of a lattice. Since ϕ sends an orthonormal basis of $I^{1,N}$ to an orthonormal basis of Pic(S), the isomorphism ϕ is an isomorphism of lattices. The image $K_S^{\perp}$ of $\mathbf{E}_N$ is isomorphic to the lattice $\mathbf{E}_N$.

The image of an exceptional vector is the divisor class E such that $E^2 = E \cdot K_S = -1$. By Riemann-Roch, E is an effective divisor class. Write it as a sum of irreducible components $E = R_1 + \ldots + R_k$. Intersecting with K_S, we obtain that there exists a unique component, say R_1 such that $R_1 \cdot K_S = -1$. For all other components we have $R_i \cdot K_S = 0$. It follows from the adjunction formula that any such component is a (-2)-curve. So, if S is a Del Pezzo surface, the image of any exceptional divisor is a (-1)-curve on S, and we have a bijection between the set of exceptional vectors in $\mathbf{E}_N$ and (-1)-curves on S. If S is a weak Del Pezzo surface, we use the following.

Lemma 8.2.18. *Let D be a divisor class with $D^2 = D \cdot K_S = -1$. Then $D = E + R$, where R is a nonnegative sum of (-2)-curves, and E is either a (-1)-curve or $K_S^2 = 1$ and $E \in |-K_S|$ and $E \cdot R = 0, R^2 = -2$. Moreover D is a (-1)-curve if and only if for each (-2)-curve R_i on S we have $D \cdot R_i \geq 0$.*

Proof. Let $e_0 = \pi^*(\ell)$, where $\pi : S \to \mathbb{P}^2$ is a birational morphism and ℓ is a line. We know that $e_0^2 = 1, e_0 \cdot K_S = -3$. Thus $((D \cdot e_0)K_S + 3D) \cdot e_0 = 0$ and hence

$$\big((D \cdot e_0)K_S + 3D\big)^2 = -6D \cdot e_0 - 9 + (D \cdot e_0)^2 K_S^2 < 0.$$

Thus $-6D \cdot e_0 - 9 < 0$ and hence $D \cdot e_0 > -9/6 > -2$. This shows that $(K_S - D) \cdot e_0 = -3 - D \cdot e_0 < 0$, and since e_0 is nef, we obtain that $|K_S - D| = \emptyset$. Applying Riemann-Roch we get $\dim |D| \geq 0$. Write an effective representative of D as a sum of irreducible components and use that $D \cdot (-K_S) = 1$. Since $-K_S$ is nef, there is only one component E entering with coefficient 1 and satisfying $E \cdot K_S = -1$, all other components are (-2)-curves. If $D \sim E$, then $D^2 = E^2 = -1$ and E is a (-1)-curve. Let $\pi : S' \to S$ be a birational morphism of a weak Del Pezzo surface of degree 1 (obtained by blowing up $8 - k$ points on S in general position not lying on E). We identify E with its preimage in S'. Then $(E + K_{S'}) \cdot K_{S'} = -1 + 1 = 0$, hence, by Hodge Index Theorem, either $S' = S$ and $E \in |-K_S|$, or

$$(E + K_{S'})^2 = E^2 + 2E \cdot K_{S'} + K_{S'}^2 = E^2 - 1 < 0.$$

Since $E \cdot K_S = -1$, E^2 is odd. Thus, the only possibility is $E^2 = -1$. If $E \in |-K_S|$, we have $E \cdot R_i = 0$ for any (-2)-curve R_i, hence $E \cdot R = 0, R^2 = -2$.

Assume $R \neq 0$. Since $-1 = E^2 + 2E \cdot R + R^2$ and $E^2 \leq 1$, $R^2 \leq -2$, we get $E \cdot R \geq 0$, where the equality take place only if $E^2 = 1$. In both cases we get

$$-1 = (E + R)^2 = (E + R) \cdot R + (E + R) \cdot E \geq (E + R) \cdot R.$$

Thus if $D \neq E$, we get $D \cdot R_i < 0$ for some irreducible component of R. This proves the assertion. ☐

The number of (-1)-curves on a Del Pezzo surface is given in Table 8.2.15. It is also can be found in Table 8.4. It is the number of vertices of the Gosset polytope. Other faces give additional information about the combinatorics of the set of (-1)-curves. For example, the number of k-faces of type α is equal to the number of sets of k non-intersecting (-1)-curves.

We can also see the geometric realization of the fundamental weights:

$$w_1 = e_0,\ w_2 = e_0 - e_1,\ w_3 = 2e_0 - e_1 - e_2,\ w_i = e_1 + \ldots + e_N,\ i = 4, \ldots, N.$$

The image of w_1 under a geometric marking represents the divisor class of $\pi^*(\mathcal{O}_{\mathbb{P}^2}(1))$, where $\pi : S \to \mathbb{P}^2$ is the blowing-down morphism defining the geometric marking. The image of w_2 represents the linear system $\pi^*(|\mathcal{O}_{\mathbb{P}^2}(1) - p_1|)$ defining a structure of a conic bundle on S. The image of w_3 is the pre-image of the homaloidal linear system of conics through the points p_1, p_2, p_3. Finally, the images of the remaining fundamental weights represent isolated linear system of disjoint (-1)-curves.

Recall the usual attributes of the minimal model program. Let $\mathrm{Eff}(S)$ be the *effective cone* of a smooth projective surface S, i.e. the open subcone in $\mathrm{Pic}(S) \otimes \mathbb{R}$ spanned by effective divisor classes. Let $\overline{\mathrm{Eff}}(S)$ be its closure. The Cone Theorem [257] states that

$$\overline{\mathrm{Eff}}(S) = \overline{\mathrm{Eff}}(S)_{K_S \geq 0} + \sum_i \mathbb{R}[C_i],$$

where $\overline{\mathrm{Eff}}(S)_{K_S \geq 0} = \{x \in \overline{\mathrm{Eff}}(S) : x \cdot K_S \geq 0\}$ and $[C_i]$ are *extremal rays* spanned by classes of smooth rational curves C_i such that $-C_i \cdot K_X \leq 3$.

Recall that a subcone τ of a cone K is extremal if there exists a linear function ϕ such that $\phi(K) \geq 0$ and $\phi^{-1}(0) \cap K = \tau$. In the case when K is a polyhedral cone, an extremal subcone is a face of K.

Theorem 8.2.19. *Let S be a Del Pezzo surface of degree d. Then*

$$\overline{\mathrm{Eff}}(S) = \sum_{i=1}^{k} \mathbb{R}[C_i],$$

where the set of curves C_i is equal to the set of (-1)-curves if $d \neq 8, 9$. If $d = 8$ and S is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, then $k = 2$, and the $[C_i]$'s are the classes of the two rulings on S. If $d = 8$ and $S \cong \mathbf{F}_1$, then $k = 2$ and C_1 is the class of the exceptional section, and $C_2]$ is the class of a fibre. If $d = 9$, then $k = 1$ and $[C_1]$ is the class of a line.

Proof. Since S is a Del Pezzo surface, $\overline{\mathrm{Eff}}(S)_{K_S \geq 0} = \{0\}$, so it suffices to find the extremal rays. It is clear that $E \cdot K_S = -1$ implies that any (-1)-curve generates an extremal ray. Choose a geometric marking on S to identify $\mathrm{Pic}(S)$ with $I^{1,N}$. Let C be a smooth rational curve such that $c = -C \cdot K_S \leq 3$. By adjunction formula, $C^2 = -2 + c$. If $c = 1$, C is a (-1)-curve. If $c = 2$, applying Corollary 7.4.7, we follow the proof of Proposition 8.2.15 to obtain that all vectors with $v \in I^{1,N}$ satisfying $v \cdot \mathbf{k}_N = -2$ and $(v, v) = 0$ belong to the same orbit of $W(\mathbf{E}_N)$. Thus, if $d < 8$, we may assume that $v = e_0 - e_1$, but then $v = (e_0 - e_1 - e_2) + e_2$ is equal to the sum of two exceptional vectors, hence $[C]$ is not extremal. If $c = 3$, then $C^2 = 1, C \cdot K_S = -3$. Again, we can apply Noether's inequality and the proof of Lemma 7.4.10 to obtain that all such vectors belong to the same orbit. Take $v = e_0$ and write $e_0 = (e_0 - e_1 + e_2) + e_1 + e_2$ to obtain that $[C]$ is not extremal if $d < 8$. We leave the cases $d = 8, 9$ to the reader. $\square$

Corollary 8.2.20. *Assume $d < 8$. Let $\phi : I^{1,N} \to \mathrm{Pic}(S)$ be a geometric marking of a Del Pezzo surface. Then $\phi^{-1}(\overline{\mathrm{Eff}}(S))$ is equal to the Gosset polytope.*

Recall from [257] that any extremal face F of $\overline{\mathrm{Eff}}(S)$ defines a contraction morphism $\phi_F : S \to Z$. The two types of extremal faces of a Gosset polytope define two types of contraction morphisms α_k-type and β_k-type. The contraction of α_k-type blows down the set of disjoint (-1)-curves which are the vertices of the set. The contraction of β_k-type defines a conic bundle structure on S. It is a morphism onto $\mathbb{P}^1$ with general fibre isomorphic to $\mathbb{P}^1$ and singular fibres equal to the union of two (-1)-curves intersecting transversally at one point. Thus the number of facets of type β of the Gosset polytope is equal to the number of conic bundle structures on S.

Another attribute of the minimal model program is the *nef cone* $\mathrm{Nef}(S)$ in $\mathrm{Pic}(S)\otimes\mathbb{R}$ spanned by divisor classes D such that $D\cdot C\geq 0$ for any effective divisor class C. The nef cone is the dual of $\overline{\mathrm{Eff}}(S)$. Under a geometric marking it becomes isomorphic to the dual of the Gosset polytope. It has two types of vertices represented by the normal vectors to facets. One type is represented by the Weyl group orbit of the vector e_0 and another by the vector e_0-e_1.

8.2.7 Effective roots

Let $\phi: I^{1,N}\to\mathrm{Pic}(S)$ be a geometric marking of a weak Del Pezzo surface of degree $d=9-N$. The image of a root $\alpha\in\mathbf{E}_N$ is a divisor class D such that $D^2=-2$ and $D\cdot K_S=0$. We say that α is an *effective root* if $\phi(\alpha)$ is an effective divisor class. Let $\sum_{i\in I}n_iR_i$ be its effective representative. Since $-K_S$ is nef, we obtain that $R_i\cdot K_S=0$. Since K_S^2, we also get $R_i^2<0$. Together with the adjunction formula this implies that each R_i is a (-2)-curve. Since a (-2)-curve does not move, we will identify it with its divisor class.

Proposition 8.2.21. *Let S be a weak Del Pezzo surface of degree $d=9-N$. The number r of (-2)-curves on S is less or equal than N. The sublattice $\mathcal{N}_S$ of $\mathrm{Pic}(S)$ generated by (-2)-curves is a root lattice of rank r.*

Proof. Since each nodal curve is contained in $K_S^{\perp}$ and $R_i\cdot R_j\geq 0$ for $i\neq j$, it suffices to prove that the set of (-2)-curves is linearly independent over $\mathbb{Q}$. Suppose that this is not true. Then we can find two disjoint sets of curves $R_i, i\in I$, and $R_j, j\in J$, such that

$$\sum_{i\in I}n_iR_i\sim\sum_{j\in J}m_jR_j,$$

where n_i,m_j are some non-negative rational numbers. Taking intersection of both sides with R_i we obtain that

$$R_i\cdot(\sum_{i\in I}n_iR_i)=R_i\cdot(\sum_{j\in J}m_jR_j)\geq 0.$$

This implies that

$$(\sum_{i\in I}n_iR_i)^2=\sum_{i\in I}n_i\big(R_i\cdot(\sum_{i\in I}n_iR_i)\big)\geq 0.$$

Since $(\mathbb{Z}K_S)^{\perp}$ is negative definite, this could happen only if $\sum_{i\in I}n_iR_i\sim 0$. Since all coefficients are non-negative, this happens only if all $n_i=0$. For the same reason each m_i is equal to 0. $\square$

It is clear that, if α is a nodal root, then $-\alpha$ is not a nodal root. Let $\eta=x_1+\cdots+x_N$ be the bubble cycle defined by the blowing down structure $S\to\mathbb{P}^2$ defining the geometric marking. It is clear that $\phi(\alpha_{ij})=[\mathcal{E}_i-\mathcal{E}_j]$ is effective if and only if $x_i\succ_{i-j}x_j$. It is also clear that α_{ij} is nodal if and only if $i=j+1$.

A root α_{ijk} is effective if and only if there exist points $x_{i'},x_{j'},x_{k'}$ such that $x_i\succ_{i-i'}x_{i'},x_j\succ_{j-j'}x_{j'},x_k\succ_{k-k'}x_{k'}$, and there is a line in the plane whose proper transform in S belongs to the class $e_0 - e_{i'} - e_{j'} - e_{k'}$. The root $\alpha_{i',j',k'}$ is a nodal root.

The root $2\mathbf{e}_0 - \mathbf{e}_{i_1} - \ldots - \mathbf{e}_{i_6}$ is nodal if and only if its image in Pic(S) is the divisor class of the proper transform of an irreducible conic passing through the points $x_{i_1}, \ldots, x_{i_6}$.

The root $3\mathbf{e}_0 - \mathbf{e}_1 - \ldots - \mathbf{e}_8 - \mathbf{e}_i$ is nodal if and only if its image in Pic(S) is the divisor class of the proper transform of an irreducible cubic with double points at x_i and passing through the rest of the points.

Definition 8.7. *A* Dynkin curve *is a reduced connected curve R on a projective nonsingular surface X such that its irreducible components R_i are -2-curves and the matrix $(R_i \cdot R_j)$ is a Cartan matrix. The type of a Dynkin curve is the type of the corresponding root system.*

Under a geometric marking a Dynkin curve on a weak Del Pezzo surface S corresponds to an irreducible root base in the lattice $\mathbf{E}_N$. We use the Borel-De Siebenthal-Dynkin procedure to determine all possible root bases in $\mathbf{E}_N$.

Theorem 8.2.22. *Let R be a Dynkin curve on a projective nonsingular surface X. There is a birational morphism $f : X \to Y$, where Y is a normal surface satisfying the following properties:*

(i) $f(R)$ *is a point;*

(ii) the restriction of f to $X \setminus R$ is an isomorphism;

(iii) $f^*\omega_Y \cong \omega_X$.

Proof. Let H be a very ample divisor on X. Since the intersection matrix of components of $R = \sum_{i=1}^n R_i$ has non-zero determinant, we can find rational numbers r_i such that

$$(\sum_{i=1}^n r_i R_i) \cdot R_j = -H \cdot R_j, \quad j = 1, \ldots, n.$$

It is easy to see that the entries of the inverse of a Cartan matrix are nonpositive. Thus all r_i's are nonnegative numbers. Replacing H by some multiple mH, we may assume that all r_i are nonnegative integers. Let $D = \sum r_i R_i$. Since $H + D$ is an effective divisor and $(H+D) \cdot R_i = 0$ for each i, we have $\mathcal{O}_X(H+D) \otimes \mathcal{O}_{R_i} = \mathcal{O}_{R_i}$. Consider the standard exact sequence

$$0 \to \mathcal{O}_X(H) \to \mathcal{O}_X(H+D) \to \mathcal{O}_D \to 0.$$

Replacing H by mH, we may assume, by Serre's Duality, that $h^1(\mathcal{O}_X(H)) = 0$ and $\mathcal{O}_X(H)$ is generated by global sections. Let $s_0, \ldots, s_{N-1}$ be sections of $\mathcal{O}_X(H)$ which define an embedding in $\mathbb{P}^{N-1}$. Consider them as sections of $\mathcal{O}_X(H+D)$. Let s_{N+1} be a section of $\mathcal{O}_X(H+D)$ which maps to $1 \in H^0(X, \mathcal{O}_D)$. Consider the map $f' : X \to \mathbb{P}^N$ defined by the sections $(s_0, \ldots, s_N)$. Then $f'(D) = (0, \ldots, 0, 1)$ and $f'|X \subset D$ is an embedding. So we obtain a map $f : X \to \mathbb{P}^N$ satisfying properties (i) and (ii). Since X is normal, f' factors through a map $f : X \to Y$, where Y is normal.

Let ω_Y be the canonical sheaf of Y (it is defined to be equal to the sheaf $j_*\omega_{Y\setminus f'(R)}$, where $j : Y \setminus f'(R) \to Y$ is the natural open embedding). We have

$$\omega_X = f^*\omega_Y \otimes \mathcal{O}_X(A)$$

for some divisor A. Since $K_X \cdot R_i = 0$ for each i, and $f^*\omega_Y \otimes \mathcal{O}_{R_i} = \mathcal{O}_{R_i}$ we get $A \cdot R_i = 0$. Since the intersection matrix of R is negative definite we obtain $A = 0$. $\square$

Definition 8.8. *A point $y \in Y$ of a normal variety Y is called a* canonical singularity *if there exists a resolution $\pi : X \to Y$ such that $\pi^*\omega_Y \cong \omega_X$. In the case $\dim Y = 2$, a canonical singularity is called a RDP (*rational double point*).*

We state the next well-known theorem without proof.

Theorem 8.2.23. *Let $y \in Y$ be a RDP and $\pi : X \to Y$ be a resolution such that $\pi^*\omega_Y \cong \omega_X$. Then $\pi^{-1}(y)$ is a Dynkin curve. Moreover (Y, y) is analytically equivalent to one of the following singularities*

$$\begin{aligned} A_n &: \quad z^2 + x^2 + y^{n+1} = 0, \quad n \geq 1, \\ D_n &: \quad z^2 + y(x^2 + y^{n-2} = 0, \quad n \geq 4, \\ E_6 &: \quad z^2 + x^3 + y^4 = 0, \\ E_7 &: \quad z^2 + x^3 + xy^3 = 0, \\ E_8 &: \quad z^2 + x^3 + y^5 = 0. \end{aligned} \tag{8.7}$$

The corresponding Dynkin curve is of respective type A_n, D_n, E_n.

Remark 8.2.1. The singularity of type A_1 is called in classical and modern literature a *node*. For this reason a (-2)-curve is also called *nodal* although the same term is also used for an irreducible singular curve with ordinary double points (nodes) as a singularities.

8.2.8 Cremona isometries

Definition 8.9. *An orthogonal transformation σ of* Pic(S) *is called a* Cremona isometry *if $\sigma(K_S) = K_S$ and σ sends any effective class to an effective class. The group of Cremona isometries will be denoted by* Cris(S).

It follows from Corollary 8.2.12 that Cris(S) is a subgroup of $W(S)$.

Proposition 8.2.24. *An isometry σ of* Pic(S) *is a Cremona isometry if and only if it preserves the canonical class and sends a (-2)-curve to a (-2)-curve.*

Proof. Clearly, any Cremona isometry sends the class of an irreducible curve to the class of an irreducible curve. Since it also preserves the intersection form, it sends a (-2)-curve to a (-2)-curve.

Let us prove the converse. Let D be an effective class in Pic(S) with $D^2 \geq 0$. Then $-K_S \cdot D > 0$ and $(K_S - D) \cdot D < 0$. This gives $-K_S \cdot \sigma(D) > 0,\ \sigma(D)^2 \geq 0$. Since $(K_S - \sigma(D)) \cdot (-K_S) = -K_S^2 + \sigma(D) \cdot K_S < 0$, we have $|K_S - \sigma(D)| = \emptyset$. By Riemann-Roch, $|\sigma(D)| \neq \emptyset$.

So it remains to show that σ sends any (-1)-curve to an effective divisor class. This follows from the next lemma. $\square$

Corollary 8.2.25. *Let $\mathcal{R}$ be the set of effective roots of a marked Del Pezzo surface (S,ϕ). Then the group of Cremona isometries* Cris(S) *is isomorphic to the subgroup of the Weyl group of $\mathbf{E}_N$ which leaves the subset $\mathcal{R}$ invariant.*

Let $W(S)^n$ be the subgroup of $W(S)$ generated by reflections with respect to (-2)-curves. It acts on a marking $\varphi : I^{1,N} \to \mathrm{Pic}(S)$ by composing on the left.

Lemma 8.2.26. *Let*

$$C^n = \{D \in \mathrm{Pic}(S) : D \cdot R \geq 0 \textit{ for any } (-2)\textit{-curve } R\}.$$

For any $D \in \mathrm{Pic}(S)$ there exists $w \in W(S)^n$ such that $w(D) \in C^n$. If $D \in C^n$ and $w(D) \in C^n$ for some $w \in W(S)^n$, then $w(D) = D$. In other words, C^n is a fundamental domain for the action of $W(S)^n$ in Pic(S).

Proof. The set of (-2)-curves form a root basis in the Picard lattice Pic(S) and $W(S)^n$ is its Weyl group. The set C^n is a chamber defined by the root basis. Now the assertion follows from the theory of finite reflection groups which we have already employed for a similar assertion in the case of canonical root bases in $\mathbf{E}_N$. $\square$

Corollary 8.2.27. *Fix a line E on a weak Del Pezzo surface. There is a natural bijection*

$$(-1)\textit{-curves on } S \longleftrightarrow W(S)^n \backslash W(S)/W(S)_E.$$

Let E be a (-1)-curve and $w \in W(S)$. By Lemma 8.2.18 there exists $g \in W(S)^n$ such that $g(w(E))$ is a (-1)-curve . This (-1)-curve is the unique (-1)-curve $l(E,w)$ in the orbit of $w(E)$ with respect to the action of $W(S)^n$. By Lemma 8.2.26, for any (-1)-curve E' there exists $w \in W(S)$ such that $w(E) = E'$. This shows that the map

$$\Phi_E : W(S) \to \text{set of } (-1)\text{-curves}, \quad w \mapsto l(E,w),$$

is surjective. Suppose that $l(E,w) = l(E,w')$. Then $gw(E) = w'(E)$ for some $g \in W(S)^n$. Thus, $w^{-1}gw'(E) = E$, and hence $w'^{-1}gw \in W(S)_E$ and $w' \in W(S)^n wW(S)$. Conversely, each w' in the double coset $W(S)^n wW(S)$ defines the same (-1)-curve $l(E,w)$.

Theorem 8.2.28. *For any marked weak Del Pezzo surface (S,φ), there exists $w \in W(S)^n$ such that $(S, w\circ\varphi)$ is geometrically marked weak Del Pezzo surface.*

Proof. We use induction on $N = 9 - K_S^2$. Let $e_i = \phi(\mathbf{e}_i), i = 0,\ldots,k$. It follows from the proof of Lemma 8.2.18, that each e_i is an effective class. Assume e_N is the class of a (-1)-curve E_1. Let $\pi_N : S \to S_{N-1}$ be the blowing down of E_N. Then $e_0, e_1, \ldots, e_{N-1}$ are equal to the preimages of the divisor classes $e_0', e_1', \ldots, e_{N-1}'$ on S_{N-1} which define a marking of S_{N-1}. By induction, there exists an element $w \in W(S_{N-1})^n$ such that $w(e_0'), w(e_1'), \ldots, w(e_{N-1}')$ define a geometric marking. Since $\pi_N(E_N)$ does not lie on any (-2)-curve (otherwise S is not a weak Del Pezzo surface), we see that for any (-2)-curve R on S_{N-1}, $\pi_N^*(R)$ is a (-2)-curve on S. Thus, under the canonical isomorphism $\mathrm{Pic}(S) \cong \pi_N^*(\mathrm{Pic}(S_{N-1})) \perp \mathbb{Z}e_N$, we can identify $W(S_{N-1})^n$ with a subgroup of $W(S)^n$. Applying w to $(e_0,\ldots,e_{N-1})$ we get a geometric marking of S.

If e_N is not a (-1)-curve, then we apply an element $w \in W(S)^n$ such that $w(e_N) \in C^n$. By Lemma 8.2.18, $w(e_N)$ is a (-1)-curve. Now we have a basis $w(e_0), \ldots, w(e_N)$ satisfying the previous assumption. □

Let $\varphi : I^{1,N} \to \text{Pic}(S)$ and $\varphi' : I^{1,N} \to \text{Pic}(S)$ be two geometric markings corresponding to two blowing-down structures $\pi = \pi_1 \circ \ldots \circ \pi_N$ and $\pi' = \pi'_1 \circ \ldots \circ \pi'_N$. Then $T = \pi' \circ \pi^{-1}$ is a Cremona transformation of $\mathbb{P}^2$ and $w = \varphi \circ \varphi'^{-1} \in W(\mathbf{E}_N)$ is its characteristic matrix. Conversely, if T is a Cremona transformation with F-points $x_1, \ldots, x_N$ such that their blow-up is a weak Del Pezzo surface S, a characteristic matrix of T defines a pair of geometric markings φ, φ' of S and an element $w \in W(\mathbf{E}_N)$ such that

$$\varphi = \varphi' \circ w.$$

Two different geometric markings π, π' of S define a Cremona transformation of $\pi' \circ \pi^{-1} : \mathbb{P}^2 - \to \mathbb{P}^2$.

Example 8.2.1. Let S be a Del Pezzo surface of degree 3 and $\pi : S \to \mathbb{P}^2$ be the blow-up of 6 points. Let $e_0, e_1, \ldots, e_6$ be the geometric marking and and $\alpha = 2e_0 - e_1 - \ldots - e_6$. The reflection $w = s_\alpha$ transforms the geometric marking $e_0, e_1, \ldots, e_6$ to the geometric marking $e'_0, e'_1, \ldots, e'_6$, where $e'_0 = 5e_0 - 2(e_1 + \ldots + e_6), e'_i = 2e_0 - (e_1 + \ldots + e_6) + e_i, i = 1, \ldots, 6$. The corresponding Cremona transformation is the symmetric involutorial transformation of degree 5 with characteristic matrix equal to

$$\begin{pmatrix} 5 & 2 & 2 & 2 & 2 & 2 & 2 \\ -2 & 0 & -1 & -1 & -1 & -1 & -1 \\ -2 & -1 & 0 & -1 & -1 & -1 & -1 \\ -2 & -1 & -1 & 0 & -1 & -1 & -1 \\ -2 & -1 & -1 & -1 & 0 & -1 & -1 \\ -2 & -1 & -1 & -1 & -1 & 0 & -1 \\ -2 & -1 & -1 & -1 & -1 & -1 & 0 \end{pmatrix}. \tag{8.8}$$

Corollary 8.2.29. *Assume S is a Del Pezzo surface. Then any marking $\varphi : I^{1,N} \to$ Pic(S) is a geometric marking. The Weyl group of $\mathbf{E}_N$ acts simply transitively on the set of markings by composing on the right.*

Corollary 8.2.30. *There is a bijection from the set of geometric markings on S and the set of left cosets $W(S)/W(S)^n$.*

Proof. The group $W(S)$ acts simply transitively on the set of markings. By Theorem 8.2.28, each orbit of $W(S)^n$ contains a unique geometric marking. □

Corollary 8.2.31. *The group* Cris(S) *acts on the set of geometric markings of S.*

Proof. Let $(e_0, \ldots, e_N)$ defines a geometric marking, and $\sigma \in \text{Cris}(S)$. Then there exists $w \in W(S)^n$ such that $\omega(\sigma(e_0)), \ldots, \omega(\sigma(e_N))$ defines a geometric marking. Since $\sigma(e_1)$ is a (-1)-curve E_1, it belongs to C^n. Hence, by Lemma 8.2.26, we get $w(\sigma(e_1)) = \sigma(e_1)$. This shows that $w \in W^n(\bar{S})$, where $S \to \bar{S}$ is the blow-down $\sigma(E_1)$. Continuing in this way, we see that $w \in W(\mathbb{P}^2)^n = \{1\}$. Thus $w = 1$ and we obtain that σ sends a geometric marking to a geometric marking. □

Example 8.2.2. The action of Cris(S) on geometric markings is not transitive in general. For example, consider 6 distinct points $x_1, \ldots, x_6$ in $\mathbb{P}^2$ lying on an irreducible conic C. Let S be their blow-up and ϕ be the corresponding geometric marking. This is a weak Del Pezzo surface with a (-2)-curve R equal to the proper inverse transform of the conic. Let T be the quadratic transformation with F-points at x_1, x_2, x_3. Then $C \in |2\ell - x_1 - x_2 - x_3|$ and hence is equal to $T^{-1}(\ell)$, where ℓ is a line in $\mathbb{P}^2$. This line contains the points $q_i = T(p_i), i = 4, 5, 6$. Let $q_1 = T(\overline{p_2, p_3}), q_2 = T(\overline{p_1, p_3}), q_3 = T(\overline{p_1, p_2})$. Then the blow-up of the points $q_1, \ldots, q_6$ is isomorphic to S and defines a geometric marking ϕ'. Let w be the corresponding element of the Weyl group $W(Q_6) \cong W(\mathbf{E}_6)$. We have

$$R = 2e_0 - e_1 - \ldots - e_6 = e_0' - e_4' - e_5' - e_6'.$$

However, the element $w \in W(S)$ defined by the two bases sends e_i to e_i'. Thus $w(R) \neq R$ and hence $w \notin$ Cris(S). Note that w is the reflection with respect to the root $\alpha = e_0 - e_1 - e_2 - e_3$ and $\alpha \cdot R = -1$, so that

$$r_\alpha(R) = R - \alpha = (2e_0 - e_1 - \ldots - e_6) - (e_0 - e_1 - e_2 - e_3) = e_0 - e_4 - e_5 - e_6.$$

Since the points p_4, p_5, p_6 are not collinear, this is not an effective class. The group Cris(S) in this case consists of permutations of the vectors $e_1, \ldots, e_6$ and is isomorphic to $\mathfrak{S}_6$. Its index in $W(\mathbf{E}_6)$ is equal to 72. The group $W(S)^n$ is generated by the reflection r_α. Thus we get $\frac{1}{2}\#W(\mathbf{E}_6) = 36 \cdot 6!$ geometric markings and the group Cris(S) has 36 orbits on this set.

Let S be a weak Del Pezzo surface of degree d and Aut(S) be its group of biregular automorphims. By functoriality Aut(S) acts on Pic(S) leaving the canonical class K_S invariant. Thus Aut(S) acts on the lattice $Q_X = (\mathbb{Z}K_S)^\perp$ preserving the intersection form. Let

$$\rho : \text{Aut}(S) \to \text{O}(Q_X), \quad \sigma \mapsto \sigma^*,$$

be the corresponding homomorphism.

Proposition 8.2.32. *The image of ρ is contained in the group* Cris(S). *If S is a Del Pezzo surface, the kernel of ρ is trivial if $d \leq 5$. If $d \geq 6$, then the kernel is a linear algebraic group of dimension* $2d - 10$.

Proof. Clearly, any automorphism induces a Cremona isometry of Pic(S). We know that it is contained in the Weyl group. An element in the kernel does not change any geometric basis of Pic(S). Thus it descends to an automorphism of $\mathbb{P}^2$ which fixes an ordered set of $k = 9 - d$ points in general linear position. If $k \geq 4$ it must be the identity transformation. Assume $k \leq 3$. The assertion is obvious when $k = 0$.

If $k = 1$, the surface S is the blow-up of one point. Each automorphism leaves the unique exceptional curve invariant and acts trivially on the Picard group. The group Aut(S) is the subgroup of Aut$(\mathbb{P}^2)$ fixing a point. It is a connected linear algebraic group of dimension 6 isomorphic to the semi-direct product of $\mathbb{C}^2 \rtimes \text{GL}(2)$.

If $k = 2$, the surface S is the blow-up of two distinct points p_1, p_2. Each automorphism leaves the proper inverse transform of the line $\overline{p_1, p_2}$ invariant. It either leaves the exceptional curves E_1 and E_2 invariant, or switches them. The kernel of the Weyl reprsentation consists of elements which do not switch E_1 and E_2. It is isomorphic to the subgroup of Aut$(\mathbb{P}^2)$ which fixes two points in $\mathbb{P}^2$ and is isomorphic to the group G of invertible matrices of the form

$$\begin{pmatrix} 1 & 0 & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}.$$

Its dimension is equal to 4. The image of the Weyl representation is a group of order 2. So Aut$(S) = G \rtimes C_2$.

If $k = 3$, the surface S is the blow-up of 3 non-collinear points. The kernel of the Weyl representation is isomorphic to the group of invertible diagonal 3×3 matrices modulo scalar matrices. It is isomorphic to a 2-dimension torus $(\mathbb{C}^*)^2$. $\square$

Corollary 8.2.33. *Assume that $d \leq 5$, then* Aut(S) *is isomorphic to a subgroup of the Weyl group $W(\mathbf{E}_{9-d})$.*

We will see later examples of automorphisms of weak Del Pezzo surfaces of degree 1 or 2 which act trivially on Pic(S).

8.3 Anticanonical models

8.3.1 Anticanonical linear systems

Lemma 8.3.1. *Let S be a weak Del Pezzo surface with $K_S^2 = d$. Then*

$$\dim H^0(S, \mathcal{O}_S(-rK_S)) = 1 + \tfrac{1}{2}r(r+1)d.$$

Proof. By Ramanujam's Vanishing Theorem, for any $r \geq 0$ and $i > 0$,

$$H^i(S, \mathcal{O}_S(-rK_S)) = H^i(S, \mathcal{O}_S(K_S + (-r-1)K_S)) = 0. \tag{8.9}$$

The Riemann-Roch Theorem gives

$$\dim H^0(S, \mathcal{O}_S(-rK_S)) = \tfrac{1}{2}(-rK_S - K_S)\cdot(-rK_S) + 1 = 1 + \tfrac{1}{2}r(r+1)d.$$

$\square$

Theorem 8.3.2. *Let S be a weak Del Pezzo surface of degree d and $\mathcal{N}$ be the union of (-2)-curves on S. Then*

(i) $|-K_S|$ *has no fixed part.*

(ii) If $d > 1$, then $|-K_S|$ has no base points.

(iii) If $d > 2$, $|-K_S|$ defines a regular map ϕ to $\mathbb{P}^d$ which is an isomorphism outside $\mathcal{N}$. The image surface $\bar{S}$ is a normal nondegenerate surface of degree d. The image of each connected component of $\mathcal{N}$ is a RDP of $\phi(S)$.

(iv) *If $d = 2$, $|-K_S|$ defines a regular map $\phi : S \to \mathbb{P}^2$. It factors as a birational morphism $f : S \to \bar{S}$ onto a normal surface and a finite map $\pi : \bar{S} \to \mathbb{P}^2$ of degree 2 branched along a curve of degree 4. The image of each connected component of $\mathcal{N}$ is a RDP of $\bar{S}$.*

(v) *If $d = 1$, $|-2K_S|$ defines a regular map $\phi : S \to \mathbb{P}^3$. It factors as a birational morphism $f : S \to \bar{S}$ onto a normal surface and a finite map $\pi : \bar{S} \to Q \subset \mathbb{P}^3$ of degree 2, where Q is a quadric cone. The morphism π is branched along a curve of degree 6 cut out on Q by a cubic surface. The image of each connected component of $\mathcal{N}$ under f is a RDP of $\bar{S}$.*

Proof. The assertions are easily verified if $S = \mathbf{F}_0$ or $\mathbf{F}_2$. So we assume that S is obtained from $\mathbb{P}^2$ by blowing up $k = 9 - d$ points x_i.

(i) Assume there is a fixed part F of $|-K_S|$. Write $|-K_S| = F + |M|$, where $|M|$ is the mobile part. If $F^2 > 0$, by Riemann-Roch,

$$\dim |F| \geq \tfrac{1}{2}(F^2 - F \cdot K_S) \geq \tfrac{1}{2}(F^2) > 0,$$

and hence F moves. Thus $F^2 \leq 0$. If $F^2 = 0$, we must also have $F \cdot K_S = 0$. Thus $F = \sum n_i R_i$, where R_i are (-2)-curves. Hence $[f] \in (\mathbb{Z}K_S)^\perp$ and hence $F^2 \leq -2$ (the intersection form on $(\mathbb{Z}K_S)^\perp$ is negative definite and even). Thus $F^2 \leq -2$. Now

$$\begin{aligned} M^2 &= (-K_S - F)^2 = K_S^2 + 2K_S \cdot F + F^2 \leq K_S^2 + F^2 \leq d - 2, \\ -K_S \cdot M &= K_S^2 + K_S \cdot F \leq d. \end{aligned}$$

Suppose $|M|$ is irreducible. Since $\dim |M| = \dim |-K_S| = d$, the linear system $|M|$ defines a rational map to $\mathbb{P}^d$ whose image is a nondegenerate irreducible surface of degree $\leq d-3$ (strictly less if $|M|$ has base points). This contradicts Theorem 8.1.1.

Now assume that $|M|$ is reducible, i.e. defines a rational map to a nondegenerate curve $W \subset \mathbb{P}^d$ of some degree t. By Theorem 8.1.1, we have $t \geq d$. Since S is rational, W is a rational curve, and then the preimage of a general hyperplane section is equal to the disjoint sum of t linearly equivalent curves. Thus $M \sim tM_1$ and

$$d \geq -K_S \cdot M = -tK_S \cdot M_1 \geq d(-K_S \cdot M_1).$$

Since $-K_S \cdot M = 0$ implies $M^2 < 0$ and a curve with negative self-intersection does not move, this gives $-K_S \cdot M_1 = 1, d = t$. But then $M^2 = d^2 M_1^2 \leq d - 2$ gives a contradiction.

(ii) Assume $d > 1$. We have proved that $|-K_S|$ is irreducible. A general member of $|-K_S|$ is an irreducible curve C with $\omega_C = \mathcal{O}_C(C + K_S) = \mathcal{O}_C$. If C is smooth, then it is an elliptic curve and the linear system $|\mathcal{O}_C(C)|$ is of degree $d > 1$ and has no base points. The same is true for a singular irreducible curve of arithmetic genus 1. This is proved in the same way as in the case of a smooth curve. Consider the exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(C) \to \mathcal{O}_C(C) \to 0.$$

Applying the exact sequence of cohomology, we see that the restriction of the linear system $|C| = |-K_S|$ to C is surjective. Thus we have an exact sequence of groups

$$0 \to H^0(S, \mathcal{O}_S) \to H^0(S, \mathcal{O}_S(C)) \to H^0(S, \mathcal{O}_C(C)) \to 0.$$

Since $|\mathcal{O}_C(C)|$ has no base points, we have a surjection

$$H^0(S,\mathcal{O}_C(C))\otimes\mathcal{O}_C\to\mathcal{O}_C(C).$$

This easily implies that the homomorphism

$$H^0(S,\mathcal{O}_S(C))\otimes\mathcal{O}_C\to\mathcal{O}_S(C)$$

is surjective. Hence $|C|=|-K_S|$ has no base points.

(iii) Assume $d>2$. Let $x,y\in S$ be two points outside E. Let $f:S'\to S$ be the blowing up of x and y. By Proposition 8.1.7, blowing them up, we obtain a weak Del Pezzo surface S' of degree $d-2$. We know that the linear system $|-K_{S'}|$ has no fixed components. Thus

$$\dim|-K_S-x-y|=\dim|-K_{S'}-E_x-E_y|\geq 1.$$

This shows that $|-K_S|$ separates points. Also, the same is true if $y\succ_1 x$ and x does not belong to any (-1)-curve E on S or $x\in E$ and y does not correspond to the tangent direction defined by E. Since $-K_S\cdot E=1$ and $x\in E$, the latter case does not happen.

Since $\phi:S-\to\bar{S}$ is a birational map given by a complete linear system $|-K_S|$, its image is a nondegenerate surface of degree $d=(-K_S)^2$. Since $-K_S\cdot R=0$ for any (-2)-curve, we see that ϕ blows down R to a point p. If $d=3$, then $\bar{S}$ is a cubic surface with isolated singularities (the images of connected components of $\mathcal{N}$). It is well-known that a hypersurface with no singularities in codimension 1 is a normal variety. Thus $\bar{S}$ is a normal surface. If $d=4$, then S is obtained by a blow-up one point on a weak Del Pezzo surface S' of degree 3. This point does not lie on a (-2)-curve. Thus, $\bar{S}'$ is obtained from $\bar{S}$ by a linear projection from a nonsingular point. Since $\bar{S}'$ is normal, $\bar{S}$ must be normal too (its local rings are integral extensions of local rings of $\bar{S}'$, and their fields of fractions coincide). Continuing in this way we see that $\bar{S}$ is normal for any $d>2$.

The fact that singular points of $\bar{S}$ are RDP is proven in the same way as we have proved assertion (iii) of Theorem 8.2.22.

(iv) Assume $d=2$. By (ii), the linear system $|-K_S|$ defines a regular map $\phi:S\to\mathbb{P}^2$. Since $K_S^2=2$, the map is of degree 2. Using Stein's factorization [206], it factors through a birational morphism onto a normal surface $f:S\to\bar{S}$ and a finite degree 2 map $\pi:\bar{S}\to\mathbb{P}^2$. Also we know that $f_*\mathcal{O}_S=\mathcal{O}_{\bar{S}}$. A standard Hurwitz's formula gives

$$\omega_{\bar{S}}\cong\pi^*(\omega_{\mathbb{P}^2}\otimes\mathcal{L}), \tag{8.10}$$

where $s\in H^0(\mathbb{P}^2,\mathcal{L}^{\otimes 2})$ vanishes along the branch curve W of π. We have

$$\mathcal{O}_S(K_S)=\omega_S=(\pi\circ f)^*\mathcal{O}_{\mathbb{P}^2}(-1)=f^*(\pi^*\mathcal{O}_{\mathbb{P}^2}(-1)).$$

It follows from the proof of Theorem 8.2.22 (iii) that singular points of $\bar{S}$ are RDP. Thus $f^*\omega_{\bar{S}}=\omega_S$, and hence

$$f^*\omega_{\bar{S}}\cong f^*(\pi^*\mathcal{O}_{\mathbb{P}^2}(-1)).$$

Applying f_* and using the projection formula and the fact that $f_*\mathcal{O}_X = \mathcal{O}_Y$, we get $\omega_{\bar{S}} \cong \pi^*\mathcal{O}_{\mathbb{P}^2}(-1)$. It follows from (8.10) that $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^2}(2)$ and hence $\deg W = 4$.

Proof of (v). Let $\pi : S \to \mathbb{P}^2$ be the blow-up of 8 points $x_1, \ldots, x_8$. Then $|-K_S|$ is the proper inverse transform of the pencil $|3\ell - x_1 - \ldots - x_8|$ of plane cubics passing through the points $x_1, \ldots, x_8$. Let x_9 be the ninth intersection point of two cubics generating the pencil. The point $x_9' = \pi^{-1}(x_9)$ is the base point of $|-K_S|$. By Bertini's Theorem, all fibres except finitely many, are nonsingular curves (the assumption that the characteristic is zero is important here). Let F be a nonsingular member from $| - K_S|$. Consider the exact sequence

$$0 \to \mathcal{O}_S(-K_S) \to \mathcal{O}_S(-2K_S) \to \mathcal{O}_F(-2K_S) \to 0. \tag{8.11}$$

The linear system $|\mathcal{O}_F(-2K_S)|$ on F is of degree 2. It has no base points. We know from (8.9) that $H^1(S, \mathcal{O}_S(-K_S)) = 0$. Thus the restriction map

$$H^0(S, \mathcal{O}_S(-2K_S)) \to H^0(F, \mathcal{O}_F(-2K_S))$$

is surjective. By the same argument as we used in the proof of (ii), we obtain that $| - 2K_S|$ has no base points. By Lemma 8.3.1, $\dim | - 2K_S| = 3$. Let $\phi : S \to \mathbb{P}^3$ be a regular map defined by $| - 2K_S|$. Its restriction to any nonsingular member F of $| - K_S|$ is given by the linear system of degree 2 and hence is of degree 2. Therefore, the map f is of degree $t > 1$. The image of ϕ is a surface of some degree k. Since $(-2K_S)^2 = 4 = kt$, we conclude that $k = t = 2$. Thus the image of ϕ is a quadric surface Q in $\mathbb{P}^3$ and the images of members F of $|-K_S|$ are lines l_F on Q. I claim that Q is a quadric cone. Indeed, all lines l_F intersect at the point $\phi(x_9')$. This is possible only if Q is a cone.

Let $S \xrightarrow{\pi} S' \xrightarrow{\phi'} Q$ be the Stein factorization. Note that a (-2)-curve R does not pass through the base point x_9' of $| - K_S|$ (because $-K_S \cdot R = 0$). Thus $\pi(x_9')$ is a nonsingular point q' of S'. Its image in Q is the vertex q of Q. Since ϕ' is a finite map, the local ring $\mathcal{O}_{S',q'}$ is a finite algebra over $\mathcal{O}_{Q,q}$ of degree 2. After completion, we may assume that $\mathcal{O}_{S',q'} \cong \mathbb{C}[[u,v]]$. If $u \in \mathcal{O}_{Q,q}$, then v satisfies a monic equation $v^2 + av + b$ with coefficients in $\mathcal{O}_{Q,q}$, where after changing v to $v + \frac{1}{2}a$ we may assume that $a = 0$. Then $\mathcal{O}_{Q,q}$ is equal to the ring of invariants in $\mathbb{C}[[u,v]]$ under the automorphism $u \mapsto u, v \mapsto -v$ which as easy to see isomorphic to $\mathbb{C}[[u,v^2]]$. However, we know that q is a singular point so the ring $\mathcal{O}_{Q,q}$ is not regular. Thus we may assume that $u^2 = a, v^2 = b$ and then $\mathcal{O}_{Q,q}$ is the ring of invariants for the action $(u,v) \mapsto (-u,-v)$. This action is free outside the maximal ideal (u,v). This shows that the finite map ϕ' is unramified in a neighborhood of q' with q' deleted. In particular, the branch curve Q of ϕ' does not pass through q. We leave to the reader to repeat the argument from the proof of (iv) to show that the branch curve W of ϕ belongs to the linear system $|\mathcal{O}_Q(3)|$. $\square$

8.3.2 Anticanonical model

Let X be a normal projective algebraic variety and D be a Cartier divisor on X. It defines the graded algebra

$$R(X,D) = \bigoplus_{r=0}^{\infty} H^0(S,\mathcal{O}_S(rD))$$

which depends only (up to isomorphism) on the divisor class of D in Pic(X). Assume $R(X,D)$ is finitely generated, then $X_D = \text{Proj } R(X,D)$ is a projective variety. If $s_0,\ldots,s_n$ are homogeneous generators of $R(X,D)$ of degrees $q_0,\ldots,q_n$ there is a canonical closed embedding into the weighted projective space

$$X_D \hookrightarrow \mathbb{P}(q_0,\ldots,q_n).$$

Also the evaluation homomorphism of sheaves of graded algebras

$$R(X,D)\otimes\mathcal{O}_X \to \text{Sym}^\bullet(\mathcal{L})$$

defines a morphism

$$\varphi_{\text{can}} : X = \text{Proj}(\text{Sym}^\bullet(\mathcal{L})) \to X_D.$$

For every $r > 0$ the inclusion of subalgebras Sym$^\bullet(H^0(X,\mathcal{O}_X(rD))) \to R(X,D)$ defines a rational map

$$\tau_r : X_D - \to \mathbb{P}(H^0(X,\mathcal{O}_X(rD))).$$

The rational map $\phi_{|rD|} : X- \to \mathbb{P}(H^0(X,\mathcal{O}_X(rD)))$ defined by the complete linear system $|rD|$ factors through φ

$$\phi_{|rd|} : X \overset{\varphi}{- \to} X_D \overset{\tau_r}{- \to} \mathbb{P}(H^0(X,\mathcal{O}_X(rD))).$$

A proof of the following proposition can be found in [116], 7.1.

Proposition 8.3.3. *Suppose $|rD|$ has no base points for some $r > 0$ and $D^{\dim X} > 0$. Then*

(i) $R(X,D)$ *is a finitely generated algebra;*

(ii) X_D *is a normal variety;*

(iii) $\dim X_D = \max_{r>0}\dim\phi_{|rD|}(X)$*;*

(iv) if $\dim X_D = \dim X$*, then* φ *is a birational morphism.*

We apply this to the case when $X = S$ is a weak Del Pezzo surface and $D = -K_S$. Applying the previous Proposition, we easily obtain that

$$X_{-K_S} \cong \bar{S},$$

where we use the notation of Theorem 8.3.2. The variety $\bar{S}$ is called the *anticanonical model* of S. If S is of degree $d > 2$, the map $\tau_1 : \bar{S} \to \mathbb{P}^d$ is a closed embedding, hence $R(S, -K_S)$ is generated by $d+1$ elements of order 1. If $d = 2$, the map τ_1 is the double cover of $\mathbb{P}^2$. This shows that $R(S, -K_S)$ is generated by 3 elements s_0, s_1, s_2 of degree 1 and one element s_3 of degree 2 with a relation $s_3^2 + f_4(s_0, s_1, s_2) = 0$ for some homogeneous polynomial f_4 of degree 2. This shows that $\bar{S}$ is isomorphic to a hypersurface of degree 4 in $\mathbb{P}(1, 1, 1, 2)$ given by an equation

$$t_3^2 + f_4(t_0, t_1, t_2) = 0. \tag{8.12}$$

In the case $d = 1$, by Lemma 8.3.1 we obtain that

$$\dim R(S, -K_S)_1 = 2, \ \dim R(S, -K_S)_2 = 4, \ \dim R(S, -K_S)_3 = 7.$$

Let s_0, s_1 be generators of degree 1, s_2 be an element of degree 2 which is not in $S^2(R(S, -K_S)_1)$ and let s_3 be an element of degree 3 which is not in the subspace generated by $s_0^3, s_0 s_1^2, s_0^2 s_1, s_1^3, s_2 s_0, s_2 s_1$. The subring $R(S, -K_S)'$ generated by s_0, s_1, s_2, s_3 is isomorphic to $\mathbb{C}[t_0, t_1, t_2, t_3]/(F(t_0, t_1, t_2, t_3))$, where

$$F = t_3^2 + t_2^3 + f_4(t_0, t_1)t_2 + f_6(t_0, t_1),$$

and $f_4(t_0, t_1)$ and $f_6(t_0, t_1)$ are binary forms of degrees 4 and 6. The projection $[t_0, t_1, t_2, t_3] \mapsto [t_0, t_1, t_2]$ defines a double cover of the quadratic cone $Q \subset \mathbb{P}^3$ which is isomorphic to the weighted projective plane $\mathbb{P}(1, 1, 2)$. Using Theorem 8.3.2 one can show that the rational map $\bar{S} - \to$ Proj $R(S, -K_S)'$ is an isomorphism. This shows that the anticanonical model $\bar{S}$ of a weak Del Pezzo surface of degree 1 is isomorphic to a hypersurface $V(F)$ of degree 6 in $\mathbb{P}(1, 1, 2, 3)$.

Remark 8.3.1. The singularities of the branch curves of the double cover $S \to \mathbb{P}^2$ ($d = 2$) and $S \to Q$ ($d = 1$) are *simple singularities*. This means that in appropriate analytic (or formal) coordinates they are given by one of the following equations:

$$\begin{aligned} A_n &: \quad x^2 + y^{n+1} = 0, \quad n \geq 1, \\ D_n &: \quad y(x^2 + y^{n-2} = 0, \quad n \geq 4, \\ E_6 &: \quad x^3 + y^4 = 0, \\ E_7 &: \quad x^3 + xy^3 =, \\ E_8 &: \quad x^3 + y^5 = 0. \end{aligned} \tag{8.13}$$

This easily follows from Theorem 8.2.23.

8.4 Normal surfaces of degree d in $\mathbb{P}^d$

8.4.1 Classification

We will always assume that surfaces are nondegenerate, i.e. do not lie in a hyperplane. We saw that the image of a weak Del Pezzo surface of degree $d > 2$ under the map given by the linear system $|-K_S|$ is a nondegenerate surface of degree d in $\mathbb{P}^d$. We call this surface an *anticanonical model* of S. It is a normal surface with canonical singularities.

Another example of a normal surface of degree d in $\mathbb{P}^d$ is the projection of a normal scroll of degree d in $\mathbb{P}^{d+1}$, where $d > 3$. Its hyperplane sections are Veronese curves. Since the projection of such a curve from a general point in the hyperplane is a nonsingular curve, the projected surface is normal.

We shall see that except cones over nonsingular elliptic curves of degree d in a hyperplane of $\mathbb{P}^d$, there is nothing else.

Theorem 8.4.1. *A normal nondegenerate surface of degree d in $\mathbb{P}^d$ is one of the following surfaces:*

(i) a projection of a normal surface of degree d in $\mathbb{P}^{d+1}$;

(ii) a cone over a nonsingular elliptic curve of degree d lying in a hyperplane;

(iii) an anticanonical model of a weak Del Pezzo surface.

First we need the following.

Lemma 8.4.2. *Let C be a nondegenerate nonsingular irreducible curve of degree $d > 2$ in $\mathbb{P}^{d-1}$. Then $g \leq 1$ and the equality takes place if and only if the restriction map $r : H^0(\mathbb{P}^{d-1}, \mathcal{O}_{\mathbb{P}^{d-1}}(1)) \to H^0(C, \mathcal{O}_C(1))$ is surjective.*

Proof. Let H be a hyperplane section of C. We have

$$\dim H^0(C, \mathcal{O}_C(H)) = d + 1 - g + \dim H^1(C, \mathcal{O}_C(H)) \geq d, \tag{8.14}$$

and the equality takes place if and only if C is projectively normal. Thus we obtain

$$g \leq 1 + \dim H^1(C, \mathcal{O}_C(H)) = 1 + \dim H^0(C, \mathcal{O}_C(K_C - H)). \tag{8.15}$$

If $|K_C - H| = \emptyset$, then $g \leq 1$ and the equality takes place if and only if r is surjective. Assume $|K_C - H| \neq \emptyset$. By Clifford's Theorem [206],

$$\dim H^0(C, \mathcal{O}_C(K_C - H)) \leq 1 + \tfrac{1}{2}\deg(K_C - H) = 1 + (g - 1 - \tfrac{1}{2}d) = g - \tfrac{1}{2}d,$$

unless $K_C = H$ or C is a hyperelliptic curve and $K_S - H = kg^1_2$ for some $k > 0$. If we are not in one of the exceptional cases, we obtain $g \leq 1 + g - \frac{1}{2}d$ which is a contradiction. If $K_C = H$, we get $d = 2g-2$ and (8.14) gives $g = 2g-2+1-g+1 \geq 2g-2$, hence $g = 2$ and $d = 2g-2 = 2$, a contradiction. If $K_C - H = kg^1_2$, then $H = (g-1)g^1_2 - kg^1_2 = (g-1-k)g^1_2$. Since $\deg H > 2$, we get $k < g - 2$. Thus $\dim H^0(C, \mathcal{O}_C(K_C - H)) = k+1$, and (8.15) gives $g \leq 1+k+1 < 2+g-2 = g$, a contradiction again. Thus $|K_C - H| = \emptyset$ and we are done. □

Theorem 8.4.3. *Let X be a nondegenerate normal surface of degree d in $\mathbb{P}^d$. Assume that X is not a projection of a surface of degree d in $\mathbb{P}^{d+1}$ and has at most canonical singularities. Then X is an anticanonical model of a weak Del Pezzo surface S.*

Proof. First of all V is a rational surface. By projection from a general point on the surface we obtain that X is birationally isomorphic to a surface of degree $d-1$ in $\mathbb{P}^{d-1}$. Continuing in this way we obtain that X is birationally isomorphic to a cubic surface Y in $\mathbb{P}^3$. Since X has canonical singularities, Y cannot be a cubic cone. We will see in the next Chapter that an irreducible cubic surface is rational except when it is a cone. Let $\pi : S \to X$ be a minimal resolution of singularities. Since X has only canonical singularities, $\pi^*\omega_X \cong \omega_S$. Since X is normal, $\pi_*(\mathcal{O}_S) = \mathcal{O}_X$, and by the projection formula,

$$\pi_*\omega_S \cong \omega_X.$$

This implies that the canonical homomorphism $H^1(X, \omega_X) \to H^1(S, \omega_S)$ is injective. Since S is a nonsingular rational surface, we get $H^1(S, \omega_S) \cong H^1(S, \mathcal{O}_S) = 0$. Thus

$$H^1(X, \omega_X) \cong H^1(X, \mathcal{O}_X) = 0.$$

Let C be a general hyperplane section of X. Since X is normal, it is a smooth curve. By the previous lemma, its genus is 0 or 1. The exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(1) \to \mathcal{O}_C(1) \to 0 \tag{8.16}$$

shows that the restriction homomorphism $H^0(X, \mathcal{O}_X(1)) \to H^0(C, \mathcal{O}_C(1))$ is surjective. If C is of genus 0, we have $h^0(\mathcal{O}_C(1)) = \deg \mathcal{O}_C(1) + 1 = d + 1$. This implies that $h^0(\mathcal{O}_X(1)) = d + 1$, hence $|\mathcal{O}_X(1)|$ is not complete and X is a projection of a surface in $\mathbb{P}^{d+1}$.

Thus we may assume that C is an elliptic curve. Let us identify it with its preimage under π. By the adjunction formula,

$$\mathcal{O}_C = \omega_C = \mathcal{O}_S(K_S + C) \otimes \mathcal{O}_C.$$

By Riemann-Roch,

$$\dim H^0(S, \mathcal{O}_S(K_S + C)) = \tfrac{1}{2}(K_S \cdot C + C^2) + 1 = 1.$$

Thus $|K_S+C|$ consists of an isolated curve D. Since $D \cdot \pi^*(H) = 0$ for any hyperplane section H of V not passing through the singularities, we obtain that each irreducible component R of D is contained in the exceptional curve of the resolution π. Since V has only canonical singularities, R is a (-2)-curve. Since $(K_S + C) \cdot R = K_S \cdot R + C \cdot R = 0$ for any irreducible component of a resolution, we get $D^2 = 0$. Since the sublattice of Pic(S) generated by the components of a Dynkin curve is negative definite, we get $D = 0$. Thus $K_S + C \sim 0$ and $-K_S = \pi^*\mathcal{O}_V(1)$ is nef and big. So, S is a weak Del Pezzo surface of degree $K_S^2 = d$. Clearly, S is its anticanonical model. □

Corollary 8.4.4. *Let X be a nondegenerate normal surface of degree d in $\mathbb{P}^d$. Assume that X has at most canonical singularities and is not a projection of a surface of degree d in $\mathbb{P}^{d+1}$. Then $d \leq 9$. Moreover, V is either surface of degree 8 in $\mathbb{P}^8$ isomorphic to the image of $\mathbf{F}_n$ ($n = 0, 2$) under the map defined by the linear system $|-2K_{\mathbf{F}_n}|$, or a projection of the Veronese surface $v_3(\mathbb{P}^2) \subset \mathbb{P}^9$.*

Proof. Use that a weak Del Pezzo surface of degree ≥ 3 not isomorphic to $\mathbf{F}_0$ or $\mathbf{F}_2$ is the blow-up of $N \leq 8$ bubble points $x_1, \ldots, x_N$ in $\mathbb{P}^2$ and the linear system $|-K_S| = |3\ell - x_1 - \ldots - x_N|$. It is a subsystem of the complete linear system $|3\ell|$ defining a Veronese map $v_3 : \mathbb{P}^2 \to \mathbb{P}^9$. □

Now everything is ready to prove Theorem 8.4.1. It remains to show that a normal surface S of degree d in $\mathbb{P}^d$ is not a projection of a surface of degree d in $\mathbb{P}^d$ and its singularities are not canonical singularities, then it must be isomorphic to a cone over a curve of degree d. By Lemma 8.4.2, the curve must be a nonsingular elliptic curve.

Assume that $d = 3$. If the surface S has a point of multiplicity 3, it must be a cone over a cubic curve, and the assertion is checked. Assume that a singular point is of multiplicity 2. By projecting the surface to $\mathbb{P}^3$ from this point, we find a birational morphism from a nonsingular model S onto the projective plane. Then repeating the proof of Theorem 8.4.3, we obtain that S is an anticanonical model of weak Del Pezzo surface of degree 3. Now, if $d > 3$, the projection of S from a general subspace of codimension 3 is a surface of degree 3 in $\mathbb{P}^3$. By the above it must be a cone over a cubic curve, hence S is a cone.

Recall that a nondegenerate subvariety X of a projective space $\mathbb{P}^n$ is called *projectively normal* if X is normal and the natural restriction map

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m) \to H^0(X, \mathcal{O}_X(m))$$

is surjective for all $m \geq 0$. This can be restated in terms of vanishing of cohomology

$$H^1(\mathbb{P}^n, \mathcal{I}_X(m)) = 0,\ m > 0 \text{ (resp. } m = 1),$$

where $\mathcal{I}_X$ is the ideal sheaf of X. It is known that X is projectively normal if and only if its projective coordinate ring $k[X]$ is a normal domain. One shows that the integral closure is the ring

$$\overline{k[X]} = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(m)).$$

Theorem 8.4.5. *Let X be an anticanonical model of a weak Del Pezzo surface of degree $d \geq 4$. Then X is projectively normal.*

Proof. The linear normality follows immediately from exact sequence (8.16) (use that $h^0(\mathcal{O}_C(1)) = \deg \mathcal{O}_C(1) = d$). Let H be a general hyperplane. Tensoring the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n}(m-1) \to \mathcal{O}_{\mathbb{P}^n}(m) \to \mathcal{O}_H(m) \to 0$$

with $\mathcal{I}_X$ we get an exact sequence

$$0 \to \mathcal{I}_X(m-1) \to \mathcal{I}_X(m) \to \mathcal{I}_{H \cap X}(m) \to 0. \tag{8.17}$$

We know from the proof of Lemma 8.4.2 that $E = H \cap X$ is an irreducible linearly normal curve of genus 1. Let H' be a general hyperplane in $H = \mathbb{P}^{d-1}$. The exact sequence

$$0 \to \mathcal{I}_E(m-1) \to \mathcal{I}_E(m) \to \mathcal{I}_{H' \cap E}(m) \to 0$$

gives a surjection $H^1(E,\mathcal{I}_E(1)) \to H^1(E,\mathcal{I}_E(2))$. By linear normality of E we obtain $H^1(E,\mathcal{I}_E(1)) = H^1(E,\mathcal{I}_E(2)) = 0$. Continuing in this way we prove the projective normality of $E \subset H$. The same cohomology game with exact sequence (8.17) gives a surjective map $r_m : H^1(X,\mathcal{I}_X(m-1)) \to H^1(X,\mathcal{I}_X(m))$ for $m \geq 1$. The exact sequence

$$0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{P}^d} \to \mathcal{O}_X \to 0$$

together with vanishing of $H^1(X,\mathcal{O}_X)$ and $H^1(\mathbb{P}^d,\mathcal{O}_{\mathbb{P}^d})$ show that $H^0(X,\mathcal{I}_X) = 0$. The surjectivity of r_1 implies $H^1(X,\mathcal{I}_X(1)) = 0$. Now the surjectivity of r_2 implies $H^1(X,\mathcal{I}_X(2)) = 0$, and so on. □

8.4.2 Rational normal scrolls of degree d in $\mathbb{P}^d$

Now we know that a normal surface of degree d in $\mathbb{P}^d$ which is not an anticanonical weak Del Pezzo surface is either a projection of a Veronese surface in $\mathbb{P}^4$ or a projection of a rational normal scroll $S_{a,d+1}$ of degree d in $\mathbb{P}^{d+1}$.

Proposition 8.4.6. *Let $S = S_{a,d+1}$ be a rational normal scroll of dimension 2 and degree d in $\mathbb{P}^{d+1}$. Then $S_{a,d+1}$ is isomorphic to the image of a minimal ruled surface $\mathbf{F}_{d-2a}$ under the regular map defined by the complete linear system $|e+(d-a)f|$, where e is the divisor class of a section E_0 with $e^2 = -(d-2a)$ and f is the divisor class of a fibre.*

Proof. Let C be a general member of the linear system $|e+(d-a)f|$. It is easy to see that the linear system has no fixed components, so C is an irreducible curve. We have $C^2 = d$ and $C \cdot f = 1$. Thus C is a section of $\mathbf{F}_{d-2a}$. The restriction of the linear system to C is a linear system of degree d. By Riemann-Roch, its dimension is equal to $\dim|e+(d-a)f| - 1 = d$. Thus a hyperplane section of the image surface S is a smooth Veronese curve, hence S is a normal surface of degree d in $\mathbb{P}^{d+1}$. The image of a section E is a Veronese curve R_1 of degree a in the subspace defined by the linear system $|(d-a)f|$. The image of a section $E \in |e+(d-2a)f|$, disjoint from E_0, is a Veronese curve R_2 of degree $d-a$ in a subspace of dimension $d-a$. The image of a fibre is a line joining a point in R_1 with a point on R_2. The two curves are identified with the base of the ruled surface. So, we obtain the definition of $S_{a,d+1}$ as the join of two Veronese curves.

We postpone the proof of the converse until the later chapter where we will study ruled surfaces in projective spaces. □

We have already noted that $S_{a,3}$ in $\mathbb{P}^3$ is either a nonsingular quadric $S_{1,3}$ or an irreducible quadric cone $S_{0,3}$. It corresponds to the linear system $|e+f|$ in $\mathbf{F}_0$ in the first case, and the linear system $|e+2f|$ on $\mathbf{F}_2$.

A rational normal scroll of degree 3 in $\mathbb{P}^4$ is either $S_{0,4}$ or $S_{1,4}$. The first surface is the cone over a Veronese curve of degree 3 in $\mathbb{P}^3$, the second surface is the image of $\mathbf{F}_1$ under the map defined by the linear system $|e+2f|$. Its projection from a general point in $\mathbb{P}^4$ is a non-normal cubic surface in $\mathbb{P}^3$. We will describe them later.

A rational normal scroll of degree 4 in $\mathbb{P}^5$ is either $S_{0,5}$ or $S_{1,5}$, or $S_{2,5}$. The first surface is the cone over a Veronese curve of degree 4 in $\mathbb{P}^4$, the second surface is the image of $\mathbf{F}_2$ under the map defined by the linear system $|e+3f|$, the third surface is the image of $\mathbf{F}_0$ under the map defined by the linear system $|e+2f|$.

There is another way to get a rational normal scroll $S_{a,d+1}$. We leave a proof of the following proposition to the reader.

Proposition 8.4.7. *Let $L = |\mathcal{O}_{\mathbb{P}^2}(d) - (d-1)p_1 - p_2 - \ldots - p_d|$ be the linear system of plane curves of degree d with a $d-1$-multiple point at p_1 and simple base points at $p_1, \ldots, p_d$ (maybe infinitely near). Then the image of $\mathbb{P}^2$ under the rational map given by L is isomorphic to $S_{a,d+1}$, where $d-1-a$ is the number of points $p_i, i > 0$ infinitely near to p_1. Conversely, each $S_{a,d+1}$ can be obtained in this way.*

8.4.3 Surfaces of degree ≥ 7

A weak Del Pezzo surface of degree 9 is isomorphic to $\mathbb{P}^2$ and its anticanonical model is isomorphic to the Veronese surface $v_3(\mathbb{P}^2)$ of degree 9 in $\mathbb{P}^9$. It does not contain lines.

A Del Pezzo surface of degree 8 is isomorphic either to $\mathbb{P}^1 \times \mathbb{P}^1$ or to $\mathbf{F}_1$, the blow-up of one point in the plane. The anticanonical model of the first surface is a hyperplane section of the Veronese 3-fold $v_2(\mathbb{P}^3) \subset \mathbb{P}^9$. It does not contain lines.

The anticanonical model of $\mathbf{F}_1$ is a projection of the Veronese surface $v_3(\mathbb{P}^2)$ from its point. It contains one line.

A weak Del Pezzo surface of degree 8 is isomorphic to the ruled surface $\mathbf{F}_2$. Its bicanonical model is a section of the Veronese 3-fold by a tangent hyperplane.

The anticanonical model of a weak Del Pezzo surface of degree 7 is a projection of the Veronese surface from a secant line of the surface. If the secant line is a tangent line, the projection acquires a singular point of type A_1 .

Let $\rho : \mathrm{Aut}(S) \to \mathrm{O}(\mathrm{Pic}(S))$ be the representation of the automorphism group in the Picard group of a Del Pezzo surface S. We described its kernel in section 8.2.8. If $S = \mathbf{F}_0$, then the image is the group of order 2 which permutes the divisor classes of the two rulings on $\mathbf{F}_0$. If S is of degree 2, then again the image is of order 2. It acts by permuting the classes of the two exceptional curves. In the remaining cases, the image is trivial.

8.4.4 Surfaces of degree 6 in $\mathbb{P}^6$

Let X be a nondegenerate surface of degree 6 in $\mathbb{P}^6$ with at most canonical singularities. By Theorem 8.4.3 its minimal resolution $\sigma : S \to X$ is a weak Del Pezzo surface of degree 6 and X is isomorphic to its anticanonical model. Let $\pi : S \to \mathbb{P}^2$ be the blowing-down structure on S. It is the blow-up of three points x_1, x_2, x_3 in an almost general position. In this case it means that the corresponding bubble cycle $\eta = x_1 + x_2 + x_3$, up to admissible order, is one of the following

(i,i') x_1, x_2, x_3 are three proper non-collinear (collinear) proper points;

(ii, ii') $x_2 \succ x_1, x_3$ are non-collinear (collinear) proper points;

(iii, iii') $x_3 \succ x_2 \succ x_1$ are non-collinear (collinear) r points.

In cases (i),(ii) and (iii) the homaloidal net $|\mathcal{O}_{\mathbb{P}^2}(2)-\eta|$ with $\eta = x_1+x_2+x_3$ defines a quadratic Cremona transformation ϕ. Thus S is a resolution (π,σ) of ϕ with the birational morphism $\sigma : S \to \mathbb{P}^2$ defined by blowing down 3 exceptions configurations with the divisor classes $e_0-e_1-e_2, e_0-e_1-e_3, e_0-e_2-e_3$, where e_0,e_1,e_2,e_3 is a geometric basis defined by π. Let Γ_ϕ be the graph of ϕ. The canonical map $\alpha : S \to \Gamma_\phi$ is a resolution of singularities. Let $\tau : S \to \mathbb{P}^8$ be the composition

$$\Phi : S \to \Gamma_\phi \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^2 \stackrel{s}{\hookrightarrow} \mathbb{P}^8,$$

where the last map is the Segre map. This map is given by the linear system $|e_0+e_0'|$, where $e_0' = 2e_0-e_1-e_2-e_3$ and $|e_0'|$ is the homaloidal net defining the quadratic map ϕ. Since $e_0+e_0' = 3e_0-e_1-e_2-e_3 = -K_S$ we obtain that Φ is defined by the anticanonical linear system on S and hence its image is a surface of degree 6 in $\mathbb{P}^6$. This shows that X is isomorphic to the intersection of the Segre variety $s(\mathbb{P}^2\times\mathbb{P}^2)$ with a linear subspace of codimension 2. It also implies that $X \cong \Gamma_\phi$.

Since there are three possible quadratic transformations up to composition with a projective automorphism, we obtain three non-isomorphic surfaces X;

- X_1 corresponding to three non-collinear proper points x_1,x_2,x_3;
- X_2 corresponding to three non-collinear points $x_2 \succ x_1, x_3$;
- X_3 corresponding to three non-collinear points $x_3 \succ x_2 \succ x_1$.

The surface X_i is isomorphic to the graph of the quadratic transformation τ_i from section 7.2.1. The surface X_1 is nonsingular and has 6 lines on it. They form a hexagon. The surface X_2 has one RDP of type A_1. It has 3 lines, two of them pass through the singular point. The surface X_3 has one RDP of type A_2. It has 1 line passing through the singular point.

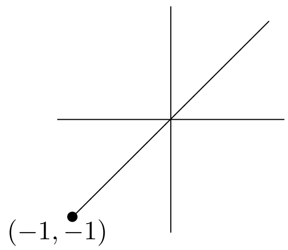
Let X_1', X_2', X_3' be the surfaces corresponding to cases (i)', (ii)' and (iii)', respectively. They are not linear sections of the Segre variety. We leave to check the following properties of the surfaces to the reader.

The surface X_1' has a unique RDP of type A_1 and has three lines passing through this point.

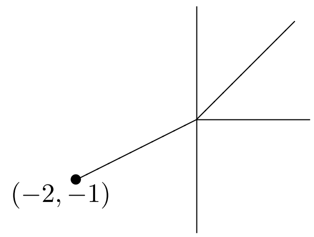
The surface X_2' has two RDPs of type A_1. It has 2 lines, one of them joins the two singular points.

The surface X_3' has 2 singular points of types A_2 and A_1. It has one line passing through the point of type A_2.

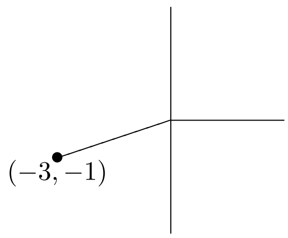
Remark 8.4.1. The surfaces X_1, X_2, X_2', X_3' are examples of *toric surfaces*. They contain an open Zariski subset isomorphic to a complex torus $(\mathbb{C}^\vee)^2$ which acts on X extending its action on itself by translations. If $x_1 = [1,0,0], x_2 = [0,1,0], x_3 = [0,0,1]$, then the torus in X_1 is the preimage in X_1 of the complement of the coordinate axes $t_i = 0$. Its complement in X_1 is the hexagon of lines. The same description of the torus is true in the case X_3 (resp. X_2) if we choose x_2 to be the tangent direction $t_2 = 0$ (resp. and $x_3 = [0,0,1]$). The corresponding fans defining the toric surfaces are the following.



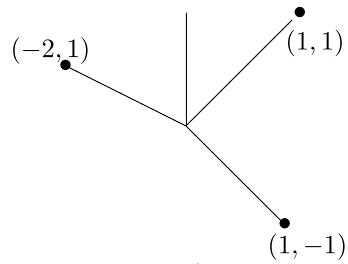
X_1



X_2



X_2'



X_3'

Let $\mathrm{Sec}(S)$ be the secant variety of S. Its expected dimension is equal to 5. In fact, we have

Proposition 8.4.8. *Let S be a weak Del Pezzzo surface whose anticanonical model X is contained in the Severi variety $\mathcal{S}_{2,2} = s(\mathbb{P}^2 \times \mathbb{P}^2) \subset \mathbb{P}^8$. Then*

$$\dim \mathrm{Sec}(S) = 5.$$

Proof. It is well know that the secant variety of $\mathcal{S}_{2,2}$ is the determinantal cubic hypersurface D of degenerate 3×3 matrices whose entries are 9 unknowns in $\mathbb{P}^8$. Let z be a general point in D and consider the *entry variety* $Y_z = \mathbb{T}_z D \cap \mathcal{S}_{2,2}$ representing the points on Y lying on the secants through z. The cone over Y_z with vertex at z is the union of secants passing through z. Since the projection of a cubic from its point is a quadric and $\mathbb{T}_z X$ is a plane, it is easy to see that Y_z is a conic. Since X is the intersection of $\mathcal{S}_{2,2}$ by a linear subspace of codimension 2, the entry varieties of $\mathrm{Sec}(X)$ must be either conics or pairs of points. Since X contains only one-dimensional family of conics (represented by lines through points x_1, x_2, x_3), we get that through any general point of $\mathrm{Sec}(X)$ passes only one conic. This easily gives that $\dim \mathrm{Sec}(X) = 5$, as is expected. □

Theorem 8.4.9. *Assume X is a nonsingular surface of degree 6 in $\mathbb{P}^6$. Then X is projectively equivalent to the subvariety given by equations expressing the rank condition*

$$\mathrm{rank}\begin{pmatrix} t_0 & t_1 & t_2 \\ t_3 & t_0 & t_4 \\ t_5 & t_6 & t_0 \end{pmatrix} \leq 2.$$

The secant variety $\mathrm{Sec}(X)$ *is the cubic hypersurface defined by the determinant of this matrix.*

Proof. We know that X is isomorphic to the blow-up of three non-collinear proper points. We may assume that $x_1 = [1,0,0], x_2 = [0,1,0], x_2 = [0,0,1]$. The linear system $|-K_X|$ is the proper transform of the linear system of cubics through the three points. It is generated by the cubics $V(f)$, where f is a monomial in coordinates z_0, z_1, z_2 in $\mathbb{P}^2$ different from z_0^3, z_1^3, z_2^3. Let us order them in the following way

$$z_0z_1z_2,\ z_1^2z_2,\ z_1z_2^2,\ z_0z_2^2,\ z_0^2z_2,\ z_0z_1^2,\ z_0^2z_1.$$

The surface X is projectively equivalent to the image of $\mathbb{P}^2$ under the rational map given by these 7 homogeneous polynomials $t_0, \dots, t_6$. The relations between these polynomials are the minors of the matrix. This shows that X is contained in the intersection of 9 quadrics given by the minors. On the other hand, by Lemma 8.3.1, we have $h^0(X, \mathcal{O}_X(-2K_S)) = 19$. Since X is projectively normal, the dimension of the linear system of quadrics containing X is equal to 8. This shows that the nine minors generate the linear system of quadrics containing X [225]. Let B be the base scheme of the linear system of quadrics. Suppose $\dim B > \dim X$. A general hyperplane section of S is a projectively normal elliptic curve of degree 6 in $\mathbb{P}^5$. It is known that it is given by 9 linearly independent quadrics. This shows that $\dim B = \dim X$. It follows from the theory of determinant varieties [9], Chapter 2, Prop. 4.1, that B is a Cohen-Macaulay surface containing a nonsingular surface X. It must coincide with X.

Since the rank of the sum of two matrices of rank ≤ 1 is at most 2, we see that Sec(X) is contained in the determinantal cubic hypersurface. By the previous proposition, it must be its irreducible component. However, by loc. cit., the determinantal cubic is irreducible. □

Let us describe the group of automorphisms of a Del Pezzo surface of degree 6. The surface is obtained by blowing up 3 non-collinear points p_1, p_2, p_3. We may assume that their coordinates are $[1,0,0], [0,1,0], [0,0,1]$. We know from section 8.2.8 that the kernel of the representation $\rho : \mathrm{Aut}(S) \to \mathrm{O}(\mathrm{Pic}(S))$ is a 2-dimensional torus. The root system is of type $A_2 + A_1$, so the Weyl group is isomorphic to $\mathfrak{S}_3 \times C_2$. Let us show that the image of the Weyl representation is the whole group.

The subgroup $\mathfrak{S}_3$ is the image of the group of automorphisms of S induced by automorphisms of the projective plane which which permute the coordinates. The generator of the cyclic group of order 2 is induced by the standard Cremona transformation. It is easy to see this as follows. The surface S is isomorphic to the blow-up of 2 points on $\mathbb{P}^1 \times \mathbb{P}^1$ not lying on the same fibre of any ruling. By coordinate change, we may assume that the points have coordinates $x_1 = ([1,0],[1,0])$ and $x_2 = ([0,1],[0,1])$. The torus is represented by homotheties on each factor of the product. The standard Cremona transformation is represented by the automorphism given in inhomogeneous coordinates on the factors by $(x,y) \mapsto (1/x, 1/y)$. The subgroup $\mathfrak{S}_3$ is generated by two elements of order 2 defined by a switch and the automorphism of S induced by the product of elementary transformations $\mathrm{elm}_{x_1} \circ \mathrm{elm}_{x_2}$.

We leave to the reader to verify the following.

Theorem 8.4.10. *Let S be a Del Pezzo surface of degree 6. Then*

$$\mathrm{Aut}(S) \cong (\mathbb{C}^\vee)^2 \rtimes (\mathfrak{S}_3 \times \mathfrak{S}_2).$$

If we represent the torus as the quotient group $(\mathbb{C}^\vee)^3$ by the diagonal subgroup $\Delta(\mathbb{C}^\vee)$, then the subgroup $\mathfrak{S}_3$ acts permutations of factors, and the cyclic subgroup $\mathfrak{S}_2$ acts by the negation.

Note that the Weyl group $\mathfrak{S}_3 \times \mathfrak{S}_2$ realized by automorphisms acts on the set of lines on S. Its incidence graph is a hexagon. The group $\mathfrak{S}_3$ is isomorphic to the dihedral group D_6 and acts on the graph in the same way as the dihedral group acts by symmetries of a regular hexagon. The generator of $\mathfrak{S}_2$ is also a symmetry but it is not induced by a motion of the plane.

Finally we mention that the Gosset polytope $\Sigma_3 = -1_{21}$ corresponding to a Del Pezzo surface of degree 6 is an octahedron. This agrees with the structure of $W(\mathbb{E}_3)$ isomorphic to the octahedron group. The surface has 2 blowing-down morphisms $S \to \mathbb{P}^2$ corresponding to two α-facets and three conic bundle structures corresponding to the pencils of lines through three points on the plane.

8.4.5 Surfaces of degree 5

Let X be a nondegenerate surface of degree 5 in $\mathbb{P}^5$ with at most canonical singularities. By Theorem 8.4.3 its minimal resolution $\sigma : S \to X$ is a weak Del Pezzo surface of degree 5 and X is isomorphic to its anticanonical model.

Proposition 8.4.11. *Let X be a nonsingular Del Pezzo surface of degree 5 in $\mathbb{P}^5$. Then X is isomorphic to a linear section of the Grassmann variety $G(2,5)$ of lines in $\mathbb{P}^4$.*

Proof. We use some elementary facts about Grassmannians which we recall in a later Chapter. It is known that the degree of $G = G(2,5)$ in the Plücker embedding is equal to 5 and $\dim G = 6$. Also is known that the canonical sheaf is equal to $\mathcal{O}_G(-5)$. By the adjunction formula, the intersection of G with a general linear subspace of codimension 4 is a nonsingular surface X with $\omega_X \cong \mathcal{O}_X(-1)$. This must be a Del Pezzo surface of degree 5. Since all Del Pezzo surfaces of degree 5 are isomorphic, the assertion follows. $\square$

Remark 8.4.2. Let E be the restriction of the tautological rank 2 quotient bundle on G to X. Then E is isomorphic to a rank 2 bundle on S generated by 5 sections with Chern classes $c_1 = -K_S$ and $c_2 = 2$. One can show that this vector bundle is given by an extension

$$0 \to \mathcal{O}_S \to E \to \mathcal{I}_Z(3) \to 0,$$

where $\mathcal{I}_Z$ is the ideal sheaf of the closed subscheme defined by any two points x, y such that $x_1, \ldots, x_4, x, y$ are in an almost general position.

Corollary 8.4.12. *Let X be a Del Pezzo surface of degree 5 in $\mathbb{P}^5$. Then its homogeneous ideal is generated by 5 linearly independent quadrics.*

Proof. Since X is projectively normal, applying Lemma 8.3.1, we obtain that the linear system of quadrics containing X has dimension equal to 4. It is known that the homogeneous ideal of the Grassmannian $G(2,5)$ is generated by 5 quadrics. So, restricting this linear system to its linear section, we obtain the quadrics containing X define X scheme theoretically. $\square$

Let X be an anticanonical model of a Del Pezzo surface of degree 5. The linear system of cubics in $\mathbb{P}^5$ containing X has dimension 24. Let us see that any nonsingular cubic fourfold containing X is rational (the rationality of a general cubic fourfold is unknown at the moment).

Lemma 8.4.13. *Let X be an anticanonical model of weak Del Pezzo surface S of degree 5 in $\mathbb{P}^5$. For any general point $z \in \mathbb{P}^5$ there exists a unique secant of X containing z.*

Proof. It is known that $\mathrm{Sec}(X) = \mathbb{P}^5$. This follows from Severi's Theorem that any nondegenerate surface in $\mathbb{P}^5$ with secant variety of dimension 4 is a Veronese surface (see [429]). Let $x, y \in X$ such that $z \in \ell = \overline{x, y}$. We may assume that they are distinct nonsingular points on X. Consider the projection $p_\ell : X - \to \mathbb{P}^3$ with center equal to ℓ. Its image is a cubic surface isomorphic to the anticanonical model of the blow-up of S at the preimages x', y' of x, y on S. Here we use that the points x', y' do not lie on (-2)-curves on S, hence the blow-up of x', y' is a weak Del Pezzo surface of degree 3. The map p_ℓ is an isomorphism outside x, y. Suppose z belongs to another secant $\ell' = \overline{x', y'}$. Then the plane generated by ℓ and ℓ' defines a point on the cubic surface such that the preimage under the projection map π_ℓ contains x', y'. This contradiction proves the assertion. □

Theorem 8.4.14. *Let F be an irreducible cubic fourfold containing a nondegenerate nonsingular surface of degree 5 in $\mathbb{P}^5$. Then F is a rational variety.*

Proof. Consider the linear system $|\mathcal{I}_X(2)|$ of quadrics containing X. It defines a morphism $Y \to \mathbb{P}^4$ whose fibres are proper transforms of secants of X. This shows that the subvariety of $G(2, 6)$ formed by secants of X is isomorphic to $\mathbb{P}^4$. Let take a general point z in F. By the previous Lemma, there exists a unique secant of X passing through z. By Bezout's Theorem, no other point outside X lies on this secant. This gives rational injective map $F - \to \mathbb{P}^4$ defined outside X. Since a general secant intersects F at three points, with two of them on X, we see that the map is birational. □

Remark 8.4.3. According to a result of A. Beauville [25], Proposition 8.2, any smooth cubic fourfold containing X is a pfaffian cubic hypersurface, i.e. is given by the determinant of a skew-symmetric matrix with linear forms as its entries. Conversely, any pfaffian cubic fourfold contains a nondegenerate surface of degree 5, i.e. an anticanonical weak Del Pezzo or a scroll.

Let us look at singularities and lines on a weak Del Pezzo surface of degree 5.

Let $\pi : S \to \mathbb{P}^2$ be the blowing down structure on S. It is the blow-up of four points non-collinear points x_1, x_2, x_3, x_4 in an almost general position. In this case it means that the corresponding bubble cycle $\eta = x_1 + x_2 + x_3 + x_4$, up to admissible order, is one of the following:

(i) x_1, x_2, x_2, x_3, x_4 are proper points;

(ii) $x_2 \succ x_2, x_3, x_4$;

(iii) $x_3 \succ x_2 \succ x_1, x_4$;

(iv) $x_2 \succ x_1, x_4 \succ x_3, x_2 \succ x_1$;

(v) $x_4 \succ x_3 \succ x_2 \succ x_1$.

As we have seen in section 8.2.7 the singularities of X correspond to root bases in the lattice $\mathbf{E}_4$. The possibilities are

$$A_1, A_1 + A_1, A_2, A_1 + A_2, A_3, A_4.$$

All these cases are realized:

$$A_1 : x_2 \succ x_1, x_2, x_3, \text{ no three points are collinear;}$$

$$A_1 + A_1 : x_2 \succ x_1, x_4 \succ x_3, \text{ no three points are collinear;}$$

$$A_2 : x_3 \succ x_2 \succ x_1, x_4, \text{ no three points are collinear;}$$

$$A_1 + A_2 : x_3 \succ x_2 \succ x_1, x_4; x_1, x_2, x_3 \text{ are collinear;}$$

$$A_3 : x_4 \succ x_3 \succ x_2 \succ x_1,\ x_4, x_1, x_2, x_3 \text{are not collinear;}$$

$$A_4 : x_4 \succ x_3 \succ x_2 \succ x_1,\ x_1, x_2, x_3 \text{are collinear.}$$

Since any set of four points in general position is projectively equivalent to the set

$$x_1 = [1, 0, 0],\ x_1 = [1, 0, 0],\ x_1 = [1, 0, 0],\ x_1 = [1, 0, 0],$$

we obtain that all Del Pezzo surfaces of degree 6 or 5 are isomorphic.

A Del Pezzo surface of degree 5 has 10 lines. The union of them is a divisor in $|-2K_S|$. The incidence graph of the set of 10 lines is the famous *Petersen graph*.

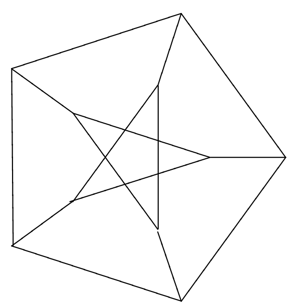


Table 8.5: Petersen graph

The number of lines on a weak Del Pezzo surface depends on the structure of its Dynkin curves, or, equivalently, singularities of its anticanonical model. It is easy to derive the following table for the number of lines on a singular anticanonical model of a weak Del Pezzo surface of degree 5.

The Weyl group $W(\mathbf{E}_4)$ is isomorphic to the Weyl group $W(A_4) \cong \mathfrak{S}_5$. If S has only one (-2)-curve, then the group of Cremona isometries is isomorphic to $\mathfrak{S}_4$. It acts on the set of 10 elements with 7 orbits. There are 3 orbits with stabilizer of order 12 and 4 orbits with stabilizer of order 24.

A_1	A_1+A_1	A_2	A_1+A_2	A_3	A_4
7	4	3	1	1	1

Table 8.6: Lines on a Del Pezzo surface of degree 5

The Gosset polytope $\Sigma_4 = 0_{21}$ has 5 facets of type α corresponding to contractions of 5 disjoint lines on S and 5 pencils of conics corresponding to the pencils of lines through a point in the plane and the pencil of conics through the four points.

Let us study automorphisms of a Del Pezzo surface of degree 5.

Theorem 8.4.15. *Let S be a Del Pezzo surface of degree 5. Then*

$$\mathrm{Aut}(S) \cong \mathfrak{S}_5.$$

Proof. The group $\mathfrak{S}_5$ is generated by its subgroup isomorphic to $\mathfrak{S}_4$ and an element of order 5. The subgroup $\mathfrak{S}_4$ is realized by projective transformations permuting the points $x_1,\ldots,x_4$. The action is realized by the standard representattion of $\mathfrak{S}_4$ in the hyperplane $z_1+\cdots+z_4 = 0$ of $\mathbb{C}^4$ identified with $\mathbb{C}^3$ by the projection to the first 3 coordinates. An element of order 5 is realized by a quadratic transformation with fundamental points x_1,x_2,x_3 defined by the formula

$$T : [t_0,t_1,t_2] \mapsto [t_0(t_2-t_1), t_2(t_0-t_1), t_0t_2]. \tag{8.18}$$

It maps the line $t_0 = 0$ to the point x_2, the line $t_1 = 0$ to the point x_4, the line $t_2 = 0$ to the point x_1, the point x_4 to the point x_3. □

Note that the group of automorphisms acts on the Petersen graph of 10 lines and defines an isomorphism with the group of symmetries of the graph.

Let S be a Del Pezzo surface of degree 5. The group $\mathrm{Aut}(S) \cong \mathfrak{S}_5$ acts on linearly on the space $V = H^0(S,\mathcal{O}_S(-K_S)) \cong \mathbb{C}^6$. Let us compute the character of this representation. Choose the following basis in the space V:

$$(t_0^2t_1-t_0t_1t_2, t_0^2t_2-t_0t_1t_2, t_1^2t_0-t_0t_1t_2, t_1^2t_2-t_0t_1t_2, t_2^2t_0-t_0t_1t_2, t_2^2t_1-t_0t_1t_2). \tag{8.19}$$

Let $s_1 = (12), s_2 = (23), s_3 = (34), s_4 = (45)$ be the generators of $\mathfrak{S}_5$. It follows from the proof of Theorem 8.4.15 that s_1,s_2,s_3 generate the subgroup of $\mathrm{Aut}(S)$ which is realized by projective transformations permuting the points p_1,p_2,p_3,p_4. The last generator is realized by a quadratic transformation T. Choose the following representatives of the conjugacy classes in $\mathfrak{S}_5$ different from the conjugacy class of the identity element id:

$$g_1 = (12),\ g_2 = (123) = s_2s_1, g_3 = (1234) = s_3s_2s_1,$$

$$g_4 = (12345) = s_4s_3s_2s_1, g_5 = (12)(34) = s_1s_3, g_6 = (123)(45) = s_3s_2s_1s_4.$$

The subgroup generated by s_1,s_2 acts by permuting the coordinates t_0,t_1,t_2. The generator s_3 acts as the projective transformation

$$s_3 : (y_1,\ldots,y_6) \mapsto (-y_1, y_1+y_2, -y_3, y_3+y_4, -y_1-y-2+y_4+y_6, y_2-y_3-y_4+y_5),$$

where $(y_1,\ldots,y_6)$ is the basis from (8.19). Finally s_4 acts by formula (9.57). The simple computation gives the character vector of the representation

$$\chi = (\chi(\mathrm{id}),\chi(g_1),\chi(g_2),\chi(g_3),\chi(g_4),\chi(g_5),\chi(g_6)) = (6,0,0,0,1,-2,0).$$

Using the character table of $\mathfrak{S}_5$ we find that χ is the character of an irreducible representation isomorphic to the second exterior power of the standard 4-dimensional representation of $\mathfrak{S}_5$ (see [175], p. 28).

Now let us consider the linear representation of $\mathfrak{S}_5$ on the symmetric square $S^2(V)$. Its character χ_{S^2V} can be easily found using the standard facts about linear representation of finite groups. Using the formula

$$\chi_{S^2V}(g) = \tfrac{1}{2}(\chi(g)^2+\chi(g^2)),$$

we get $\chi_{S^2V} = (21,3,0,-1,1,5,0)$. Taking the inner product with the character of the trivial representation we get 1. This shows that the subspace of invariant vectors $\dim S^2V^{\mathfrak{S}_5}$ is one-dimensional. Similarly, we find that $\dim S^2V$ contains one copy of the one-dimensional sign representation of $\mathfrak{S}_5$. The equation of the union of 10 lines is

$$F = t_0t_1t_2t_3(t_0^2-t_1^2)(t_0^2-t_2^2)(t_1^2-t_2^2) = 0.$$

It is easy to check that F transforms under $\mathfrak{S}_5$ as the sign representation. It is less trivial but straightforward to find a generator of the vector space $S^2V^{\mathfrak{S}_5}$. It is equal to

$$G = 2\sum t_i^4t_j^2 - 2\sum t_i^4t_jt_k - \sum t_i^3t_j^2t_k + 6t_0^2t_1^2t_2^2.$$

Its singular points are the reference points. In another coordinate system, the equation looks even better:

$$t_0^6+t_1^6+t_2^6+(t_0^2+t_1^2+t_2^2)(t_0^4+t_1^4+t_2^4)-12t_0^2t_1^2t_2^2 = 0.$$

(see [151]). The singular points are $([1,-1,-1],[-1,1,-1],[-1,-1,1],[1,1,1]$. The $\mathfrak{S}_5$-invariant plane sextic $W = V(G)$ is called the *Wiman sextic*. Its proper transform on S is a smooth curve of genus 6 in $|-2K_S|$. All curves in the pencil of sextics spanned by $V(\lambda F+\mu G)$ (the *Wiman pencil*) are $\mathfrak{A}_5$-invariant. It contains two $\mathfrak{S}_5$-invariant members $V(F)$ and $V(G)$.

Remark 8.4.4. It is known that a Del Pezzo surface of degree 5 is isomorphic to the GIT-quotient of the space $(\mathbb{P}^1)^5$ by the group $\mathrm{SL}(2)$ (see [130]). The group $\mathfrak{S}_5$ is realized naturally by the permutation of factors. The isomorphism is defined by assigning to any point x on the surface the five ordered points $(p_1,\ldots,p_5 = x)$, where $p_1,\ldots,p_4$ are the tangent directions of the conic in the plane passing through the points x_1,x_2,x_3,x_4,x.

8.5 Quartic Del Pezzo surfaces

8.5.1 Equations

Here we study in more details weak Del Pezzo surfaces of degree 4. They are obtained by blowing up 5 points in $\mathbb{P}^2$ and hence vary in a family. Surfaces of degree 3 will be studied in the next Chapter.

Theorem 8.5.1. *Let X be an anticanonical model of a weak Del Pezzo surface S of degree 4. Then S is a complete intersection of two quadrics in $\mathbb{P}^4$. Moreover, if X is nonsingular, then the equations of the quadrics can be reduced, after a linear change of variables, to the diagonal forms:*

$$\sum_{i=0}^{4} t_i^2 = \sum_{i=0}^{4} a_i t_i^2 = 0,$$

where $a_i \neq a_j$ for $i \neq j$.

Proof. By Theorem 8.4.5, X is projectively normal in $\mathbb{P}^4$. This gives the exact sequence

$$0 \to H^0(\mathbb{P}^4, \mathcal{I}_X(2)) \to H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}) \to H^0(X, \mathcal{O}_X(2)) \to 0.$$

By Lemma 8.3.1,

$$\dim H^0(X, \mathcal{O}_X(2)) = \dim H^0(S, \mathcal{O}_S(-2K_S)) = 13.$$

This implies that X is the base locus of a pencil $t_0Q_0 + t_1Q_1 = 0$ of quadrics. Assume X is nonsingular. The determinant equation shows that it contains 5 singular quadrics counting with multiplicities. A linear pencil of quadrics is defined by a line ℓ in the projective space $|\mathcal{O}_{\mathbb{P}^d}(2)|$ of quadrics in $\mathbb{P}^d$. Let Δ be the discriminant hypersurface. Its singular locus consists of quadrics of corank > 1. None of these quadrics is contained in our pencil since otherwise the base locus is obviously singular. For any nonsingular point $Q \in \Delta$, the tangent space of Δ at Q can be identified with the linear space of quadrics passing through the singular point of Q (see Example 1.2.1). Again, since the base locus X of our pencil is a nonsingular surface, we obtain that X does not contain a singular point of a singular quadric from the pencil. This shows that the line ℓ intersects Δ transversally, and hence contains exactly $5 = \deg \Delta$ singular quadrics.

Now we are in business. Let $p_1, \ldots, p_5$ be the singular points of singular quadrics from the pencil. We claim that the points span $\mathbb{P}^4$. In fact, otherwise we obtain a pencil $\mathcal{P}$ of quadrics in some hyperplane H with ≥ 5 singular members, hence all quadrics in this pencil must be singular. By Bertini's Theorem, there is a point $q \in H$ singular for all quadrics, hence the base locus of the pencil consists of the union of 4 lines through q taken with multiplicities. The base locus of $\mathcal{P}$ is the hyperplane section $H \cap X$ of X. Since X is nonsingular and its hyperplane section is a curve of arithmetic genus 1, it is easy to see that it does not contain the union of 4 lines with a common point. So, the points $p_1, \ldots, p_5$ span $\mathbb{P}^4$. Choose coordinates such that $p_1 = [1, \ldots, 0]$ and so on. Let $Q_t = V(f_t)$ be a quadric from the pencil. Its first polar with respect to p_i is given by the equation $l_t = \frac{\partial f_t}{\partial x_i} = 0$. Since one of the quadrics in the pencil has a singular point at p_i, the polar does not depend on t. So we may assume that $l_t = a_i(t_0, t_1)l_i$, where l_i is a linear function in the variables $x_0, \ldots, x_4$ and $a_i(t_0, t_1) = a_it_0 + b_it_1$ is a linear function in variables t_0, t_1. The linear functions l_i are obviously linearly independent, since otherwise all quadrics have a common singular point contradicting the nonsingularity of the base locus. Again let us choose coordinates to assume that

$l_i = x_i$. By Euler's formula,

$$f_t = \sum_{i=0}^{4} a_i(t_0,t_1)x_i^2 = t_0(\sum_{i=0}^{4} a_i x_i^2) + t_1(\sum_{i=0}^{4} b_i x_i^2).$$

Note that the singular quadrics from the pencil correspond to $[t_0,t_1] = [b_i, -a_i]$. Their singular points are $[1,0,\ldots,0]$ and so on. After a linear change of variables in the coordinates of the pencil, we may assume that $b_i = 1, i = 0,\ldots,4$. This gives the equations from the assertions of the theorem. Since all points $[b_i,a_i] = [1,a_i]$ are distinct, we see that $a_i \neq a_j$ for $i \neq j$. $\square$

Let X be an anticanonical model of a Del Pezzo surface S of degree 4. It is a nonsingular quartic surface given by the equations from Theorem 8.5.1. Following the classical terminology an anticanonical model of a weak Del Pezzo surface of degree 4 is called a *Segre quartic surface*.

One can say more about equations of singular weak Del Pezzo quartics. Let $\mathcal{Q}$ be a pencil of quadrics in $\mathbb{P}^n$. We view it as a line in the space of symmetric matrices of size $n+1$ spanned by two matrices A, B. Assume that $\mathcal{Q}$ contains a nonsingular quadric, so that we can choose B to be a nonsingular matrix. Consider the λ-matrix $A+\lambda B$ and compute its elementary divisors. Let $\det(A+\lambda B) = 0$ has r distinct roots $\alpha_1,\ldots,\alpha_r$. For every root α_i we have elementary divisors of the matrix $A+\lambda B$

$$(\lambda-\alpha_i)^{e_i^{(1)}},\ldots,(\lambda-\alpha_i)^{e_i^{(s_i)}},\quad e_i^{(1)} \leq \ldots \leq e_i^{(s_i)}.$$

The *Segre symbol* of the pencil $\mathcal{Q}$ is the collection

$$[(e_1^{(1)}\ldots e_1^{(s_1)})(e_2^{(1)}\ldots,e_2^{(s_2)})\ldots(e_r^{(1)}\ldots,e_r^{(s_r)})].$$

It is a standard result in linear algebra (see, for example, [176] or [221]) that one can simultaneously reduce the pair of matrices (A,B) to the form (A',B') (i.e. there exists an invertible matrix C such that $CAC^t = A', CBC^t = B'$) such that the corresponding quadratic forms Q_1', Q_2' have the following form

$$Q_1' = \sum_{i=1}^{r}\sum_{j=1}^{s_i} p(\alpha_i, e_i^{(j)}),\quad Q_2' = \sum_{i=1}^{r}\sum_{j=1}^{s_i} q(e_i^{(j)}), \tag{8.20}$$

where

$$\begin{aligned} p(\alpha,e) &= \alpha\sum_{i=1}^{e} t_i t_{e+1-i} + \sum_{i=1}^{e-1} t_{i+1}t_{e+1-i}, \\ q(e) &= \sum_{i=1}^{e} t_i t_{e+1-i}. \end{aligned}$$

It is understood here that each $p(\alpha,e)$ and $q(e)$ are written in disjoint sets of variables. This implies the following.

Theorem 8.5.2. *Let X and X' be two complete intersections of quadrics and $\mathcal{P}, \mathcal{P}'$ be the corresponding pencils of quadrics. Assume that $\mathcal{P}$ and $\mathcal{P}'$ contains a nonsingular quadric. Let H and H' be the set of singular quadrics in $\mathcal{P}$ and $\mathcal{P}'$ considered as sets marked with the corresponding part of the Segre symbol. Then X is projectively equivalent to X' if and only if the Segre symbols of $\mathcal{P}$ and $\mathcal{P}'$ coincide and there exists a projective isomorphism $\phi : \mathcal{P} \to \mathcal{P}'$ such that $\phi(H) = H'$ and the marking is preserved.*

Applying this to our case $n = 4$, we obtain the following possible Segre symbols:

r = 5 $[11111]$;

r = 4 $[(11)111],\ [2111]$;

r = 3 $[(11)(11)1],\ [(11)21], [311], [221], [(12)11]$;

r = 2 $[14],\ [(13)1],\ [3(11)],\ [32]; [(12)2], [(12)(11)]$;

r = 1 $[5], [(14)]$.

Here r is the number of singular quadrics in the pencil. Note that the case $[(1,1,1,1,1)]$ leads to linearly dependent matrices A, B, so it is excluded for our purpose. Also in cases $[(111)11], [(1111)1], [(112)1], [(22)1]$, there is a reducible quadric in the pencil, so the base locus is a reducible. Finally, the cases $[(23)], [(23)], [(113)],\ [(122)]$, and $[(1112)]$ correspond to cones over a quartic elliptic curve.

8.5.2 Cyclid quartics

Let X be a nonsingular quartic surface in $\mathbb{P}^4$. Let us project X to $\mathbb{P}^3$. First assume that the center of the projection p lies on X. Then the image of the projection is a cubic surface Y in $\mathbb{P}^3$ isomorphic to the blow-up of X at the point p. Let $\pi : X \cong S \to \mathbb{P}^2$ be the blowing-down map. Since S is a Del Pezzo surface, the inverse of π is the blow-up of 5 distinct points $p_1, \ldots, p_5$ no three of which are collinear. Let $p_6 = \pi(p)$. The cubic surface Y is an anticanonical model of the blow-up of the bubble cycle $p_1 + \cdots + p_6$. If $\pi(p) \notin \{p_1, \ldots, p_5\}$, no line $\overline{p_i, p_j}, i < j \leq 5$, contains p_6 and, moreover, p_6 does not lie on the conic through $p_1, \ldots, p_5$, we obtain a nonsingular cubic surface. The conditions are of course equivalent to that p does not lie on any of 16 lines on X. If it does we obtain a singular cubic surface with double rational points. The types depend on the set of lines containing p.

Now let us assume that the center of the projection p does not lie on X. Let Q_p be the unique quadric from the pencil which contains p. We assume that Q is a nonsingular quadric. We will see that the projection of X from p is a quartic surface singular along a nonsingular conic. In classical literature such a quartic surface is called a *cyclide quartic surface*.

Theorem 8.5.3. *Assume that the quadric Q_p is nonsingular. Then the projection Y of X from p is a quartic surface in $\mathbb{P}^3$ which is singular along a nonsingular conic. Any irreducible quartic surface in $\mathbb{P}^3$ which is singular along a nonsingular conic arises in this way from a Segre quartic surface X in $\mathbb{P}^4$. The surface X is nonsinguar if and only if Y is nonsingular outside the conic.*

Proof. First of all let us see that Y is indeed a quartic surface. If not, the projection is a finite map of degree 2 onto a quadric. In this case the restriction of the pencil of quadrics containing X to a line through p intersecting X has two base points. This implies that there is a quadric in the pencil containing this line and hence containing the point p. Since Q_p is the unique quadric containing p, we see that it contains all lines connecting p with some point on X. Since the lines through a point on a nonsingular quadric are contained in the tangent hyperplane at this point, we wee that X is contained in a hyperplane which contradicts the non-degeneracy of the surface.

Let H be the tangent hyperplane of Q_p at p and $C = H \cap X$. The intersection $H \cap Q_p$ is an irreducible quadric in H with singular point at p. The curve C lies on this quadric and is cut out by a quadric $Q' \cap H$ for some quadric $Q' \neq Q$ from the pencil. Thus the projection from p defines a degree 2 map from C to a nonsingular conic C equal to the projection of the cone $H \cap Q_p$. It spans the plane in $\mathbb{P}^3$ equal to the projection of the hyperplane H. Since the projection defines a birational isomorphism from X to Y which is not an isomorphism over the conic K, we see that Y is singular along C. It is also nonsingular outside C (since we assume that X is nonsingular).

Conversely, let C be a nonsingular conic in $\mathbb{P}^3$. Consider the linear system $|\mathcal{I}_C(2)|$ of quadrics through C. Choose coordinates to assume that C is given by equations $t_0 = t_1^2+t_2^2+t_3^2 = 0$. Then $|\mathcal{I}_C(2)|$ is spanned by quadrics $V(t_0t_i)$ and $V(t_1^2+t_2^2+t_3^2)$. It defines a rational map $f : \mathbb{P}^3 - \to \mathbb{P}^4$ given by the formula

$$[y_0, \ldots, y_4] = [t_0^2, t_0t_1, t_0t_2, t_0t_3, t_1^2 + t_2^2 + t_3^2].$$

Its image is the nonsingular quadric Q_1 given by the equation

$$y_1^2 + y_2^2 + y_3^2 - y_0y_4 = 0. \tag{8.21}$$

The inverse rational map is of course the projection from the point $[0, \ldots, 0, 1]$ (the image of the plane $t_0 = 0$). Let Y be an irreducible quartic surface in $\mathbb{P}^3$ singular along C. Its equation must be of the form

$$(t_1^2 + t_2^2 + t_3^2)^2 + 2t_0(t_1^2 + t_2^2 + t_3^2)g_1(t_1, t_2, t_3) + t_0^2 g_2(t_0, t_1, t_2, t_3) = 0, \tag{8.22}$$

where g_1 and g_2 are homogeneous polynomials of degree 1 and 2, respectively. Its image lies on Q_1 and is cut out by the quadric Q_2 with equation

$$y_4^2 + y_4 g_1(y_1, y_2, y_3) + g_2(y_0, y_1, y_2, y_3) = 0. \tag{8.23}$$

Thus the image of Y is a Segre quartic X in $\mathbb{P}^4$ defined by equations (8.21) and (8.22). Obviously, X does not contain the point $p = [0, 0, 0, 0, 1]$. Then map f is an isomorphism outside the plane $t_0 = 0$. Hence X is nonsingular if and only if Y has no singular points outside C.

□

Remark 8.5.1. After some obvious linear change of variables we can reduce the equation of a cyclide surface to the equation

$$(t_1^2 + t_2^2 + t_3^2)^2 + t_0^2 g_2(t_0, t_1, t_2, t_3) = 0, \tag{8.24}$$

The analog of a quartic cyclide surface in $\mathbb{P}^2$ is a quartic curve with two double points (a *cyclide curve*). Let ℓ be the line through the nodes. We may assume that its equation is $x_0 = 0$ and the coordinates of the points are $[0,1,i],[0,1,-i]$. Then the equation of a cyclide curve can be reduced to the form

$$(x_1^2+x_2^2)^2 + x_0^2 g_2(x_0,x_1,x_2) = 0.$$

A conic passing through singular points of the cyclide is the projectivized circle

$$(x_1-ax_0)^2+(x_2-bx_0)^2-cx_0^2 = 0.$$

The linear system of circles maps $\mathbb{P}^2$ to $\mathbb{P}^3$ with the image a quadric in $\mathbb{P}^3$. The coordinates in $\mathbb{P}^3$ corresponding to a choice of four linearly independent circles are called in the classical literature the *tetrahedral coordinates*. An equation of a circle in this coordinates is a linear equation. A cyclide curve is given by a quadric equation in tetrahedral coordinates.

Similarly, quadric surfaces in $\mathbb{P}^3$ which contain the singular conic of the cyclide surface with equation (8.23) are projectivized balls

$$(t_1-at_0)^2+(t_2-bt_0)^2+(t_3-t_0)^2-ct_0^2 = 0$$

in $\mathbb{P}^3$. The linear system of such balls is 4-dimensional and maps $\mathbb{P}^3$ to $\mathbb{P}^4$ with the image a quadric hypersurface. A choice of 5 linearly independent balls defines a *pentaspherical coordinates* in $\mathbb{P}^4$. The equation of a ball in these coordinates is a linear equation. By choosing a special basis formed by "orthogonal balls", one my assume that the quadric hypersurface in $\mathbb{P}^4$ is given by the sum of squares of the coordinates. In these special pentaspherical coordinates many geometric relationships between balls are expressed easier in terms of their linear equations (see [246]). For example, in pentaspherical coordinates the equation of a cyclide quartic surface is the intersection of two quadrics. This explains the relation with quartic surfaces in $\mathbb{P}^4$.

It remains to consider the projection of a nonsingular Segre surface from a point p lying on a singular quadric Q from the pencil. First we may assume that p is not the singular point of Q. Then the tangent hyperplane H of Q at p intersects Q along the union of two planes. Thus H intersects X along the union of two conics intersecting at two points. This is a degeneration of the previous case. The projection is a *degenerate cyclide surface*. It is an irreducible quartic surface singular along the union of two lines intersecting at one point. Its equation can be reduced to the form

$$t_1^2t_2^2 + t_0^2 g_2(t_0,t_1,t_2,t_3) = 0.$$

Finally let us assume that the center of the projection is the singular point of a cone from the pencil. We have already observed that in this case we have a degree 2 map $X \to V$, where V is a nonsingular quadric in $\mathbb{P}^3$. The branch locus of this map is a nonsingular quartic elliptic curve of bidegree $(2,2)$. If we choose the diagonal equations of X as in Theorem 8.5.1, and take point $p = (1,0,0,0,0)$, then cone with vertex at p is given by the equation

$$(a_2-a_1)t_1^2+(a_3-a_1)t_2^2+(a_3-a_1)t_3^2+(a_4-a_1)t_4^2 = 0.$$

It is projected to the quadric with the same equations in coordinates $[t_1, \ldots, t_4]$ in $\mathbb{P}^3$. The branch curve is cut out by the quadric with the equation

$$t_1^2 + t_2^2 + t_3^2 + t_4^2 = 0.$$

A more general cyclid quartic surfaces are obtained by projection from singular quartic surfaces in $\mathbb{P}^3$. They have been all classified by C. Segre [376].

8.5.3 Lines and singularities

Applying the procedure of Borel-De Sibenthal-Dynkin, we obtain the following list of types of root bases in $\mathbf{E}_5$:

$$D_5, A_3 + 2A_1, D_4, A_4, 4A_1, A_2 + 2A_1, A_3 + A_1, A_3, 3A_1, A_2 + A_1, A_2, 2A_1, A_1.$$

All of these types can be realized as the types of root bases defined by (-2)-curves. First we give the answer in terms of the blow-up model of X.

$D_5 : x_5 \succ x_4 \succ x_3 \succ x_2 \succ x_1, x_1, x_2, x_3$ are collinear;

$A_3 + 2A_1 : x_3 \succ x_2 \succ x_1, x_5 \succ x_4, x_1, x_4, x_5$ are collinear;

$D_4 : x_4 \succ x_3 \succ x_2 \succ x_1, x_1, x_2, x_5$ are collinear;

$A_4 : x_5 \succ x_4 \succ x_3 \succ x_2 \succ x_1;$

$4A_1 : x_2 \succ x_1, x_4 \succ x_3, x_1, x_2, x_5$ and x_3, x_4, x_5 are collinear;

$2A_1 + A_2 : x_2 \succ x_1, x_4 \succ x_3, x_1, x_2, x_5$ and x_3, x_4, x_5 are collinear;

$A_1 + A_3 : x_3 \succ x_2 \succ x_1, x_5 \succ_4, x_1, x_4, x_5$ are collinear;

$A_3 : x_4 \succ x_2 \succ x_1;$ or $x_3 \succ x_2 \succ x_1, x_1, x_4, x_5$ are collinear;

$A_1 + A_2 : x_3 \succ x_2 \succ x_1, x_5 \succ x_4,\ x_1, x_4, x_5$ are collinear;

$3A_1 : x_2 \succ x_1,\ x_4, \succ x_3, x_1, x_3, x_5$ are collinear;

$A_2 : x_3 \succ x_2 \succ x_1;$

$2A_1 : x_2 \succ x_1, x_3 \succ x_2,$ or $x_1, x_2, x_3, x_1, x_4, x_5$ are collinear;

$A_1 : x_1, x_2, x_3$ are collinear.

This can be also stated in terms of equations indicated in the next table. The number of lines is also easy to find by looking at the blow-up model. We have the following table (see [407]).

D_5	A_3+2A_1	D_4	A_4	$4A_1$	A_2+2A_1	A_3+A_1	A_3
[(41)]	[(21)(11)]	[(31)1]	[5]	[(11)(11)1]	[3(11)]	[(21)2]	[(21)11]
1	2	2	3	4	4	3	4
A_3	A_2+A_1	$3A_1$	A_2	$2A_1$	$2A_1$	A_1	
[41]	[32]	[(11)21]	[311]	[(11)111]	[221]	[2111]	
5	6	6	8	8	9	12	

Table 8.7: Lines and singularities on a weak Del Pezzo surface of degree 4

Example 8.5.1. The quartic surfaces with singular points of type $4A_1$ or $2A_1 + A_3$ have a remarkable property that they admit a double cover ramified only at the singular points. We refer to [96] for more details about these quartics surfaces. The projections of these surfaces to $\mathbb{P}^3$ are cubic symmetroid surfaces discussed in the next Chapter. The cover is the quadric surface $\mathbf{F}_0$ in the first case and the quadric cone Q in the second case.

The Gosset polytope $\Sigma_5 = 1_{21}$ has 16 facets of type α and 10 facets of type β. They correspond to contractions of 5 disjoint lines and pencils of conics arising from the pencils of lines through one of the 5 points in the plane and pencils of conics through four of the 5 points.

8.5.4 Automorphisms

By Theorem 8.5.1 a Del Pezzo surface of degree 4 is isomorphic to a nonsingular surface of degree 4 in $\mathbb{P}^4$ given by equations

$$f_1 = \sum_{i=0}^{4} t_i^2 = 0, \quad f_2 = \sum_{i=0}^{4} a_i t_i^2 = 0,$$

where the coefficients a_i are all distinct.

We know that the representation of Aut(S) in $W(S) \cong W(D_5)$ is injective.

Proposition 8.5.4.

$$W(D_5) \cong 2^4 \rtimes \mathfrak{S}_5,$$

where 2^k *denotes the elementary abelian group* $(\mathbb{Z}/2\mathbb{Z})^k$.

Proof. Of course, this is a well-known fact from the theory of reflection groups. However, we give a geometric proof exhibiting the action of $W(D_5)$ on Pic(S). Fix a geometric basis $e_0, \ldots, e_5$ corresponding to a blow-up model of S and consider 5 pairs of pencils of conics defined by the linear systems

$$L_i = |e_0 - e_i|, \quad L_i' = |-K_S - (e_0 - e_i)| = |2e_0 - e_1 - e_2 - e_3 - e_4 - e_5 + e_i|, i = 1, \ldots, 5.$$

Let $\alpha_1, \ldots, \alpha_5$ be the canonical root basis defined by the geometric basis and r_i be the corresponding reflections. Then $r_2, \ldots, r_5$ generate $\mathfrak{S}_5$ and act by permuting the 5 pairs of pencils. Consider the product $r_1 \circ r_5$. It is immediately checked that it switches L_4 with L_4' and L_5 with L_5' leaving L_i, L_i' invariant for $i = 1, 2, 3$. Similarly, a conjugate of $r_1 \circ r_5$ in $W(D_5)$ does the same for some other pair pair of the indices. The subgroup generated by the conjugates is isomorphic to 2^4. Its elements switch the L_i with L_i' in an even number of pairs of pencils. This defines a surjective homomorphism $W(D_5) \to \mathfrak{S}_5$ with kernel containing 2^4. Comparing the orders of the groups we see that the kernel is 2^4 and we have an isomorphism of groups asserted in the proposition. □

The image of the addition map $|L_i| \times |L_i'| \to |-K_S|$ defines a 3-dimensional linear system contained in $|-K_S|$. It defines the projection $\psi_i : S \to \mathbb{P}^3$. Since $D_i \cdot D_i' = 2$ for $D_i \in L_i, D_i' \in L_i'$, the degree of the map is equal to 2. So the image of ψ is a quadric in $\mathbb{P}^3$. This shows that the center of the projection is the vertex of one of the five singular quadric cones in the pencil of quadrics containing the anticanonical model X of S. The deck transformation $g_i, i = 1, \ldots, 5$, of the cover is an automorphism and these five automorphisms a subgroup H of Aut(S) isomorphic to 2^4. One can come to the same conclusion by looking at the equations of X. The group of projective automorphims generated by the transformations which switch t_i to $-t_i$ realizes the subgroup 2^4.

Let G be the subgroup of $W(D_5)$ realized by permutations of the set $\{e_1, \ldots, e_5\}$. It is isomorphic to $\mathfrak{S}_5$ and $W(S)$ is equal to the semi-direct product $H \rtimes G$. Now suppose that Aut(S) contains an element $g \notin H$. Composing it with elements from H, we may assume that $g \in G$. Since elements of G leave e_0 invariant, g is realized by a projective transformation of $\mathbb{P}^2$ leaving the set of points $x_1, \ldots, x_5$ invariant. Since there is a unique conic through these points, the group is isomorphic to a finite group of PSL(2) leaving invariant a binary quintic without multiple roots. All these groups can be easily found. It follows from the classification of finite subgroups of SL(2) and their algebra of invariants that the only possible groups are the cyclic groups C_2, C_3, C_4, C_5, the permutation group $\mathfrak{S}_3$, and the dihedral group D_{10} of order 10. The corresponding binary forms are projectively equivalent to the following binary forms:

(i) $C_2 : u_0(u_0^2 - u_1^2)(u_0^2 + au_1^2),\ a \neq -1, 1;$

(ii) $C_4 : u_0(u_0^2 - u_1^2)(u_0^2 + u_1^2);$

(iii) $C_3, C_6 : u_0u_1(u_0 - u_1)(u_0 - \eta u_1)(u_0 - \eta^2 u_1^2),\ \eta = e^{2\pi i/3};$

(iv) $C_5, D_5 : (u_0 - u_1)(u_0 - \epsilon u_1)(u_0 - \epsilon^2 u_1)(u_0 - \epsilon^3 u_1)(u_0 - \epsilon^4 u_1),$

where $\epsilon = e^{2\pi i/5}$. The corresponding surfaces are projectively equivalent to the following surfaces:

(i) $C_2 : t_0^2 + t_2^2 + a(t_1^2 + t_3^2) + t_4^2 = t_0^2 - t_2^2 + b(t_1^2 - t_3^2) = 0,\ a \neq -b, b;$

(ii) $C_4 : t_0^2 + t_1^2 + t_2^2 + t_3^2 + t_4^2 = t_0^2 + it_1^2 - t_2^2 - it_3^2 = 0;$

(iii) $\mathfrak{S}_3 : t_0^2 + \eta t_1^2 + \eta^2 t_2^2 + t_3^2 = t_0^2 + \eta^2 t_1^2 + \eta t_2^2 + t_4^2 = 0;$

(iv) $D_{10} : t_0^2 + \epsilon t_1^2 + \epsilon^2 t_2^2 + \epsilon^3 t_3^2 + \epsilon^4 t_4^2 = \epsilon^4 t_0^2 + \epsilon^3 t_1^2 + \epsilon^2 t_2^2 + \epsilon t_3^2 + t_4^2 = 0.$

8.6 Del Pezzo surfaces of degree 2

8.6.1 Lines and singularities

Let S be a weak Del Pezzo surface of degree 2. Recall that the anticanonical linear system defines a birational morphism $\phi' : S \to X$, where X is the anticanonical model of S isomorphic to the double cover of $\mathbb{P}^2$ branched along a plane quartic curve C with at most simple singularities (see section 6.3.3. Let $\phi : S \to \mathbb{P}^2$ be the composition of ϕ and the double cover map $\sigma : X \to \mathbb{P}^2$. The restriction of ϕ to a (-1)-curve E is a map of degree $-K_S \cdot E = 1$. Its image in the plane is a line ℓ. The preimage of ℓ is the union of E and a divisor $D \in |-K_S - E|$. Since $-K_S \cdot D = 1$, the divisor D is equal to $E' + R$, where E' is a (-1)-curve and R is the union of (-2)-curves. Also we immediately find that $E \cdot D = 2, D^2 = -1$. There are three possible cases:

(i) $E \neq E',\ E \cdot E' = 2$;

(ii) $E \neq E', E \cdot E' = 1$;

(iii) $E \neq E', E = E'$.

In the first case, the image of E is a line ℓ tangent to C at two nonsingular points. The image of $D - E'$ is a singular point of C. By Bezout's Theorem, ℓ cannot pass through the singular point. Hence $D = E'$ and ℓ is a bitangent of W.

In the second case, $E \cdot D - E' = 1$. The line ℓ passes through the singular point $\phi(D - E')$ and is tangent to C at a nonsingular point.

Finally, in the third case, ℓ is a component of W.

Of course, when S is a Del Pezzo surface, the quartic C is nonsingular, and we have 56 lines paired into 28 pairs corresponding to 28 bitangents of C. Let $\pi : S \to \mathbb{P}^2$ be the blow-up of seven points $x_1, \ldots, x_7$ in general position. Then 28 pairs of lines are the proper inverse transforms of the isolated pairs of curves:

21 pairs: a line through x_i, x_j and the conic through the complementary five points;

7 pairs: a cubic with a double point at x_i and passing through other points plus the exceptional curve $\pi^{-1}(x_i)$.

We use the procedure of Borel-De Siebenthal-Dynkin to compile the list of root bases in $\mathbf{E}_7$. It is convenient first to compile the list of maximal (by inclusions) root bases of type A, D, E (see [240], §12).

Type	rank $n-1$	rank n
A_n	$A_k + A_{n-k-1}$	
D_n	A_{n-1}, D_{n-1}	$D_k + D_{n-k}, k \geq 2$
E_6	D_5	$A_1 + A_5, A_2 + A_2 + A_2$
E_7	E_6	$A_1 + D_6, A_7, A_2 + A_5$
E_8		$D_8, A_1 + E_7, A_8, A_2 + E_6, A_4 + A_4$

Table 8.8: Maximal root bases

Here $D_2 = A_1 + A_1$ and $D_3 = A_3$.

From this easily find the following table of root bases in $\mathbf{E}_7$. Note that there are two roots bases of types $A_1 + A_5$, $A_2 + 2A_1$, $3A_1$, $A_1 + A_3$ and $4A_1$ which are not equivalent with respect to the Weyl group.

The simple singularities of plane quartics were classified by P. Du Val [145], Part III.

A_1: one node;

$2A_1$: two nodes;

A_2: one cusp;

$3A_1$: irreducible quartic with three nodes;

r	Types
7	$E_7, A_1 + D_6, A_7, 3A_1 + D_4, A_1 + 2A_3, A_5 + A_2, 7A_1$
6	$E_6, D_5 + A_1, D_6, A_6, A_1 + A_5, 3A_2, 2A_1 + D_4, 2A_3,$ $3A_1 + A_3, 6A_1, A_1 + A_2 + A_3, A_2 + A_4$
5	$D_5, A_5, A_1 + D_4, A_1 + A_4, A_1 + 2A_2, 2A_1 + A_3,$ $3A_1 + A_2, A_2 + A_3, 5A_1$
≤ 4	$D_4, A_{i_1} + \cdots + A_{i_k}, i_1 + \cdots + i_k \leq 4$

Table 8.9: Root bases in the $\mathbf{E}_7$-lattice

$3A_1$: a cubic and a line;

$A_1 + A_2$: one node and one cusp;

A_3: one tacnode (two infinitely near ordinary double points);

$4A_1$: a nodal cubic and a line;

$4A_1$: two conics intersecting at 4 points;

$2A_1 + A_2$: two nodes and one cusp;

$A_1 + A_3$: a node and a tacnode;

$A_1 + A_3$: cubic and a tangent line;

A_4: one rhamphoid cusp (two infinitely near cusps);

$2A_2$: two cusps;

D_4: an ordinary triple point;

$5A_1$: a conic and two lines;

$3A_1 + A_2$: a cuspidal cubic and a line;

$2A_1 + A_3$: two conics tangent at one point;

$2A_1 + A_3$: a nodal cubic and its tangent line;

$A_1 + A_4$: a rhamphoid cusp and a node;

$A_1 + 2A_2$: a cusp and two nodes;

$A_2 + A_3$: a cusp and a tacnode;

A_5: one oscnode (two infinitely near cusps);

A_5: a cubic and its flex tangent;

D_5: nodal cubic and a line tangent at one branch;

$A_1 + D_4$: a nodal cubic and line through the node;

E_6: an irreducible quartic with one e_6-singularity;

D_6: triple point with one cuspidal branch;

$A_1 + A_5$: two conics intersecting at two points with multiplicities 3 and 1;

$A_1 + A_5$: a nodal cubic and its flex tangent;

$6A_1$: four lines in general position;

$3A_2$: a three cuspidal quartic;

$2A_1 + D_4$: two lines and conic through their intersection point;

$D_5 + A_1$: cuspidal cubic and a line through the cusp;

$2A_3$: two conics intersecting at two points with multiplicities 2;

$3A_1 + A_3$: a conic plus its tangent line plus another line;

$A_1 + A_2 + A_3$: cuspidal cubic and its tangent;

A_6: one oscular rhamphoid cusp (three infinitely near $x_1 \succ x_2 \succ x_1$ cusps);

$A_2 + A_4$: one rhamphoid cups and a cusp;

E_7: cuspidal cubic and its cuspidal tangent;

$A_1 + D_6$: conic plus tangent line and another line through point of contact;

$D_4 + 3A_1$: four lines with three concurrent;

A_7: two irreducible conics intersecting at one point;

$A_5 + A_2$: cuspidal cubic and a flex tangent;

$2A_3 + A_1$: conic and two tangent lines.

Note that all possible root bases are realized except $7A_1$ (this can be realized in characteristic 2). One can compute the number of lines but this rather tedious. For example, in the case A_1 we have 44 lines and the nodal Weyl group $W(S)^{\mathrm{n}}$ acts on the set $W(S)/W(S)_E$ with 6 orbits of cardinality 2 and 44 orbits of cardinality 1. This gives that a one-nodal quartic has 21 bitangents (i.e. lines with two nonsingular points of tangency).

The Gosset polytope $\Sigma_7 = 3_{21}$ has 575 facets of type α and 126 facets of type β. They correspond to contractions of 7 disjoint (-1)-curves and pencils of conics arising from 7 pencils of lines through one of the 7 points in the plane, 35 pencils of conics through 4 points, 42 pencils of cubic curves through 6 points with a node at one of these points, 35 pencils of 3-nodal quartics through the 7 points, and 7 pencils of quintics through the 7 points with 6 double points.

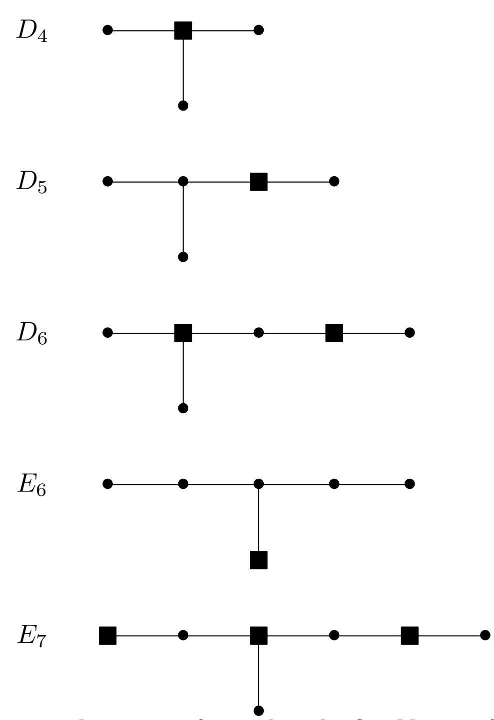
8.6.2 The Geiser involution

Let S be a weak Del Pezzo surface of degree 2. Consider the degree 2 regular map $\phi : S \to \mathbb{P}^2$ defined by the linear system $|-K_S|$. In the blow-up model of S, the linear system $|-K_S|$ is represented by the net of cubic curves $\mathcal{N}$ with seven base bubble points $x_1, \ldots, x_7$ in $\mathbb{P}^2$. It is an example of a Laguerre net considered in Remark 7.1.3. Thus we can view S as the blow-up of 7 points in the plane $\mathbb{P}^2$ which is canonically identified with $|-K_S|$. The target plane $\mathbb{P}^2$ can be identified with the dual plane $|-K_S|^\vee$ of $|-K_S|$. The plane quartic curve C belongs to $|-K_S|^\vee$.

If S is a Del Pezzo surface, then ϕ is a finite map of degree 2 and any subpencil of $|-K_S|$ has no fixed component. Any pencil contained in $\mathcal{N}$ has no fixed components and has 2 points outside the base points of the net. Assigning the line through these points, we will be able to identify the plane $\mathbb{P}^2$ with the net $\mathcal{N}$, or with $|-K|$. This is the property of a Laguerre net. The inverse map is defined by using the coresidual points of Sylvester. For every nonsingular member $D \in \mathcal{N}$, the restriction of $|-K_S|$ to D defines a g_2^1 realized by the projection from the coresidual point on D. This map extends to an isomorphism $\mathcal{N} \to \mathbb{P}^2$.

Let $X \subset \mathbb{P}(1,1,1,2)$ be an anticanonical model of S. The map ϕ factors through a birational map $\sigma : S \to X$ that blows down the Dynkin curves and a degree 2 finite map $\bar{\phi} : X \to \mathbb{P}^2$ ramified along a plane quartic curve C with simple singularities. The deck transformation γ of the cover $\bar{\phi}$ is a birational automorphism of S called the *Geiser involution*. In fact, the Geiser involution is a biregular automorphism of S. Since σ is a minimal resolution of singularities of X, this follows from the existence of a equivariant minimal resolution of singularities of surfaces [269] and the uniqueness of a minimal resolution of surfaces.

Proposition 8.6.1. *The Geiser involution γ has no isolated fixed points. Its locus of fixed points is the disjoint union of smooth curves $W + R_1 + \cdots + R_k$, where $R_1, \ldots, R_k$ are among irreducible components of Dynkin curves. The curve W is the normalization of the branch curve of the double cover $\phi : S \to \mathbb{P}^2$. A Dynkin curve of type A_{2k} has no fixed components, a Dynkin curve of type A_{2k+1} has one fixed component equal to the central component. A Dynkin curve of type D_4, D_5, D_6, E_6, E_7 have fixed components marked by square on their Coxeter-Dynkin diagrams.*



Assume that S is a Del Pezzo surface. Then the fixed locus of the Geiser involution is a smooth irreducible curve W isomorphic to the branch curve of the cover. It belongs to the linear system $|-2K_S|$ and hence its image in the plane is a curve of degree 6 with double points at $x_1, \ldots, x_7$. It is equal to the jacobian curve of the net of cubics, i.e. the locus of singular points of singular cubics from the set. It follows from the Lefschetz Fixed-Point-Formula that the trace of γ in Pic$(S) \cong H^2(S, \mathbb{Z})$ is equal to $e(W) - 2 = -6$. This implies that the trace of σ on $Q_S = (K_S)^\perp$ is equal to -7.

Since rank $Q_S = 7$ this implies that γ acts as the minus identity on Q_S. It follows from the theory of finite reflection groups that the minus identity isogeny of the lattice $\mathbf{E}_7$ is represented by the element w_0 in $W(\mathbf{E}_7)$ of maximal length as a word in simple reflections. It generates the center of $W(\mathbf{E}_7)$.

We can also consider the Geiser involution as a Cremona involution of the plane. It coincides with the Geiser involution described in Chapter 7. The characteristic matrix of a Geiser involution with respect to the bases $e_0, \ldots, e_7$ and $\sigma^*(e_0), \ldots, \sigma^*(e_7)$ is the following matrix:

$$\begin{pmatrix} 8 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ -3 & -2 & -1 & -1 & -1 & -1 & -1 & -1 \\ -3 & -1 & -2 & -1 & -1 & -1 & -1 & -1 \\ -3 & -1 & -1 & -2 & -1 & -1 & -1 & -1 \\ -3 & -1 & -1 & -1 & -2 & -1 & -1 & -1 \\ -3 & -1 & -1 & -1 & -1 & -2 & -1 & -1 \\ -3 & -1 & -1 & -1 & -1 & -1 & -2 & -1 \\ -3 & -1 & -1 & -1 & -1 & -1 & -1 & -2 \end{pmatrix}. \tag{8.25}$$

We can consider this matrix as the matrix of the element $w_0 \in \mathrm{O}(I^{1,7})$ in the basis $\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_7$. It is immediately checked that its restriction to $\mathbf{E}_7$ is equal to the minus identity transformation. As an element of the Weyl group $W(\mathbf{E}_7)$ it is usually denoted by w_0. This is element of maximal length as a word in simple reflections. The group $\langle w_0 \rangle$ is equal to the center of $W(\mathbf{E}_7)$.

The element w_0 acts on the Gosset polytope 3_{21} as the reflection with respect to the center defined by the vector $\frac{1}{2}\mathbf{k}_7 = -\frac{1}{56}\sum v_i$, where v_i are the exceptional vectors. The 28 orbits on the set of vertices correspond to 28 bitangents of a nonsingular plane quartic.

8.6.3 Automorphisms of Del Pezzo surfaces of degree 2

Let S be a Del Pezzo surface of degree 2. The Geiser involution γ belongs to the center of $W(S)$. The quotient group $\mathrm{Aut}(S)/\langle\gamma\rangle$ is the group of automorphisms of the branch cover $\phi : S \to \mathbb{P}^2$. We use the classification of automorphisms of plane quartic curves from Chapter 6. Let G' be a group of automorphisms of the branch curve $V(f)$ given by a quartic polynomial f. Let $\chi : G' \to \mathbb{C}^*$ be the character of G' defined by $\sigma^*(f) = \chi(\sigma)f$. Let

$$G = \{(g', \alpha) \in G' \times \mathbb{C}^* : \chi(g') = \alpha^2\}.$$

This is a subgroup of the group $G' \times \mathbb{C}^*$. The projection to G' defines an isomorphism $G \cong 2.G'$. The extension splits if and only if χ is equal to the square of some character of G'. In this case $G \cong G' \times 2$. The group G acts on S given by equation (8.12) by

$$(\sigma', \alpha) : [t_0, t_1, t_2, t_3] \mapsto [\sigma'^*(t_0), \sigma'^*(t_1), \sigma'^*(t_2), \alpha t_3].$$

Any group of automorphisms of S is equal to a group G as above. This easily gives the classification of possible automorphism groups of Del Pezzo surfaces of degree 2.

Type	Order	Structure	Equation	Parameters
I	336	$2 \times L_2(7)$	$t_3^2 + t_0^3t_1 + t_1^3t_2 + t_2^3t_0$	
II	192	$2 \times (4^2 : \mathfrak{S}_3)$	$t_3^2 + t_0^4 + t_1^4 + t_2^4$	
III	96	$2 \times 4\mathfrak{A}_4$	$t_3^2 + t_2^4 + t_0^4 + at_0^2t_1^2 + t_1^4$	$a^2 = -12$
IV	48	$2 \times \mathfrak{S}_4$	$t_3^2 + t_2^4 + t_1^4 + t_0^4 +$ $+a(t_0^2t_1^2 + t_0^2t_2^2 + t_1^2t_2^2)$	$a \neq \frac{-1\pm\sqrt{-7}}{2}$
V	32	$2 \times AS_{16}$	$t_3^2 + t_2^4 + t_0^4 + at_0^2t_1^2 + t_1^4$	$a^2 \neq 0, -12, 4, 36$
VI	18	18	$t_3^2 + t_0^4 + t_0t_1^3 + t_1t_2^3$	
VII	16	$2 \times D_8$	$t_3^2 + t_2^4 + t_0^4 + t_1^4 + at_0^2t_1^2 + bt_2^2t_0t_1$	$a, b \neq 0$
VIII	12	2×6	$t_3^2 + t_2^3t_0 + t_0^4 + t_1^4 + at_0^2t_1^2$	
IX	12	$2 \times \mathfrak{S}_3$	$t_3^2 + t_2^4 + at_2^2t_0t_1 + t_2(t_0^3 + t_1^3) + bt_0^2t_1^2$	
X	8	2^3	$t_3^2 + t_2^4 + t_1^4 + t_0^4$ $+at_2^2t_0^2 + bt_1^2t_2^2 + ct_0^2t_1^2$	distinct $a, b, c \neq 0$
XI	6	6	$t_3^2 + t_2^3t_0 + f_4(t_0, t_1)$	
XII	4	2^2	$t_3^2 + t_2^4 + t_2^2f_2(t_0, t_1) + f_4(t_0, t_1)$	
XIII	2	2	$t_3^2 + f_4(t_0, t_1, t_2)$	

Table 8.10: Groups of automorphisms of Del Pezzo surfaces of degree 2

We leave to a curious reader the task of classifying automorphism groups of weak Del Pezzo surfaces. Notice that in the action of Aut(S) in the Picard group they correspond to certain subgroups of the group Cris(S). Also the action is not necessary faithful, for example the Geiser involution acts trivially on Pic(S) in the case of a weak Del Pezzo surface with singularity of type E_7.

8.7 Del Pezzo surfaces of degree 1

8.7.1 Lines and singularities

Let S be a weak Del Pezzo surface of degree 1. Its anticanonical model X is a finite cover of degree 2 of a quadratic cone Q ramified over a curve B in the linear system $|\mathcal{O}_Q(3)|$ with at most simple singularities. The list of types of possible Dynkin curves is easy to compile. First we observe that all diagrams listed for the case of the $\mathbf{E}_7$-lattice are included in the list. Also all the diagrams $A_1 + T$, where T is from the previous list are included. We give only the new types.

r	Types
8	$E_8, A_8, D_8, 2A_4, A_1 + A_2 + A_5, A_3 + D_5, 2D_4,$ $A_2 + E_6, A_3 + D_5, 4A_2$
7	$D_7, A_2 + D_5, A_3 + A_4, A_3 + D_4$
6	$A_2 + D_4$

Table 8.11: Root bases in the $\mathbf{E}_8$-lattice

Note that there are two root bases of types A_7, $2A_3$, $A_1 + A_5$, $2A_1 + A_3$ and $4A_1$ which are not equivalent with respect to the Weyl group.

The following result of P. Du Val [145] will be left without proof. Note that Du Val uses the following notation:

$$A_1 = [],\ A_n = [3^{n-1}],\ n \geq 2,\ D_n = [3^{n-3,1,1}],\ n \geq 4,$$

$$E_6 = [3^{3,2,1}],\ E_7 = [3^{4,2,1}],\ E_8 = [3^{5,2,1}].$$

Theorem 8.7.1. *All types of root bases in $\mathbf{E}_8$ can be realized by Dynkin curves except the cases $7A_1, 8A_1, D_4 + 4A_1$.*

In fact, describes explicitly the singularities of the branch sextic similarly to the case of weak Del Pezzo surfaces of degree 2.

The number of lines on a Del Pezzo surface of degree 1 is equal to 240. Note the coincidence with the number of roots. The reason is simple, for any root $\alpha \in \mathbf{E}_8$, the sum $-k_8 + \alpha$ is an exceptional vector. The image of a line under the cover $\phi : S \to Q$ is a conic. The plane spanning the conic is a *tritangent plane*, i.e. a plane touching the branch sextic W at three points. There are 120 tritangent planes, each cut out a conic in Q which splits under the cover in the union of two lines intersecting at three points. Note that the effective divisor D of degree 3 on W such that $2D$ is cut out by a tritangent plane, is an odd theta characteristic on W. This gives another explanation of the number $120 = 2^3(2^4 - 1)$.

The Gosset polytope $\Sigma_8 = 4_{21}$ has 17280 facets of type α corresponding to contractions of sets of 8 disjoint (-1)-curves, and 2160 facets of type β corresponding to conic bundle structures arising from the pencils of conics $|de_0 - m_1e_1 - \ldots - m_8|$ in the plane which we denote by $(d; m_1, \ldots, m_8)$:

- 8 of type $(1; 1, 0^7)$,
- 70 of type $(2; 1^4, 0^5)$,
- 168 of type $(3; 2, 1^5, 0^2)$,
- 280 of type $(4; 2^3, 1^4, 0)$,
- 8 of type $(4; 3, 1^7)$,
- 56 of type $(5; 2^6, 1, 0)$,
- 280 of type $(5; 3, 2^3, 1^4)$,
- 420 of type $(6; 3^2, 2^4, 1^2)$,
- 280 of type $(7, 3^4, 2^3, 1)$,
- 56 of type $(7, 4, 3, 2^6)$,
- 8 of type $(8; 3^7, 1)$,
- 280 of type $(8; 4, 3^4, 2^3)$,

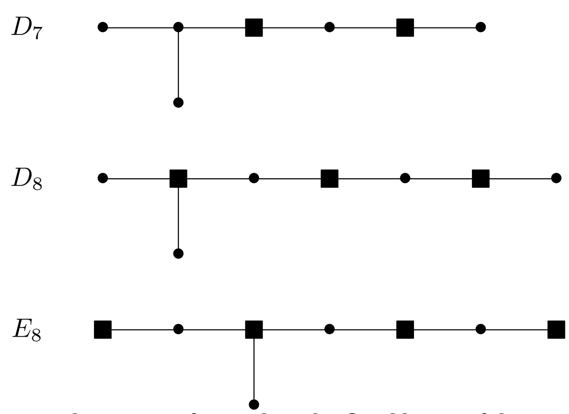
- 168 of type $(9; 4^2, 3^5, 2)$,
- 70 of type $(10; 4^4, 3^4)$,
- 8 of type $(11; 4^7, 3)$,

8.7.2 Bertini involution

Let S be a weak Del Pezzo surface of degree 1. Consider the degree 2 regular map $\phi : S \to Q$ defined by the linear system $|-2K_S|$. In the blow-up model of S, the linear system $|-2K_S|$ is represented by the web $\mathcal{W}$ of sextic curves with eight base bubble points $x_1, \ldots, x_8$ in $\mathbb{P}^2$. If S is a Del Pezzo surface, then ϕ is a finite map of degree 2.

Let $X \subset \mathbb{P}(1,1,2,3)$ be the anticanonical model of S. The map ϕ factors through the birational map $\sigma : S \to X$ that blows down the Dynkin curves and a degree 2 finite map $\bar{\phi} : X \to Q$ ramified along a curve of degree 6 cut out by a cubic surface. The deck transformation β of the cover $\bar{\phi}$ is a birational automorphism of S called the *Bertini involution*. As in the case of the Geiser involution, we prove that the Bertini involution is a biregular automorphism of S.

Proposition 8.7.2. *The Bertini involution β has one isolated fixed point, the base point of $|-K_S|$. The one-dimensional part of the locus of fixed points is the disjoint union of smooth curves $W + R_1 + \cdots + R_k$, where $R_1, \ldots, R_k$ are among irreducible components of Dynkin curves. The curve W is the normalization of the branch curve of the double cover $\phi : S \to Q$. A Dynkin curve of type A_{2k} has no fixed components, a Dynkin curve of type A_{2k+1} has one fixed component equal to the central component. A Dynkin curve of type D_4, D_7, D_8, E_8 have fixed components marked by square on their Coxeter-Dynkin diagrams. The fixed components of Dynkin curves of other types given in the diagrams from Proposition 8.6.1.*



Assume that S is a Del Pezzo surface. Then the fixed locus of the Bertini involution is a smooth irreducible curve W of genus 4 isomorphic to the branch curve of the cover and the base point of $|-K_S|$. It belongs to the linear system $|-3K_S|$ and hence its image in the plane is a curve of degree 9 with triple points at $x_1, \ldots, x_8$. It follows from the Lefschetz fixed-point-formula that the trace of β in $\mathrm{Pic}(S) \cong H^2(S, \mathbb{Z})$ is equal to $1 + e(W) - 2 = -7$. This implies that the trace of σ on $Q_S = (K_S)^\perp$ is equal to -8. Since rank $Q_S = 8$ this implies that γ acts as the minus identity on Q_S. It follows from the theory of finite reflection groups that the minus identity isogeny of the lattice $\mathbf{E}_7$ is represented by the element w_0 in $W(\mathbf{E}_8)$ of maximal length as a word in simple reflections. It generates the center of $W(\mathbf{E}_8)$.

We can also consider the Bertini involution as a Cremona involution of the plane. It coincides with the Bertini involution described in Chapter 7. The characteristic matrix of a Geiser involution with respect to the bases $e_0, \ldots, e_8$ and $\sigma^*(e_0), \ldots, \sigma^*(e_8)$ is the following matrix:

$$\begin{pmatrix} 17 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\ -6 & -3 & -2 & -2 & -2 & -2 & -2 & -2 & -2 \\ -6 & -2 & -3 & -2 & -2 & -2 & -2 & -2 & -2 \\ -6 & -2 & -2 & -3 & -2 & -2 & -2 & -2 & -2 \\ -6 & -2 & -2 & -2 & -3 & -2 & -2 & -2 & -2 \\ -6 & -2 & -2 & -2 & -2 & -3 & -2 & -2 & -2 \\ -6 & -2 & -2 & -2 & -2 & -2 & -3 & -2 & -2 \\ -6 & -2 & -2 & -2 & -2 & -2 & -2 & -3 & -2 \\ -6 & -2 & -2 & -2 & -2 & -2 & -2 & -23 & -3 \end{pmatrix}.$$

We can consider this matrix as the matrix of the element $w_0 \in \mathrm{O}(I^{1,8})$ in the basis $\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_8$. It is immediately checked that its restriction to $\mathbf{E}_9$ is equal to the minus identity transformation. As an element of the Weyl group $W(\mathbf{E}_8)$, it is usually denoted by w_0. This is element of maximal length as a word in simple reflections. The group $\langle w_0 \rangle$ is equal to the center of $W(\mathbf{E}_8)$.

The element w_0 acts on the Gosset polytope 4_{21} as the reflection with respect to the center defined by the vector $\mathbf{k}_8 = -\frac{1}{240} \sum v_i$, where v_i are the exceptional vectors. The 120 orbits on the set of vertices correspond to 120 tritangent planes of the branch curve of the Bertini involution.

8.7.3 Rational elliptic surfaces

We know that the linear system $|-K_S|$ is an irreducible pencil with one base point x_0. Let $\tau : F \to S$ be its blow-up. The proper inverse transform of $|-K_S|$ in F is a base-point-free pencil of curves of arithmetic genus 1. It defines an elliptic fibration $\varphi : F \to \mathbb{P}^1$. The exceptional curve $E = \tau^{-1}(x_0)$ is a section of the fibration. Conversely, let $\varphi : F \to \mathbb{P}^1$ be an elliptic fibration on a rational surface F which admits a section E and relative minimal in the sense that no fibre contains a (-1)-curve. It follows from the theory of elliptic surfaces that $-K_F$ is the divisor class of a fibre and E is a (-1)-curve. Blowing down E, we obtain a rational surface S with $K_S^2 = 1$. Since K_F is obviously nef, we obtain that K_S is nef, so S is a weak Del Pezzo surface of degree 1.

Let $\varphi : F \to \mathbb{P}^1$ be a rational elliptic surface with a section E. The section E defines a rational point e on a generic fibre F_η, considered as a curve over the functional field K of the base of the fibration. It is a smooth curve of genus 1, so it admits a group law with the zero equal to the point e. It follows from the theory of relative minimal models of surfaces that any automorphism of F_η over K extends to a biregular automorphism of F over $\mathbb{P}^1$. In particular, the negation automorphism $x \to -x$ extends to an automorphism of F fixing the curve E. Its descent to the blowing down of E is the Bertini involution.

Let D be a Dynkin curve on S. The point x_0 cannot lie on D. In fact, otherwise the proper transform R' of a component of D that contains x_0 is a (-3)-curve on F. However, $-K_F$ is nef on F hence $K_F \cdot R' \le 0$ contradicting the adjunction formula. This implies that the preimage $\tau^*(D)$ of D on F is a Dynkin curve contained in a fibre. The whole fibre is equal to the union of $\tau^*(D) + R$, where R is a (-2)-curve intersecting the zero section E. Kodaira's classification of fibres of elliptic fibrations shows that the intersection graph of the irreducible components of each reducible fibre is equal to one of the extended Coxeter-Dynkin diagrams.

The classification of Dynkin curves on a weak Del Pezzo surfaces of degree 1 gives the classification of all possible collections of reducible fibres on a rational elliptic surface with a section.

The equation of the anticanonical model in $\mathbb{P}(1,1,2,3)$

$$t_3^2 + t_2^3 + f_4(t_0,t_1)t_2 + f_6(t_0,t_1) = 0, \tag{8.26}$$

after dehomogenization $t = t_1/t_0, x = t_2/t_0^2, y = t_3/t_0^3$ become the *Weierstrass equation* of the elliptic surface

$$y^2 + x^3 + a(t)x + b(t) = 0.$$

The classification of all possible singular fibres of rational elliptic surfaces (not necessary reducible) in terms of the Weierstrass equation was done by several people, e.g. [311].

8.7.4 Automorphisms of Del Pezzo surfaces of degree 1

Let S be a Del Pezzo surface of degree 1. We identify it with its anticanonical model (8.26) The vertex of Q has coordinates $[0,0,1]$ and its preimage in the cover consist of one point $[0,0,1,a]$, where $a^2+1=0$ (note that $[0,0,1,a]$ and $[0,0,1,-a]$ represent the same point in $\mathbb{P}(1,1,2,3)$. This is the base point of $|-K_S|$. The members of $|-K_S|$ are isomorphic to genus 1 curves with equations $y^2+x^3+f_4(t_0,t_1)x+f_6(t_0,t_1)=0$. Our group $\bar{G}$ acts on $\mathbb{P}^1$ via a linear action on (t_0,t_1). The locus of zeros of $\Delta = f_4^3+27f_6^2$ is the set of points in $\mathbb{P}^1$ such that the corresponding genus 1 curve is singular. It consists of a simple roots and b double roots. The zeros of f_4 are either common zeros with f_6 and Δ, or represent nonsingular elliptic curves with automorphism group isomorphic to $\mathbb{Z}/6\mathbb{Z}$. The zeros of f_6 are either common zeros with f_4 and Δ, or represent nonsingular elliptic curves with automorphism group isomorphic to $\mathbb{Z}/4\mathbb{Z}$. The group $\bar{G}$ leaves both sets invariant.

Recall that $\bar{G}$ is determined up to conjugacy by its set of points in $\mathbb{P}^1$ with nontrivial stabilizers. If $\bar{G}$ is not cyclic, then there are three orbits in this set of cardinalities $n/e_1, n/e_2, n/e_3$, where $n = \#\bar{G}$ and (e_1,e_2,e_3) are the orders of the stabilizers. Let Γ be a finite noncyclic subgroup of $\mathrm{PGL}(2)$. We have the following possibilities:

(i) $\Gamma = D_{2k}, n = 2k, (e_1,e_2,e_3) = (2,2,k)$;

(ii) $\Gamma = T$, $n = 12, (e_1, e_2, e_3) = (2, 3, 3)$;

(iii) $\Gamma = O$, $n = 24, (e_1, e_2, e_3) = (2, 3, 4)$;

(iv) $\Gamma = I$, $n = 60, (e_1, e_2, e_3) = (2, 3, 5)$.

If $\bar{\Gamma}$ is a cyclic group of order n, there are 2 orbits of cardinality 1.

The polynomials f_4 and f_6 are relative invariants of $\bar{G}$. Each orbit defines a binary form (the orbital form) with the set of zeros equal to the orbit. One can show that any projective invariant is a polynomial in orbital forms. This immediately implies that $\bar{G} \not\cong \mathfrak{A}_5$ and if $\bar{G} \cong \mathfrak{S}_4$, then $f_4 = 0$.

We choose to represent $\bar{G}$ by elements of $\mathrm{SL}(2)$, i.e. we consider $\bar{G}$ as a quotient of a binary polyhedral subgroup $G \subset \mathrm{SL}(2)$ by its intersection with the center of $\mathrm{SL}(2)$. A projective invariant of $\bar{G}$ becomes a relative invariant of G. We use the description of relative invariants and the corresponding characters of G from [391]. This allows us to list all possible polynomials f_4 and f_6.

The following is the list of generators of the groups $\bar{G}$, possible relative invariants f_4, f_6 and the corresponding character.

We use that a multiple root of f_6 is not a root of f_4 (otherwise the surface is singular).

Case 1: $\bar{G}$ is cyclic of order n. Here ϵ_n denote a primitive n-th root of 1.

n	f_4	$\chi(\sigma)$	f_6	$\chi(\sigma)$
2	$at_0^4 + bt_0^2t_1^2 + ct_1^4$	1	$at_0^6 + t_0^2t_1^2(bt_0^2 + ct_1^2) + dt_1^6$	1
	$t_0t_1(at_0^2 + bt_1^2)$	-1	$t_0t_1(at_0^4 + bt_0^2t_1^2 + ct_1^4)$	1
3	$t_0(at_0^3 + bt_1^3)$	ϵ_3^2	$ct_0^6 + dt_0^3t_1^3 + et_1^6$	1
	$t_1(at_0^3 + bt_1^3)$	ϵ_3	$t_0t_1^2(ct_0^3 + dt_1^3)$	ϵ_3^2
	$t_0^2t_1^2$	1	$t_0^2t_1(ct_0^3 + dt_1^3)$	ϵ_3
4	$at_0^4 + bt_1^4$	-1	$t_0^2(ct_0^4 + dt_1^4)$	-i
	$t_0t_1^3$	i	$t_0t_1(at_0^4 + bt_1^4)$	-1
	$t_0^3t_1$	-i	$t_1^2(ct_0^4 + dt_1^4)$	i
	$t_0^2t_1^2$	1	$t_0^3t_1^3$	1
5	t_0^4	ϵ_5^2	$t_0(at_0^5 + t_1^5)$	ϵ_5^3
	t_0^4	ϵ_5^2	$t_1(at_0^5 + t_1^5)$	ϵ_5^2
	$t_0^3t_1$	ϵ_5	$t_0(t_0^5 + t_1^5)$	ϵ_5^3
	$t_0^3t_1$	ϵ_5	$t_1(t_0^5 + t_1^5)$	ϵ_5^2
	$t_0^2t_1^2$	1	$t_0(t_0^5 + t_1^5)$	ϵ_5^3
6	t_0^4	ϵ_3	$t_0^6 + t_1^6$	-1
	$t_0^3t_1$	ϵ_6	$t_0^6 + t_1^6$	-1
	$t_0^2t_1^2$	1	$t_0^6 + t_1^6$	-1
			$t_0(t_0^5 + t_1^5)$	ϵ_5^3
> 6	t_0^4	ϵ_n^2	t_1^6	ϵ_n^{-3}
			$t_0t_1^5$	ϵ_n^{-2}

Case 2: $\bar{G} = D_n$ is a dihedral group of order $n = 2k$. It is generated by two matrices

$$\sigma_1 = \begin{pmatrix} \epsilon_{2k} & 0 \\ 0 & \epsilon_{2k}^{-1} \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

(i) $k=2$:

$$f_4 = a(t_0^4+t_1^4)+bt_0^2t_1^2,\ \chi(\sigma_1)=\chi(\sigma_2)=1,$$
$$f_4 = at_0t_1(t_0^2-t_1^2),\ \chi(\sigma_1)=\chi(\sigma_2)=-1;$$
$$f_4 = a(t_0^4-t_1^4),\ \chi(\sigma_1)=1,\chi(\sigma_2)=-1,$$
$$f_4 = at_0t_1(t_0^2+t_1^2),\ \chi(\sigma_1)=-1,\chi(\sigma_2)=1;$$
$$f_6 = at_0t_1\big(a(t_0^4+t_1^4)+bt_0^2t_1^2\big),\ \chi(\sigma_1)=1,\chi(\sigma_2)=-1;$$
$$f_6 = a(t_0^6+t_1^6)+bt_0^2t_1^2(t_0^2+t_1^2),\ \chi(\sigma_1)=-1,\chi(\sigma_2)=-1;$$
$$f_6 = a(t_0^6-t_1^6)+bt_0^2t_1^2(t_0^2-t_1^2),\ \chi(\sigma_1)=-1,\chi(\sigma_2)=1;$$
$$f_6 = t_0t_1(t_0^4-t_1^4),\chi(\sigma_1)=\chi(\sigma_2)=1.$$

Note that the symmetric group $\mathfrak{S}_3$ acts on the set of 3 exceptional orbits by projective transformations. This shows, that up to linear change of variables, we have the following essentially different cases.

$$f_4 = a(t_0^4+t_1^4)+bt_0^2t_1^2,\ \chi(\sigma_1)=\chi(\sigma_2)=1;$$
$$f_4 = at_0t_1(t_0^2+t_1^2),\ \chi(\sigma_1)=-1,\chi(\sigma_2)=1;$$
$$f_6 = at_0t_1\big(a(t_0^4+t_1^4)+bt_0^2t_1^2\big),\ \chi(\sigma_1)=1,\chi(\sigma_2)=-1;$$
$$f_6 = t_0t_1(t_0^4-t_1^4),\chi(\sigma_1)=\chi(\sigma_2)=1.$$

(ii) $k=3$:

$$f_4 = t_0^2t_1^2,\ \chi(\sigma_1)=\chi(\sigma_2)=1;$$
$$f_6 = t_0^6+t_1^6+at_0^3t_1^3,\ \chi(\sigma_1)=\chi(\sigma_2)=-1;$$
$$f_6 = t_0^6-t_1^6,\ \chi(\sigma_1)=\chi(\sigma_2)=1.$$

(iii) $k=4$:

$$f_4 = t_0^4\pm t_1^4,\ \chi(\sigma_1)=\chi(\sigma_2)=\pm1;$$
$$f_6 = at_0t_1(t_0^4\pm t_1^4),\ \chi(\sigma_1)=\chi(\sigma_2)=\mp1;$$

(iv) $k=6$:

$$f_4 = at_0^2t_1^2,\quad f_6 = t_0^6\pm t_1^6,\chi(\sigma_1)=\chi(\sigma_2)=\mp1.$$

Case 3: $\bar{G}=\mathfrak{A}_4$. It is generated by matrices

$$\sigma_1=\begin{pmatrix} i & 0\\ 0 & -i\end{pmatrix},\quad \sigma_2=\begin{pmatrix} 0 & i\\ i & 0\end{pmatrix},\quad \sigma_3=\frac{1}{\sqrt{2}}\begin{pmatrix} \epsilon_8^{-1} & \epsilon_8^{-1}\\ \epsilon_8^5 & \epsilon_8\end{pmatrix}.$$

Up to the variable change $t_0\to it_0, t_1\to t_1$, we have only one case

$$f_4 = t_0^4+2\sqrt{-3}t_0^2t_1^2+t_1^4,\quad f_6 = t_0t_1(t_0^4-t_1^4).$$

Case 4: $\bar{G} = \mathfrak{S}_4$. It is generated by matrices

$$\sigma_1 = \begin{pmatrix} \epsilon_8 & 0 \\ 0 & \epsilon_8^{-1} \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} \epsilon_8^{-1} & \epsilon_8^{-1} \\ \epsilon_8^5 & \epsilon_8 \end{pmatrix}.$$

There is only one, up to a change of variables, orbital polynomial of degree ≤ 6. It is

$$f_6 = t_0 t_1 (t_0^4 - t_1^4).$$

The corresponding characters are

$$\chi(\sigma_1) = -1, \quad \chi(\sigma_2) = 1, \quad \chi(\sigma_3) = 1.$$

In this case $f_4 = 0$.

In the next Theorem we list all possible groups $G' = \mathrm{Aut}(S)/\langle\beta\rangle$ and their lifts G to subgroups of $\mathrm{Aut}(S)$. We extend the action of $\bar{G}$ on the coordinates t_0, t_1 to an action on the coordinates t_0, t_1, t_2. Note that not all combinations of (f_4, f_6) admit such an extension.

In the following list, the vector $\mathbf{a} = (a_0, a_1, a_2, a_3)$ will denote the transformation $[t_0, t_1, t_2, t_3] \mapsto [a_0 t_0, a_1 t_1, a_2 t_2, a_3 t_3]$. The Bertini transformation β corresponds to the vector $(1, 1, 1, -1)$.

1. *Cyclic groups G'*

(i) $G' = 2, G = \langle (1, -1, 1, 1), \beta \rangle \cong 2^2$

$$f_4 = at_0^4 + bt_0^2 t_1^2 + ct_1^4, \quad f_6 = dt_0^6 + et_0^4 t_1^2 + ft_0^2 t_1^4 + gt_1^6.$$

(ii) $G' = 2, G = \langle (1, -1, -1, i) \rangle$,

$$f_4 = at_0^4 + bt_0^2 t_1^2 + ct_1^4, \quad f_6 = t_0 t_1 (dt_0^4 + et_0^2 t_1^2 + ft_1^4).$$

(iii) $G' = 3$, $G = \langle (1, \epsilon_3, 1, -1) \rangle \cong 6$,

$$f_4 = t_0 (at_0^3 + bt_1^3), \quad f_6 = at_0^6 + bt_0^3 t_1^3 + ct_1^6.$$

(iv) $G' = 3, G = \langle (1, \epsilon_3, \epsilon_3, -1) \rangle$,

$$f_4 = t_0^2 t_1^2, \quad at_0^6 + bt_0^3 t_1^3 + ct_1^6.$$

(v) $G' = 3, G = 6$, $\mathbf{a} = (1, 1, \epsilon_3, -1)$,

$$f_4 = 0.$$

(vi) $G' = 4, G = \langle (i, 1, -1, i), \beta \rangle \cong 4 \times 2$,

$$f_4 = at_0^4 + bt_1^4, \quad f_6 = t_0^2 (ct_0^4 + dt_1^4).$$

(vii) $G' = 4, G = \langle (i, 1, -i, -\epsilon_8) \rangle \cong 8$,

$$f_4 = at_0^2 t_1^2, \quad f_6 = t_0 t_1 (ct_0^4 + dt_1^4),$$

(viii) $G' = 5, G = \langle(1, \epsilon_5, 1, -1)\rangle \cong 10$,

$$f_4 = at_0^4, \quad f_6 = t_0(bt_0^5 + t_1^5).$$

(ix) $G' = 6, G = \langle(1, \epsilon_6, 1, 1), \beta\rangle \cong 2 \times 6$.

$$f_4 = t_0^4, \quad f_6 = at_0^6 + bt_1^6.$$

(x) $G' = 6, G = \langle(\epsilon_6, 1, \epsilon_3^2, 1), \beta\rangle \cong 2 \times 6$,

$$f_4 = t_0^2 t_1^2, \quad f_6 = at_0^6 + bt_1^6.$$

(xi) $G' = 6, G = \langle(-1, 1, \epsilon_3, 1), \beta\rangle \cong 2 \times 6$,

$$f_4 = 0, \quad f_6 = dt_0^6 + et_0^4 t_1^2 + ft_0^2 t_1^4 + gt_1^6,$$

(xii) $G' = 10, G = \langle(1, \epsilon_{10}, -1, i)\rangle \cong 20$,

$$f_4 = at_0^4, \quad f_6 = t_0 t_1^5.$$

(xiii) $G' = 12, G = \langle(\epsilon_{12}, 1, \epsilon_3^2, -1), \beta\rangle \cong 2 \times 12$,

$$f_4 = at_0^4, \quad f_6 = t_1^6.$$

(xiv) $G' = 12, G = \langle(i, 1, \epsilon_{12}, \epsilon_8)\rangle \cong 24$,

$$f_4 = 0, \quad f_6 = t_0 t_1 (t_0^4 + bt_1^4).$$

(xv) $G' = 15, G = \langle(1, \epsilon_5, \epsilon_3, \epsilon_{30})\rangle \cong 30$,

$$f_4 = 0, \quad f_6 = t_0(t_0^5 + t_1^5).$$

2. *Dihedral groups*

(i) $G' = 2^2, G = D_8$,

$$f_4 = a(t_0^4 + t_1^4) + bt_0^2 t_1^2, \quad f_6 = t_0 t_1 [c(t_0^4 + t_1^4) + dt_0^2 t_1^2],$$

$$\sigma_1 : [t_0, t_1, t_2, t_3] \mapsto [t_1, -t_0, t_2, it_3],$$
$$\sigma_2 : [t_0, t_1, t_2, t_3] \mapsto [t_1, t_0, t_2, t_3],$$
$$\sigma_1^4 = \sigma_2^2 = 1, \sigma_1^2 = \beta, \sigma_2 \sigma_1 \sigma_2^{-1} = \sigma_2^{-1}.$$

(ii) $G' = 2^2, G = 2.D_4$,

$$f_4 = a(t_0^4 + t_1^4) + bt_0^2 t_1^2, \quad f_6 = t_0 t_1 (t_0^4 - t_1^4),$$

$$\sigma_1 : [t_0, t_1, t_2, t_3] \mapsto [t_0, -t_1, -t_2, it_3],$$
$$\sigma_2 : [t_0, t_1, t_2, t_3] \mapsto [t_1, t_0, -t_2, it_3],$$
$$\sigma_1^2 = \sigma^2 = (\sigma_1 \sigma_2)^2 = \beta.$$

(iii) $G' = D_6, G = D_{12}$,

$$f_4 = at_0^2t_1^2, \quad f_6 = t_0^6 + t_1^6 + bt_0^3t_1^3,$$

$$\sigma_1 : [t_0, t_1, t_2, t_3] \mapsto [t_0, \epsilon_3 t_1, \epsilon_3 t_2, -t_3],$$

$$\sigma_2 : [t_0, t_1, t_2, t_3] \mapsto [t_1, t_0, t_2, t_3],$$

$$\sigma_1^3 = \beta, \sigma_2^2 = 1,\ \sigma_2\sigma_3\sigma_2^{-1} = \sigma_1^{-1}.$$

(v) $G' = D_8, G = D_{16}$,

$$f_4 = at_0^2t_1^2, \quad f_6 = t_0t_1(t_0^4 + t_1^4),$$

$$\sigma_1 : [t_0, t_1, t_2, t_3] \mapsto [\epsilon_8 t_0, \epsilon_8^{-1} t_1, -t_2, it_3],$$

$$\sigma_2 : [t_0, t_1, t_2, t_3] \mapsto [t_1, t_0, t_2, t_3],$$

$$\sigma_1^4 = \beta, \sigma_2^2 = 1,\ \sigma_2\sigma_1\sigma_2^{-1} = \sigma_1^{-1}.$$

(vi) $G' = D_{12}, G = 2.D_{12}$,

$$f_4 = at_0^2t_1^2, \quad f_6 = t_0^6 + t_1^6,$$

$$\sigma_1 : [t_0, t_1, t_2, t_3] \mapsto [t_0, \epsilon_6 t_1, \epsilon_3^2 t_2, t_3],$$

$$\sigma_2 : [t_0, t_1, t_2, t_3] \mapsto [t_1, t_0, t_2, t_3], \sigma_3 = \beta.$$

We have

$$\sigma_1^6 = \sigma_2^2 = \sigma_3^3 = 1,\ \sigma_2\sigma_1\sigma_2^{-1} = \sigma_1^{-1}\sigma_3.$$

3. *Other groups*

(i) $G' = \mathfrak{A}_4, G = 2.\mathfrak{A}_4$,

$$f_4 = t_0^4 + 2\sqrt{-3}t_0^2t_1^2 + t_2^4, \quad f_6 = t_0t_1(t_0^4 - t_1^4),$$

$$\sigma_1 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ \sigma_3 = \frac{1}{\sqrt{2}}$$

$$\begin{pmatrix} \epsilon_8^{-1} & \epsilon_8^{-1} & 0 & 0 \\ \epsilon_8^5 & \epsilon_8 & 0 & 0 \\ 0 & 0 & \sqrt{2}\epsilon_3 & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{pmatrix}.$$

(ii) $G' = 3 \times D_4, G = 3 \times D_8$,

$$f_4 = 0, \quad f_6 = t_0t_1(t_0^4 + at_0^2t_1^2 + t_1^4).$$

(iii) $G' = 3 \times D_6, G = 6.D_6 \cong 2 \times 3.D_6$,

$$f_4 = 0, \quad f_6 = t_0^6 + at_0^3 t_1^3 + t_1^6.$$

It is generated by

$$\sigma_1 : [t_0, t_1, t_2, t_3] \mapsto [t_0, t_1, \epsilon_3 t_2, t_3],$$

$$\sigma_2 = [t_0, t_1, t_2, t_3] \mapsto [t_0, \epsilon_3 t_1, t_2, t_3],$$

and

$$\sigma_3 : [t_0, t_1, t_2, t_3] \mapsto [t_1, t_0, t_2, t_3].$$

We have $\sigma_3 \cdot \sigma_2 \cdot \sigma_3^{-1} = \sigma_2^{-1}\sigma_1^4$.

(iv) $G' = 3 \times D_{12}, G = 6.D_{12}$,

$$f_4 = 0, \quad f_6 = t_0^6 + t_1^6.$$

It is generated by

$$\sigma_1 : [t_0, t_1, t_2, t_3] \mapsto [t_0, t_1, \epsilon_3 t_2, t_3],$$

$$\sigma_2 = [t_0, t_1, t_2, t_3] \mapsto [t_0, \epsilon_6 t_1, t_2, t_3],$$

$$\sigma_3 : [t_0, t_1, t_2, t_3] \mapsto [t_1, t_0, t_2, t_3].$$

We have $\sigma_3 \cdot \sigma_2 \cdot \sigma_3^{-1} = \sigma_2^{-1}\sigma_1$.

(v) $G' = 3 \times \mathfrak{S}_4, G = 3 \times 2.\mathfrak{S}_4$,

$$f_4 = 0, \quad f_6 = t_0 t_1 (t_0^4 - t_1^4),$$

$$\sigma_1 = \begin{pmatrix} \epsilon_8 & 0 & 0 & 0 \\ 0 & \epsilon_8^{-1} & & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix},$$

$$\sigma_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} \epsilon_8^{-1} & \epsilon_8^{-1} & 0 & 0 \\ \epsilon_8^5 & \epsilon_8 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{pmatrix}, \quad \sigma_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \epsilon_3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The following table gives a list of the full automorphism groups of Del Pezzo surfaces of degree 1.

Type	Order	Structure	f_4	f_6	Parameters
I	144	$3 \times (\overline{T} : 2)$	0	$t_0t_1(t_0^4 - t_1^4)$	
II	72	$3 \times 2D_{12}$	0	$t_0^6 + t_1^6$	
III	36	$6 \times D_6$	0	$t_0^6 + at_0^3t_1^3 + t_1^6$	$a \neq 0$
IV	30	30	0	$t_0(t_0^5 + t_1^5)$	
V	24	$\overline{T}$	$a(t_0^4 + \alpha t_0^2t_1^2 + t_1^4)$	$t_0t_1(t_0^4 - t_1^4)$	$\alpha = 2\sqrt{-3}$
VI	24	$2D_{12}$	$at_0^2t_1^2$	$t_0^6 + t_1^6$	$a \neq 0$
VII	24	2×12	t_0^4	t_1^6	
VIII	20	20	t_0^4	$t_0t_1^5$	
IX	16	D_{16}	$at_0^2t_1^2$	$t_0t_1(t_0^4 + t_1^4)$	$a \neq 0$
X	12	D_{12}	$t_0^2t_1^2$	$t_0^6 + at_0^3t_1^3 + t_1^6$	$a \neq 0$
XI	12	2×6	0	$g_3(t_0^2, t_1^2)$	
XII	12	2×6	t_0^4	$at_0^6 + t_1^6$	$a \neq 0$
XIII	10	10	t_0^4	$t_0(at_0^5 + t_1^5)$	$a \neq 0$
XIV	8	Q_8	$t_0^4 + t_1^4 + at_0^2t_1^2$	$bt_0t_1(t_0^4 - t_1^4)$	$a \neq 2\sqrt{-3}$
XV	8	2×4	$at_0^4 + t_1^4$	$t_0^2(bt_0^4 + ct_1^4)$	
XVI	8	D_8	$t_0^4 + t_1^4 + at_0^2t_1^2$	$t_0t_1(b(t_0^4 + t_1^4) + ct_0^2t_1^2)$	$b \neq 0$
XVII	6	6	0	$f_6(t_0, t_1)$	
XVIII	6	6	$t_0(at_0^3 + bt_1^3)$	$ct_0^6 + dt_0^3t_1^3 + t_1^6$	
XIX	4	4	$g_2(t_0^2, t_1^2$	$t_0t_1f_2(t_0^2, t_1^2)$	
XX	4	2^2	$g_2(t_0^2, t_1^2$	$g_3(t_0^2, t_1^2)$	
XXI	2	2	$f_4(t_0, t_1)$	$f_6(t_0, t_1)$	

Table 8.12: Groups of automorphisms of Del Pezzo surfaces of degree 1

Exercises

8.1 Prove that $H^1(S, \mathcal{O}_S) = 0$ for a weak Del Pezzo surface S without using the Ramanujam Vanishing Theorem.

8.2 Let $f : X' \to X$ be a resolution of a surface with canonical singularities. Show that $R^1f_*(\mathcal{O}_{X'}) = 0$.

8.3 Describe all possible types of simple singularities which may occur on a plane curve of degree 4.

8.4 Let $G(2,5)$ be the Grassmannian of lines in $\mathbb{P}^4$ embedded in $\mathbb{P}^9$ by the Plücker embedding. Show that the intersection of $G(2,5)$ with a general linear subspace of codimension 4 is an anticanonical model of a weak Del Pezzo surface of degree 5.

8.5 Let S be a weak Del Pezzo surface of degree 6. Show that its anticanonical model is isomorphic to a hyperplane section of the Segre variety $s(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$ in $\mathbb{P}^7$.

8.6 Let S be a weak Del Pezzo surface of degree 5. Show that its anticanonical model contains 5 pencils of conics and the group of automorphisms Aut(X) on this set of pencils defines an isomorphism Aut$(S) \to \mathfrak{S}_5$.

8.7 Prove that any nondegenerate surface of degree 5 in $\mathbb{P}^5$ is isomorphic to an anticanonical model of a Del Pezzo surface or a scroll.

8.8 Describe all weak Del Pezzo surfaces which are toric varieties (i.e. contain an open Zariski subset isomorphic to the torus $(\mathbb{C}^\vee)^2$ such that each translation of the torus extends to an automorphism of the surface).

8.9 Describe all possible singularities on a weak Del Pezzo surface of degree $d \geq 5$.

8.10 A *Dupont cyclide surface* is a quartic cyclide surface with 4 isolated singular points. Find an equation of such a surface.

8.11 Show that a weak quartic Del Pezzo surface is isomorphic to a minimal resolution of the double cover of the plane branched along the union of two conics. Show that the surface is a Del Pezzo surface if and only if the conics intersect transversally.

8.12 Let S be a Del Pezzo surface of degree 4 obtained by blowing up 5 points in the plane. Show that there exists a projective isomorphism from the conic containing the five points and the pencil of quadrics whose base locus is an anticanonical model of S such that the points are sent to singular quadrics.

8.13 Show that the anticanonical model of a Del Pezzo surface of degree 8 isomorphic to a quadric is given by the linear system of plane quartic curves with two fixed double points.

8.14 Prove that a Del Pezzo surface of degree 6 in $\mathbb{P}^6$ has the property that all hyperplanes intersecting the surface along a curve with a singular point of multiplicity ≥ 3 have a common point in $\mathbb{P}^6$ (according to [413] this distinguishes this surface among all other smooth projections of the Veronese surface $v_3(\mathbb{P}^2) \subset \mathbb{P}^9$ to $\mathbb{P}^6$).

8.15 Show that the linear system of quadrics with $8-d$ base points in general map $\mathbb{P}^3$ onto a 3-fold in $\mathbb{P}^{d+1}$ of degree d. Show that an anticanonical model of a Del Pezzo surface of degree $8 > d \geq 3$ is projectively equivalent to a hyperplane section of this threefold.

8.16 Show that the projection of an anticanonical Del Pezzo surfasce of degree $d \geq 3$ for a general point is a surface of degree d in $\mathbb{P}^{d-1}$ with the double curve of degree $d(d-3)/2$.

8.17 Show that the Wiman pencil of 4-nodal plane sextics contains two 6-nodal rational curves and 2 unions of 5 conics [153].

Historical Notes

As the name suggests, P. Del Pezzo was the first who laid the foundation of the theory. In his paper of 1887 [120] he proves that a non-ruled nondegenerate surface of degree d in $\mathbb{P}^d$ can be birationally projected to a cubic surface in $\mathbb{P}^3$ from $d-3$ general points on it. He showed that the images of the tangent planes at the points are skew lines on the cubic surface and deduced from this that $d \leq 9$. He also gave a blow-up model of Del Pezzo surfaces of degree $d \geq 3$, found the number of lines and studied some singular surfaces. J. Steiner was probably the first who related 7 points in the plane with curves of genus 3 by proving that the locus of singular points of the net of cubic curves is a plane sextic with nodes at the seven points [395]. A. Clebsch should be considered as a founder of the theory of Del Pezzo surfaces of degree 2. In his memoir [82] on rational double plane he considers a special case of double planes branched along a plane quartic curve. He shows that the preimages of lines are cubic curves passing through a fixed set of 7 points. He identifies the branch curve with the Steiner sextic and relates the Aronhold set of 7 bitangents with the seven base points. Although C. Geiser was the first to discover the involution defined by the double cover, he failed to see the double plane construction.

E. Bertini in [27], while describing his birational involution of the plane, proves that the linear system of curves of degree 6 with eight double base points has the property that any curve from the linear system passing through a general point x must also pass through a unique point x' (which are in the Bertini involution). He mentions that the same result was proved independently by L. Cremona. This can be interpreted by saying that the linear system defines a rational map of degree 2 onto a quadric surface. Bertini also shows that the set of fixed points of the involution is a curve of degree 9 with triple points at the base points.

The quartic cyclides in $\mathbb{P}^3$ with a nodal conic were first studied in 1864 by G. Darboux[113] and M. Moutard [287] and a year later by E. Kummer [263]. The detailed exposition of Darboux' work can be found in [114], [115]. Some special types of these surfaces were considered much earlier by Ch. Dupin [142]. Kummer was the first to observe the existence of five quadratic cones whose tangent planes cut out two conics on the surface (the *Kummer cones*). They correspond to the five singular quadrics in the pencil defining the corresponding quartic surface in $\mathbb{P}^4$. A. Clebsch finds a plane representation of a quartic cyclide by considering a web of cubics through five points in the plane [79]. He also finds in this way the configuration of 16 lines previously discovered by Darboux and proves that the Galois group of the equation for the 16 lines is isomorphic to $2^4 \rtimes \mathfrak{S}_5$. An 'epoch-making memoir' (see [384], p. 141) of C. Segre [376] finishes the classification of quartic cyclides by considering them as projections of a quartic surface in $\mathbb{P}^4$. Jessop's book [236] contains a good exposition of the theory of singular quartic surfaces including cyclides. At the same time he classified the anticanonical models of singular Del Pezzo surfaces of degree 4 in terms of pencil of quadrics they are defined by. The Segre symbol describing a pencil of quadratic forms was introduced earlier by A. Weiler [425]. The theory of canonical forms of pencils of quadrics based was developed by K. Weierstrass [424] based an earlier work of J. Sylvester [402].

One easily deduces from his classification the classification of singular points on weak anticanonical models of weak Del Pezzo surfaces. The classification of lines was found by other method by G. Timms [407].

The classification of double singular points on algebraic surfaces in $\mathbb{P}^3$ started from the work of G. Salmon [348] who introduced the following notation C_2 for an ordinary node, B_k for *binode* (the tangent cone is the union of two different planes) which depend on how the intersection of the planes intersect the surface, an *unode* U_k with tangent cone being a double plane. The indices here indicates the difference k between the degree of the dual surface and the dual of the nonsingular surface of the same degree. This nomenclature can be applied to surfaces in spaces of arbitrary dimension if the singularity is locally isomorphic to the above singularities. For anticanonical Del Pezzo surfaces the defect k cannot exceed 8 and all corresponding singularities must be rational double points of types $A_1 = C_2$, $A_{k-1} = B_k$, $D_{k-2} = U_k, k = 6, 7$, $E_6 = U_8$. Much later P. Du Val [145] have characterized these singularities as ones which do not affect the conditions on adjunctions, the conditions which can be applied to any normal surface. He showed that each RDP is locally isomorphic to either a node C_2, or binode B_k, or an unode U_k, or other unnodes $U_8* = E_6, U_8^* = E_7$ and $E_{10}^* = E_8$ (he renamed U_8 with E_8^*. A modern treatment of RDP singularities was given by M. Artin [12].

In the same series of papers Du Val classifies all possible singularities of anticanonical models of weak Del Pezzo surfaces of any degree and relates them to Coxeter's classification of finite reflection groups. For Del Pezzo surfaces of degree 1 and 2 this classification have been rediscovered in terms of possible root bases in the corresponding root lattices by T. Urabe [416]. The relationship of this classification to the study of the singular fibres of a versal deformation of a simple elliptic singularities was found by H. Pinkham [314] J. Mérindol [279] and E. Looijenga (unpublished).

The Weyl group $W(\mathbf{E}_6)$ and $W(\mathbf{E}_7)$ as the Galois group of 27 lines on a cubic surface and the group of 28 bitangents on a plane quartic were first studied by C. Jordan [238]. These groups are discussed in many classical text-books in algebra (e.g. [423], B. II, [124]). S. Kantor [241] realized the Weyl group $W(\mathbf{E}_n)$ as groups of linear transformations preserving a quadratic form of signature $(1, n)$ and a linear form. A Coble [85], Part II, was the first who showed that the group is generated by the permutations group and one additional involution. So we should credit him the discovery of the Weyl groups as reflection groups. Apparently independently of Coble, this fact was rediscovered by P. Du Val [144]. We refer to [37] for the history of Weyl groups, reflection groups and root systems. Apparently these parallel directions of study of Weyl groups had been reconciled only recently.

The Gosset polytopes were discovered in 1900 by T. Gosset [190]. The notation n_{21} belongs to him. They had been later rediscovered by E. Elte and H.S.M. Coxeter (see [101]) but only Coxeter realized that their groups of symmetries are reflection groups. The relationship between the Gosset polytopes n_{21} and curves on Del Pezzo surfaces of degree $5-n$ was found by Du Val [144]. In the case of $n = 2$, it goes back to [363]. The fundamental paper of Du Val is the origin of a modern approach to the study of Del Pezzo surfaces by means of root systems of finite-dimensional Lie algebras [121], [277].

We refer to modern texts on Del Pezzo surfaces [384], [277], [121], [258].

Chapter 9

Cubic surfaces

9.1 Lines on a nonsingular cubic surface

9.1.1 More about the E_6-lattice

Let us study the $\mathbf{E}_6$-lattice in more details. A *sixer* in $\mathbf{E}_6$ is a set of 6 mutually orthogonal exceptional vectors in $I^{1,6}$. An example of a sixer is the set $\{\mathbf{e}_1, \ldots, \mathbf{e}_6\}$.

Lemma 9.1.1. *Let $\{v_1, \ldots, v_6\}$ be a sixer. Then there exists a unique root α such that*

$$(v_i, \alpha) = 1, \quad i = 1, \ldots, 6.$$

Moreover, $(w_1, \ldots, w_6) = (r_\alpha(v_1), \ldots, r_\alpha(v_6))$ is a sixer satisfying

$$(v_i, w_j) = 1 - \delta_{ij}.$$

The root associated to $(w_1, \ldots, w_6)$ is equal to $-\alpha$.

Proof. The uniqueness is obvious since $v_1, \ldots, v_6$ are linearly independent, so no vector is orthogonal to all of them. Let

$$v_0 = \frac{1}{3}(-\mathbf{k}_6 + v_1 + \cdots + v_6) \in \mathbb{R}^{1,6}.$$

First we show that $v_0 \in I^{1,6}$. Since $M^\vee = M$ it is enough to show that, for any $x \in I^{1,6}$, $(v_0, x) \in \mathbb{Z}$. Consider the sublattice N of $I^{1,6}$ spanned by $v_1, \ldots, v_6, \mathbf{k}_6$. We have $(v_0, v_i) = 0, i > 0$, and $(v_0, \mathbf{k}_6) = -3$. Thus $(v_0, M) \subset 3\mathbb{Z}$. By computing the discriminant of N, we find that it is equal to 9. By Lemma 8.2.1 N is a sublattice of index 3 in $I^{1,6}$. Hence for any $x \in I^{1,6}$ we have $3x \in N$. This shows that

$$(v_0, x) = \frac{1}{3}(v_0, 3x) \in \mathbb{Z}.$$

Now let us set

$$\alpha = 2v_0 - v_1 - \ldots - v_6.$$

We check that α is a root, and $(\alpha, v_i) = 1, i = 1, \ldots, 6$.

It remains to check the second assertion. Since r_α preserves the symmetric bilinear form, $\{w_1, \ldots, w_6\}$ is a sixer. We have

$$(v_i, w_j) = \big(v_i, r_\alpha(v_j)\big) = \big(v_i, v_j + (v_j, \alpha)\alpha\big) = (v_i, v_j) + (v_i, \alpha)(v_j, \alpha)$$
$$= (v_i, v_j) + 1 = 1 - \delta_{ij}.$$

Finally we check that

$$(r_\alpha(v_i), -\alpha) = \big(r_\alpha^2(v_i), -r_\alpha(\alpha)\big) = -(v_i, \alpha) = 1.$$

$\square$

The two sixers with opposite associated roots form a *double-six* of exceptional vectors.

We recall the list of exceptional vectors in $\mathbf{E}_6$ in terms of the standard orthonormal basis in $I^{1,6}$.

$$\mathbf{a}_i = \mathbf{e}_i, \quad i = 1, \ldots, 6; \tag{9.1}$$
$$\mathbf{b}_i = 2\mathbf{e}_0 - \mathbf{e}_1 - \ldots - \mathbf{e}_6 + \mathbf{e}_i, \quad i = 1, \ldots, 6; \tag{9.2}$$
$$\mathbf{c}_{ij} = \mathbf{e}_0 - \mathbf{e}_i - \mathbf{e}_j, \quad 1 \le i < j \le 6. \tag{9.3}$$

Theorem 9.1.2. *The following is the list of 36 double-sixers with corresponding associated roots:*

1 of type D

$$\begin{matrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 & \mathbf{a}_6 & \quad \boldsymbol{\alpha}_{\max} \\ \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 & \mathbf{b}_5 & \mathbf{b}_6 & \quad -\boldsymbol{\alpha}_{\max} \end{matrix},$$

15 of type D_{ij}

$$\begin{matrix} \mathbf{a}_i & \mathbf{b}_i & \mathbf{c}_{jk} & \mathbf{c}_{jl} & \mathbf{c}_{jm} & \mathbf{c}_{jn} & \quad \boldsymbol{\alpha}_{ij} \\ \mathbf{a}_j & \mathbf{b}_j & \mathbf{c}_{ik} & \mathbf{c}_{il} & \mathbf{c}_{im} & \mathbf{c}_{in} & \quad -\boldsymbol{\alpha}_{ij} \end{matrix},$$

20 of type D_{ijk}

$$\begin{matrix} \mathbf{a}_i & \mathbf{a}_j & \mathbf{a}_k & \mathbf{c}_{lm} & \mathbf{c}_{mn} & \mathbf{c}_{ln} & \quad \boldsymbol{\alpha}_{ijk} \\ \mathbf{c}_{jk} & \mathbf{c}_{ik} & \mathbf{c}_{ij} & \mathbf{b}_n & \mathbf{b}_l & \mathbf{b}_m & \quad -\boldsymbol{\alpha}_{ijk} \end{matrix}.$$

The reflection with the respect to the associated root interchanges the rows preserving the order.

Proof. We have constructed a map from the set of sixers (resp. double-sixers) to the set of roots (resp. pairs of opposite roots). Let us show that no two sixers $\{v_1, \ldots, v_6\}$ and $\{w_1, \ldots, w_6\}$ can define the same root. Since $w_1, \ldots, w_6, \mathbf{k}_6$ span a sublattice of finite index in $I^{1,6}$, we can write

$$v_i = \sum_{j=1}^{6} a_j w_j + a_0 \mathbf{k}_6 \tag{9.4}$$

with some $a_j \in \mathbb{Q}$. Assume that $v_i \neq w_j$ for all j. Intersecting both sides with α, we get

$$1 = a_0 + \cdots + a_6. \tag{9.5}$$

Intersecting both sides with $-\mathbf{k}_6$, we get $1 = a_1 + \cdots + a_6 - 3a_0$, hence $a_0 = 0$. Intersecting both sides with w_j we obtain $-a_j = (v_i, w_j)$. Applying Proposition 8.2.17, we get $a_j \leq -1$. This contradicts (9.5). Thus each v_i is equal to some w_j.

The verification of the last assertion is straightforward.

□

Proposition 9.1.3. *The group $W(\mathbf{E}_6)$ acts transitively on sixers and double-sixers. The stabilizer subgroup of a sixer (resp. double-six) is of order $6!, 2 \cdot 6!$.*

Proof. We know that the Weyl group $W(\mathbf{E}_N)$ acts transitively on the set of roots and the number of sixers is equal to the number of roots. This shows that all sixers form one orbit. The stabilizer subgroup of the sixer $(\mathbf{a}_1, \ldots, \mathbf{a}_6)$ (and hence of a root) is the group $\mathfrak{S}_6$. The stabilizer of the double-sixer D is the subgroup $\langle \mathfrak{S}_6, s_{\boldsymbol{\alpha}_0} \rangle$ of order 2.6!. □

It is easy to see that two different double-sixes can share either 4 or 6 exceptional vectors. More precisely, we have

$$\#D \cap D_{ij} = 4, \ \#D \cap D_{ijk} = 6,$$

$$\#D_{ij} \cap D_{kl} = \begin{cases} 4 & \text{if } \#\{i,j\} \cap \{k,l\} = 0 \\ 6 & \text{otherwise} \end{cases},$$

$$\#D_{ij} \cap D_{klm} = \begin{cases} 4 & \text{if } \#\{i,j\} \cap \{k,l,m\} = 0,2 \\ 6 & \text{otherwise} \end{cases},$$

$$\#D_{ijk} \cap D_{lmn} = \begin{cases} 4 & \text{if } \#\{i,j\} \cap \{k,l\} = 1 \\ 6 & \text{otherwise} \end{cases}.$$

A pair of double-sixers is called a *syzygetic duad* (resp. *azygetic duad*) if they have 4 (resp. 6) exceptional vectors in common.

The next Lemma is an easy computation.

Lemma 9.1.4. *Two double-sixers with associated roots α, β form a syzygetic duad if and only if $(\alpha, \beta) \in 2\mathbb{Z}$.*

This can be interpreted as follows. Consider the vector space

$$\bar{Q} = Q/2Q \cong \mathbb{F}_2^6 \tag{9.6}$$

equipped with the quadratic form

$$q(x + 2Q) = \tfrac{1}{2}(x, x) \mod 2.$$

Notice that the lattice $\mathbf{E}_6$ is an *even lattice*, i.e. its quadratic form $x \mapsto x^2$ takes only even values. So the definition makes sense. The associated symmetric bilinear form is the symplectic form

$$(x + 2Q, y + 2Q) = (x, y) \mod 2.$$

Each pair of opposite roots $\pm\alpha$ defines a vector v in $\bar{Q}$ with $q(v) = 1$. It is easy to see that the quadratic form q has Arf-invariant (see Chapter 5, Part I) equal to 1 and hence vanishes on 28 vectors. The remaining 36 vectors correspond to 36 pairs of opposite roots or, equivalently, double-sixers.

Note that we have a natural homomorphism of groups

$$W(\mathbf{E}_6) \cong O(6, \mathbb{F}_2)^- \tag{9.7}$$

obtaned from the action of $W(\mathbf{E}_6)$ on $Q/2Q$. It is an isomorphism. This is checked by verifying that the automorphism $v \mapsto -v$ of the lattice Q does not belong to the Weyl group W and then comparing the known orders of the groups. 3 It follows from above that an syzygetic pair of double-sixers corresponds to orthogonal vectors v, w. Since $q(v+w) = q(v) + q(w) + (v, w) = 0$, we see that each nonzero vector in the isotropic plane spanned by v, w comes from a double-sixer.

A triple of pairwise syzygetic double-sixers is called a *syzygetic triad* of double-sixers. They span an isotropic plane. Similarly, we see that a pair of azygetic double-sixers span a non-isotropic plane in $\bar{Q}$ with three nonzero vectors corresponding to a triple of double-sixers which are pairwise azygetic. It is called an *azygetic triad* of double-sixers.

We say that three azygetic triads form a *Steiner compex of triads of double-sixers* if the corresponding planes in $\bar{Q}$ are mutually orthogonal. It is easy to see that an azygetic triad contains 18 exceptional vectors and thus defines a set of 9 exceptional (the omitted ones). The set of 27 exceptional vectors omitted from three triads in a Steiner complex is equal to the set of 27 exceptional vectors in the lattice $I^{1,6}$. There are 40 Steiner complexes of triads:

10 of type

$$\Gamma_{ijk,lmn} = (D, D_{ijk}, D_{lmn}),\ (D_{ij}, D_{ik}, D_{jk}),\ (D_{lm}, D_{ln}, D_{mn}),$$

30 of type

$$\Gamma_{ij,kl,mn} = (D_{ij}, D_{ikl}, D_{jkl}),\ (D_{kl}, D_{kmn}, D_{lmn}),\ (D_{mn}, D_{mij}, D_{nij}).$$

Theorem 9.1.5. *The Weyl group $W(\mathbf{E}_6)$ acts transitively on the set of triads of azygetic double-sixers with stabilizer subgroup isomorphic to the group $\mathfrak{S}_3 \times (\mathfrak{S}_3 \wr \mathfrak{S}_2)$ of order 432. It also acts transitively on Steiner complexes of triads of double-sixers. A stabilizer subgroup is a maximal subgroup of $W(\mathbf{E}_6)$ of order 1296 isomorphic to the wreath product $\mathfrak{S}_3 \wr \mathfrak{S}_3$.*

Proof. We know that a triad of azygetic double-sixers corresponds to a pair of roots (up to replacing the root with its negative) α, β with $(\alpha, \beta) = \pm 1$. This pair spans a root sublattice Q of $\mathbf{E}_6$ of type A_2. Fix a root basis. Since the Weyl group acts transitively on the set of roots, we find $w \in W$ such that $w(\alpha) = \boldsymbol{\alpha}_{\max}$. Since $(w(\beta), \boldsymbol{\alpha}_{\max}) = (\beta, \alpha) = 1$, we see that $w(\beta) = \pm\boldsymbol{\alpha}_{ijk}$ for some i, j, k. Applying elements from $\mathfrak{S}_6$, we may assume that $w(\beta) = -\boldsymbol{\alpha}_{123}$. Obviously, the roots $\boldsymbol{\alpha}_{12}, \boldsymbol{\alpha}_{23}, \boldsymbol{\alpha}_{45}, \boldsymbol{\alpha}_{56}$ are orthogonal to $w(\alpha)$ and $w(\beta)$. These roots span a root sublattice of type $2A_2$. Thus we obtain that the orthogonal complement of Q in $\mathbf{E}_6$ contains a sublattice of type $2A_2 \perp A_2$. Since $|\mathrm{disc}(A_2)| = 3$, it follows easily from Lemma 8.2.1 that $Q^\perp$ is a root lattice of type $2A_2$. Obviously, any automorphism which leaves the two roots α, β invariant leaves invariant the sublattice Q and its orthogonal complement $Q^\perp$. Thus the stabilizer contains a subgroup isomorphic to $W(A_2) \times W(A_2) \times W(A_2)$ and the permutation of order 2 which switches the two copies of A_2 in $Q^\perp$. Since $W(A_2) \cong \mathfrak{S}_3$ we obtain that a stabilizer subgroup contains a subgroup of order $2 \cdot 6^3 = 432$. Since its index is equal to 120, it must coincide with the stabilizer group.

It follows from above that a Steiner complex corresponds a root sublattice of type $A_2 \perp A_2 \perp A_2$ contained in $\mathbf{E}_6$. The group $W(A_2) \wr \mathfrak{S}_3$ of order $3 \cdot 432$ is contained in the stabilizer. Since its index is equal to 40, it coincides with the stabilizer. $\square$

Remark 9.1.1. The notions of syzygetic (azygetic) pairs, triads and a Steiner complex of triads of double-sixer is analogous to the notions of syzygetic (azygetic) pairs, triads, and a Steiner complex of bitangents of a plane quartic (see Chapter 6). In both cases we deal with a 6-dimensional quadratic space $\mathbb{F}_2^6$. However, the difference is that the quadratic forms are of different types.

A triple v_1, v_2, v_3 of exceptional vectors is called a *tritangent trio* if

$$v_1 + v_2 + v_3 = -\mathbf{k}_6.$$

If we view exceptional vectors as cosets in $I^{1,6}/\mathbb{Z}\mathbf{k}_6$, this is equivalent to saying that the cosets add up to zero.

It is easy to list all tritangent trios.

Lemma 9.1.6. *There 45 tritangent trios:*

30 of type

$$\mathbf{a}_i, \mathbf{b}_j, \mathbf{c}_{ij}, \quad i \neq j,$$

15 of type

$$\mathbf{c}_{ij}, \mathbf{c}_{kl}, \mathbf{c}_{mn}, \quad \{i, j\} \cup \{k, l\} \cup \{m, n\} = \{1, 2, 3, 4, 5, 6\}.$$

Theorem 9.1.7. *The Weyl group acts transitively on the set of tritangent trios.*

Proof. We know that the permutation subgroup $\mathfrak{S}_6$ of the Weyl group acts on tritangent trios by permuting the indices. Thus it acts transitively on the set of tritangent trios of the same type. Now consider the reflection with respect to the root $\boldsymbol{\alpha}_{123}$. We have

$$\begin{aligned}
r_{\boldsymbol{\alpha}_{123}}(\mathbf{a}_1) &= \mathbf{e}_1 + \boldsymbol{\alpha}_{123} = \mathbf{e}_0 - \mathbf{e}_3 - \mathbf{e}_4 = \mathbf{c}_{34}, \\
r_{\boldsymbol{\alpha}_{123}}(\mathbf{b}_2) &= (2\mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_3 - \mathbf{e}_4 - \mathbf{e}_5 - \mathbf{e}_6) - \boldsymbol{\alpha}_{123} = \mathbf{e}_0 - \mathbf{e}_5 - \mathbf{e}_6 = \mathbf{c}_{56}, \\
r_{\boldsymbol{\alpha}_{123}}(\mathbf{c}_{12}) &= \mathbf{e}_0 - \mathbf{e}_1 - \mathbf{e}_2 = \mathbf{c}_{12}.
\end{aligned}$$

Thus w transforms the tritangent trio $(\mathbf{a}_1, \mathbf{b}_2, \mathbf{c}_{12})$ to the tritangent trio $(\mathbf{c}_{34}, \mathbf{c}_{56}, \mathbf{c}_{12})$. This proves the assertion. $\square$

Remark 9.1.2. The stabilizer subgroup of a tritangent trio is a maximal subgroup of $W(\mathbf{E}_6)$ of index 45 isomorphic to the Weyl group of the root system of type F_4.

Let $\Pi_1 = \{v_1, v_2, v_3\}$ and $\Pi_2 = \{w_1, w_2, w_3\}$ be two tritangent trios with no common elements. We have

$$(v_i, w_1 + w_2 + w_3) = -(v_i, \mathbf{k}_6) = 1,$$

and by Proposition 8.2.17, $(v_i, w_j) \geq 0$. This implies that there exists a unique j such that $(v_i, w_j) = 1$. After reordering, we may assume $j = i$. Let $u_i = -k_6 - v_i - w_i$. Since $u_i^2 = -1, (u_i, k_6) = -1$, the vector u_i is an exceptional vector. Since

$$u_1 + u_2 + u_3 = \sum_{i=1}^{3}(-k_6 - v_i - w_i) = -3k_6 - \sum_{i=1}^{3} v_i - \sum_{i=1}^{3} w_i = -\mathbf{k}_6,$$

we get a new tritangent trio $\Pi_3 = (u_1, u_2, u_3)$. The union $\Pi_1 \cup \Pi_2 \cup \Pi_3$ contains 9 lines $v_i, w_i, u_i, i = 1, 2, 3$. There is a unique triple of tritangent trios which consists of the same 9 lines. It is formed by tritangent trios $\Pi_i' = (v_i, w_i, u_i), i = 1, 2, 3$. It is easy to see that any pair of triples of tritangents trios which consist of the same set of 9 lines is obtained in this way. Such a pair of triples of tritangent trios is called a pair of *conjugate triads of tritangent trios*.

We can easily list all conjugate pairs of triads of tritangent trios:

$$(I)\ \begin{matrix} \mathbf{a}_i & \mathbf{b}_j & \mathbf{c}_{ij} \\ \mathbf{b}_k & \mathbf{c}_{jk} & \mathbf{a}_j \\ \mathbf{c}_{ik} & \mathbf{a}_k & \mathbf{b}_i \end{matrix}\ , \quad (II)\ \begin{matrix} \mathbf{c}_{ij} & \mathbf{c}_{kl} & \mathbf{c}_{mn} \\ \mathbf{c}_{ln} & \mathbf{c}_{im} & \mathbf{c}_{jk} \\ \mathbf{c}_{km} & \mathbf{c}_{jn} & \mathbf{c}_{il} \end{matrix}\ , \quad (III)\ \begin{matrix} \mathbf{a}_i & \mathbf{b}_j & \mathbf{c}_{ij} \\ \mathbf{b}_k & \mathbf{a}_l & \mathbf{c}_{kl} \\ \mathbf{c}_{ik} & \mathbf{c}_{jl} & \mathbf{c}_{mn} \end{matrix}\ .$$

Here a triad is represented by the columns of the matrix and its conjugate triad by the rows of the same matrix. Altogether we have $20 + 10 + 90 = 120$ different triads.

There is a bijection from the set of pairs of conjugate triads to the set of azygetic triads of double-sixers. The 18 exceptional vectors contained in the union of the latter is the complementary set of the set of 9 exceptional vectors defined by a triad in the pair. Here is the explicit bijection.

$$\begin{matrix} \mathbf{a}_i & \mathbf{b}_j & \mathbf{c}_{ij} \\ \mathbf{b}_k & \mathbf{c}_{jk} & \mathbf{a}_j \\ \mathbf{c}_{ik} & \mathbf{a}_k & \mathbf{b}_i \end{matrix} \quad \leftrightarrow \quad D_{ij}, D_{ik}, D_{jk};$$

$$\begin{matrix} \mathbf{c}_{ij} & \mathbf{c}_{kl} & \mathbf{c}_{mn} \\ \mathbf{c}_{ln} & \mathbf{c}_{im} & \mathbf{c}_{jk} \\ \mathbf{c}_{km} & \mathbf{c}_{jn} & \mathbf{c}_{il} \end{matrix} \quad \leftrightarrow \quad D, D_{ikn}, D_{jlm};$$

$$\begin{matrix} \mathbf{a}_i & \mathbf{b}_j & \mathbf{c}_{ij} \\ \mathbf{b}_k & \mathbf{a}_l & \mathbf{c}_{kl} \\ \mathbf{c}_{ik} & \mathbf{c}_{jl} & \mathbf{c}_{mn} \end{matrix} \quad \leftrightarrow \quad D_{mn}, D_{jkm}, D_{jkn}.$$

Recall that the set of exceptional vectors omitted from each triad entering in a Steiner complex of triads of azygetic double-sixers is the set of 27 exceptional vectors. Thus a

Steiner complex defines three pairs of conjugate triads of tritangent trios which contains all 27 exceptional vectors. We have 40 such triples of conjugate pairs.

Theorem 9.1.8. *The Weyl group acts transitively on the set of 120 conjugate pairs of triads of tritangent trios. A stabilizer subgroup H is contained in the maximal subgroup of $W(\mathbf{E}_6)$ of index 40 realized as a stabilizer of a Steiner complex. The quotient group is a cyclic group of order 3.*

Proof. This follows from the established bijection between pairs of conjugate triads and triads of azygetic double-sixers and Theorem 9.1.5. In fact it is easy to see directly the transitivity of the action. It is clear that the permutation subgroup $\mathfrak{S}_6$ acts transitively on the set of pairs of conjugate triads of the same type. Since the Weyl group acts transitively on the set of tritangent trios, we can send a tritangent trio $(\mathbf{c}_{ij}, \mathbf{c}_{kl}, \mathbf{c}_{mn})$ to a tritangent trio $(\mathbf{a}_i, \mathbf{b}_j, \mathbf{c}_{ij})$. As is easy to see from inspection that this sends a conjugate pair of type III to a pair of conjugate triads of type I. Also it sends a conjugate pair of type II to type I or III. Thus all pairs are W-equivalent. □

Remark 9.1.3. Note that each monomial entering into the expression of the determinant of the matrix expressing a conjugate pair of triads represents three orthogonal exceptional vectors. If we take only monomials corresponding to even permutations (resp. odd) we get a partition of the set of 9 exceptional vectors into the union of 3 triples of orthogonal exceptional vectors such that each exceptional vector from one triple has non-zero intersection with two exceptional vectors from any other triple.

9.1.2 Lines and tritangent planes

Let S be an nonsingular cubic surface in $\mathbb{P}^3$. Fix a geometric marking $\phi : I^{1,6} \to \mathrm{Pic}(S)$. We can transfer all the notions and the statements from the previous section to the Picard lattice $\mathrm{Pic}(S)$.. The image of an exceptional vector is the divisor class e with $e^2 = e \cdot K_S = -1$. Under the anti-canonical embedding it defines the class of a line. So, we will identify exceptional vectors with lines on S. We have 27 lines. A tritangent trio of exceptional vectors define a set of three coplanar lines. The plane containing them is called a *tritangent plane*. We have 45 tritangent planes.

Now we can translate all the notions and the statements from the previous section to the geometric language, replacing the word an exceptional vector with the word line. Thus we have 72 sixes of lines, 36 double-sixes and 40 Steiner complexes of triads of double-sixes. If $e_0, e_1, \ldots, e_6$ define a geometric marking, then we can identify the classes of the e_i of the exceptional curves of the blow-up $S \to \mathbb{P}^2$ with exceptional vectors $\mathbf{a}_i$. We identify the proper transforms of the conic through the six points excluding the p_i with the exceptional vector $\mathbf{b}_i$. Finally we identify the line through the points p_i and p_j with the exceptional vector c_{ij}. Under the geometric marking the Weyl group $W(\mathbf{E}_6)$ becomes isomorphic to the index 2 subgroup of the isometry group of $\mathrm{Pic}(S)$ leaving the canonical class invariant (see Corollary 8.2.13). It acts transitively on the set of lines, sixes, double-sixes, tritangent planes, and on the set of conjugate pairs of triples of tritangent planes.

We do not know any elementary geometric proof of the fact any nonsingular cubic surface contains 27 lines. The first proofs of A. Cayley and G. Salmon apply only to general nonsingular cubic surfaces. Without the assumption of genericity, any proof I know uses the representation of the surface as the blow-up of 6 points. For completeness sake, let us reproduce the original proof of Cayley [50].

Theorem 9.1.9. *A general nonsingular cubic surface contains 27 lines and 45 tritangent planes.*

Proof. First of all, let us show that any cubic surface contains a line. Let $\mathrm{Hyp}(3;3)$ be the projective space of cubic surfaces in $\mathbb{P}^3$ and $G = G_1(\mathbb{P}^3)$ be the Grassmann variety of lines in $\mathbb{P}^3$. Consider the incidence varIety

$$X = \{(S, \ell) \in \mathrm{Hyp}(3;3) \times G : \ell \subset S\}.$$

The assertion follows if we show that the first projection is surjective. It is easy to see that the fibres of the second projections are linear subspaces of codimension 4. Thus $\dim X = 4 + 15 = 19 = \dim \mathrm{Hyp}(3,3)$. To show the surjectivity of the first projection, it is enough to find a cubic surface with only finitely many lines on it. Let us consider the surface S given by the equation

$$t_1 t_2 t_3 - t_0^3 = 0.$$

Suppose a line ℓ lies on S. Let $[a_0, a_1, a_2, a_3] \in \ell$. If $a_0 \neq 0$, then $a_i \neq 0, i \neq 0$. On the other hand, every line hits the planes $t_i = 0$. This shows that ℓ is contained in the plane $t_0 = 0$. But there are only three lines on S contained in this plane: $t_i = t_0 = 0, i = 1, 2$ and 3. Therefore S contains only 3 lines. This proves the first assertion.

We already know that every cubic surface $S = V(f)$ has at least one line. Pick up such a line ℓ. Without loss of generality, we may assume that it is given by the equation:

$$t_2 = t_3 = 0.$$

Thus

$$f = t_2 Q_0(t_0, t_1, t_2, t_3) + t_3 Q_1(t_0, t_1, t_2, t_3) = 0, \tag{9.8}$$

where Q_0 and Q_1 are quadratic forms. The pencil of planes $\Pi_{\lambda,\mu} = V(\lambda t_2 - \mu t_3)$ through the line ℓ cuts out a pencil of conics on S. The equation of the conic in in the plane $\Pi_{\lambda,\mu}$ is of the form

$$A_{00}(\lambda,\mu)t_0^2 + A_{11}(\lambda,\mu)t_1^2 + A_{22}(\lambda,\mu)t_2^2 +$$

$$2A_{01}(\lambda,\mu)t_0t_1 + 2A_{12}(\lambda,\mu)t_1t_2 + 2A_{02}(\lambda,\mu)t_0t_2 = 0,$$

where A_{00}, A_{11}, A_{01} are binary forms of degree 1, A_{02}, A_{12} are binary forms of degree 2 and A_{22} is a binary form of degree 3. The discriminant equation of this conic is equal to

$$\begin{vmatrix} A_{00} & A_{01} & A_{02} \\ A_{01} & A_{11} & A_{12} \\ A_{02} & A_{12} & A_{22} \end{vmatrix} = 0.$$

This is a homogeneous equation of degree 5 in variables λ, μ. Thus we expect 5 roots of this equation which gives us 5 reducible conics. This is the tricky point because we do not know whether the equation has 5 distinct roots. First we can exhibit a nonsingular cubic surface and a line on it and check that the equation has indeed 5 distinct roots. For example, let us consider the cubic surface

$$2t_0t_1t_2 + t_3(t_0^2 + t_1^2 + t_2^2 + t_3^2) = 0.$$

The equation becomes $\lambda(\lambda^4 - \mu^4) = 0$. It has 5 distinct roots. This implies that, for general nonsingular cubic surface, we have 5 reducible residual conics. Note that no conic is a double line since otherwise the cubic surface is singular (its equations is reduced to $t_0^2t_1 + t_3Q = 0$ which is a surface with singular point defined by $t_0 = t_3 = Q = 0$).

Thus each solution of the quintic equation defines a tritangent plane Π_i of S consisting of three lines, one of them is ℓ. Thus we found 11 lines on X: the line ℓ and 5 pairs of lines ℓ_i, ℓ_i' lying in the plane Π_i. Pick up some plane, say Π_1. We have 3 lines ℓ, ℓ_1, ℓ_2 in Π_1. Replacing ℓ by ℓ_1, and then by ℓ_2, and repeating the construction, we obtain 4 planes through ℓ_1 and 4 planes through ℓ_2 not containing ℓ and each containing a pair of additional lines. Altogether we found $3 + 8 + 8 + 8 = 27$ lines on S. To see that all lines are accounted for, we observe that any line intersecting either ℓ, or ℓ_1, or ℓ_2 lies in one of the planes we have considered before. So it has been accounted for. Now let L be any line. We find a plane Π through L that contains three lines L, L' and L'' on S. This plane intersects the lines ℓ, ℓ', and ℓ'' at some points p, p' and p'' respectively. We may assume that these points are distinct. Otherwise we find three non-coplanar lines in S passing through one point. As we shall see later this implies that S is singular at this point. Since neither L' nor L'' can pass through two of these points, one of these points lie on L. Hence L is coplanar with one of the lines ℓ, ℓ_1, ℓ_2'. Therefore L has been accounted for.

It remains to count tritangent planes. Each line belongs to 5 tritangent planes, each tritangent plane contains 3 lines. This easily gives that the number of tritangent planes is equal to 45. □

Remark 9.1.4. To make the argument work for any nonsingular cubic surface we may use that the number of singular conics in the pencil of conics residual to a line determines the topological Euler-Poincaré characteristic of the surface. Using the additivity of the Euler-Poincaré characteristic of a CW-complex, we obtain the formula

$$\chi(X) = \chi(B)\chi(F) + \sum_{b \in B}(\chi(F_b) - \chi(F)), \tag{9.9}$$

where $f : X \to B$ is any regular map of an algebraic variety onto a curve B with general fibre F and fibres F_b over points $b \in B$. In our case $\chi(B) = \chi(F) = 2$ and $\chi(F_b) = 3$ for a singular conic-fibre. This gives $\chi(S) = 4 + s$, where s is the number of singular conics. Since any two nonsingular surfaces are homeomorphic (they are parameterized by an open subset of a projective space), we obtain that s is the same for all nonsingular surfaces. We know that $s = 5$ for the example in above, hence $s = 5$ for all nonsingular surfaces. Also we obtain $\chi(S) = 9$ which of course agrees with the fact that S is the blow-up of 6 points in the plane.

The closure of the effective cone $\overline{\mathrm{Eff}}(S)$ of a nonsingular cubic surface is isomorphic to the Gosset polytope $\Sigma_6 = 2_{21}$. It has 72 facets corresponding to sixers and 27 faces corresponding to conic bundles on S. In a geometric basis $e_0, e_1, \ldots, e_6$ they are expressed by the linear systems of types $|e_0 - e_1, |2e_0 - e_1 - e_2 - e_3 - e_4|, |3e_0 - 2e_1 - e_2 - \ldots - e_6|$. The center of $\overline{\mathrm{Eff}}(S)$ is equal to $O = -\frac{1}{3}K_S = (E_1 + \ldots + E_{27})/27$, where $E_1, \ldots, E_{27}$ are the divisor classes of lines. A double-six represents two opposite facets whose centers lie on a line passing through O. In fact, if we consider the double-six $(e_i, e_i' = 2e_0 - e_1 - \ldots - e_6 + e_i), i = 1, \ldots, 6$, then

$$\frac{1}{12}(\sum_{i=1}^{6} e_i) + \frac{1}{12}\sum_{i=1}^{6} e_i' = -\frac{1}{3}K_S = O.$$

The line joining the opposite face is perpendicular to the facets. It is spanned by the root corresponding to the double-six. The three lines E_i, E_j, E_k in a tritangent plane add up to $-K_S$. This can be interpreted by saying that the center of the triangle with vertices E_i, E_j, E_k is equal to the center O. This easily implies that the three lines joining the center O with E_i, E_j, E_k are coplanar.

Remark 9.1.5. Let a_i, b_i, c_{ij} denotes the set of 27 lines on a nonsingular cubic surface. Consider them a 27 unknowns and cubic form F equal to the sum of 45 monomials $a_i b_j c_{ij}, c_{ij} c_{kl} c_{mn}$ corresponding to tritangent planes. It was shown by E. Cartan in his dissertation that the group of projective automorphisms of the cubic hypersurface $V(F)$ in $\mathbb{P}^{26}$ is isomorphic to the simple complex Lie group of type E_6. We refer to [273] for integer models of this cubic.

9.1.3 Schur's quadrics

Let $q \in S^2 E^\vee$ be a quadratic form on a finite-dimensional vector space V. Recall that the apolarity map defines a linear map

$$\mathrm{ap}_1^q : E \to E^\vee,\ v \mapsto P_v(q),$$

which we identify with q. For any linear subspace $L \subset E$, we have the *polar subspace* with respect to q

$$L_q^\perp = \{x \in E : b_q(x, y) = 0, \forall y \in L\} = q(L)^\perp = \cap_{v \in L} P_v(q)^\perp. \tag{9.10}$$

If q is nondegenerate, then

$$L_q^\perp = q^{-1}(L^\perp).$$

If M is a linear subspace of $E^\vee$, we define its *polar subspace* with respect to q by

$$M_q^\perp = q(M^\perp).$$

If q is nondegenerate, then $M_q^\perp$ is the orthogonal complement of $M^\vee$ with respect to the dual quadratic form $\check{q}$ on $E^\vee$ defined by the linear map $q^{-1} : E^\vee \to E$.

All of this can be extended to the projective space $|E|$ and a quadric hypersurface $Q = V(q)$ in $|E|$. For example, for any linear subspace $L \subset |E|$, the dual subspace $L^\perp$

is a linear subspace of $|E|^\vee$ spanned by the hyperplanes (considered as points in $|E^\vee|$) containing L. Also, if Q is a quadric hypersurface in $|E|$ and L is a linear subspace of $|E|$, then the polar subspace of L with respect to Q is equal to

$$L_Q^\perp = \cap_{a\in L} P_a(Q). \tag{9.11}$$

Also, for any linear subspace W of $|E^\vee|$,

$$W_Q^\perp = (\cap_{a\in W^\perp} P_a(Q))^\perp. \tag{9.12}$$

Obviously, it is enough to do the intersection for a spanning set of the subspace.

Let $\{\ell_1,\ldots,\ell_6\}$ be a set of skew lines on a nonsingular cubic surface $S \subset \mathbb{P}^3 = |E|$. A nonsingular quadric Q in $\mathbb{P}^3$ defines six skew lines $\ell_1',\ldots,\ell_6'$, where ℓ_i' is polar to ℓ_i with respect to Q.

The following beautiful result of Ferdinand Schur [367] shows that there exists a unique nonsingular quadric Q such that the ordered set $(\ell_1',\ldots,\ell_6')$ is an ordered set of skew lines on S which together with the ordered set $(\ell_1,\ldots,\ell_6)$ makes a double-sixer.

Theorem 9.1.10. *Let $(l_1,\ldots,l_6),(l_1',\ldots,l_6')$ be a double-sixer of lines on a nonsingular cubic surface S. There exists a unique nonsingular quadric in $\mathbb{P}^3$ such that l_i' is the polar line of l_i with respect to Q for each $i=1,\ldots,6$.*

Proof. Fix an ordered double-sixer $(\ell_1,\ldots,\ell_6),(\ell_1',\ldots,\ell_6)$ on a nonsingular cubic surface S. Choose a geometric marking $\phi : I^{1,6} \to \mathrm{Pic}(S)$ such that $\phi(\mathbf{e}_i) = e_i = [\ell_i], i=1,\ldots,6$. Then the linear system $\big|\phi(\mathbf{e}_0)\big|$ defines a birational map $\pi : S \to \mathbb{P}^2$ which blows the lines ℓ_i to the points p_i. The image of the lines ℓ_i' is the conic C_i passing through all p_j except p_i. The preimage of l_i' with respect to ϕ is the exceptional vector b_i. Let $\phi' : I^{1,6} \to \mathrm{Pic}(S)$ be the geometric marking such that $\phi'(\mathbf{e}_i) = \ell_i'$. It is obtained from ϕ by composing ϕ with the reflection $s_{\boldsymbol{\alpha}_{\max}} \in O(I^{1,6})$. We have $\phi'(\mathbf{e}_0) = \phi(s_{\boldsymbol{\alpha}_{\max}}(\mathbf{e}_0)) = \phi(5\mathbf{e}_0 - 2\mathbf{e}_1 - \ldots - 2\mathbf{e}_6)$. Thus the linear system $|e_0'| = |5\mathbf{e}_0 - 2\mathbf{e}_1 - \ldots - 2\mathbf{e}_6|$ defines a birational map $\pi' : S \to \mathbb{P}^2$ which blows down the lines l_i' to points q_i. Note that there is no canonical identification of two $\mathbb{P}^2$'s. One views them as different planes ${}_1\mathbb{P}^2$ and ${}_2\mathbb{P}^2$.

For any line ℓ in ${}_1\mathbb{P}^2$, its full preimage in S belongs to the linear system $|e_0|$. Since $e_0 \cdot (-K_S) = 3$, the curves in $|e_0|$ are rational curves of degree 3 (maybe reducible). Similarly, the preimages of lines ℓ' in ${}_2\mathbb{P}^2$ are rational curves of degree 3 on S. Now

$$\pi^*(\ell) + \pi'^*(\ell') \in |e_0 + 5e_0 - 2e_1 - \ldots - 2e_6| = |-2K_S|.$$

Thus the union of two rational curves $\pi^*(\ell)$ and $\pi'^*(\ell')$ is cut out by a quadric $Q_{l,l'}$ in $\mathbb{P}^3 = |-K_S|^\vee$. Note that the intersection of a quadric and a cubic is a curve of arithmetic genus 4. Our curves are reducible curves of arithmetic genus 4. When we vary ℓ and ℓ', the corresponding quadrics span a hyperplane $\mathcal{H}$ in $|-2K_S|$. The map

$${}_1\check{\mathbb{P}}^2 \times {}_2\check{\mathbb{P}}^2 \to \mathcal{H}, \quad (l,l') \mapsto Q_{l,l'}, \tag{9.13}$$

is isomorphic to the Segre map.

Recall that our surface S lies in $\mathbb{P}^3 \cong |-K_S|^\vee$. Consider the dual space $\check{\mathbb{P}}^3 = |-K_S|$ and let $\check{Q}$ be the quadric in this space which is apolar to all quadrics in the hyperplane $\mathcal{H}$, i.e. orthogonal to $\mathcal{H}$ with respect to the apolarity map

$$S^2(H^0(S, \mathcal{O}_S(-K_S)) \times S^2(H^0(S, \mathcal{O}_S(-K_S))^\vee) \to \mathbb{C}.$$

In particular, if a quadric from $\mathcal{H}$ is a pair of planes $\Pi_1 \cup \Pi_2$ corresponding to points $a = \Pi_1^\perp$ and $b = \Pi_2^\perp$ in $\check{\mathbb{P}}^3$, then $P_{a,b}(\check{Q}) = 0$.

Now choose 3 special lines $\overline{p_i, p_j}, \overline{p_i, p_k}, \overline{p_j, p_k}$ in the first plane and similar lines $\overline{q_i, q_j}, \overline{q_i, q_k}, \overline{q_j, q_k}$ in the second plane. Then $R_{ij} = \pi^*(\overline{p_i, p_j}) = l_{ij} + l_i + l_j$, where $\ell_{ij} = \phi(c_{ij})$. Similarly, $R'_{jk} = \pi'^*(\overline{q_j, q_k}) = l'_j + l'_k + l'_{jk}$, where

$$l'_{jk} \sim e'_0 - l'_j - l'_k = (5e_0 - 2\sum_{i=1}^{6} e_i) - (2e_0 - \sum_{i=1}^{6} e_i + l_j) - (2e_0 - \sum_{i=1}^{6} e_i + e_k)$$
$$= e_0 - e_j - e_k \sim l_{ij}.$$

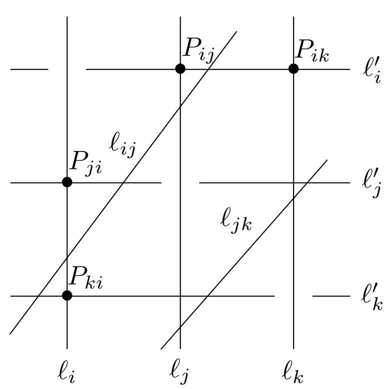
Thus the lines ℓ'_{jk} and ℓ_{ij} coincide.

Now notice that the curve $R_{ij} + R'_{jk}$ is cut out by the reducible quadric $H_{ij} \cup H_{jk}$, where H_{ij} is the tritangent plane containing the lines $\ell_i, \ell'_j, \ell_{ij}$ and H_{jk} is the tritangent plane containing the lines $\ell_j, \ell'_k, \ell_{jk}$.

Let $a \in \mathbb{P}(E) = |E^\vee|$ and $H_a = a^\perp$ be the corresponding hyperplane in $|E|$. If $\langle a, b\rangle_{\check{Q}} = 0$ for $a, b \in \mathbb{P}(E)$, then

$$\check{Q}(a) = (H_a)^\perp_{\check{Q}} \in H_b.$$

Let $P_{ij} = (H_{ij})^\perp_{\check{Q}}$. Since each pair of planes H_{ab}, H_{bc}, considered as points in the dual space, are orthogonal with respect to $\check{Q}$, the point P_{ij} belongs to $H_{ki} \cap H_{jk} \cap H_{ji}$. It is easy to see that this point is $a_j \cap b_i$. Since $a_i \cap b_j \in H_{ij}$, the points P_{ij} and P_{ji} are polar to each other with respect to $\check{Q}$. Similarly, we find that the points (P_{ki}, P_{ik}) are polar with respect to $\check{Q}$, hence the lines ℓ_i and ℓ'_i are polar with respect to Q.



Let us show that $\check{Q}$ is a nondegenerate quadric. Suppose $\check{Q}$ is degenerate, then its set of singular points Sing$(\check{Q})$ is a linear space of positive dimension equal to the kernel of the symmetric bilinear form associated to $\check{Q}$. Thus, for any subspace L of $\mathbb{P}^3$, the polar subspace $L^{\perp}_{\check{Q}}$ with respect to $\check{Q}$ lies in $\mathrm{Sing}(\check{Q})^{\perp}$. Therefore, the points P_{ij} lie in a proper subspace of $\mathbb{P}^3$. But this is obvioulsy impossible, since some of these points lie on a pair of skew lines and span $\mathbb{P}^3$. Thus we can define the dual quadric Q of $\check{Q}$ and obtain that the lines ℓ_i and ℓ_i' are polar with respect to Q.

Let us show the uniqueness of Q. Suppose we have two quadrics Q_1 and Q_2 such that $\ell_i' = (\ell_i)^{\perp}_{Q_i}, i = 1, \ldots, 6$. Let Q be a singular quadric in the pencil spanned by Q_1 and Q_2. Let K be its space of singular points.

Assume first that $\dim K = 0$. Without loss of generality, we may assume that $K \notin l_1 \cup l_2$. Then $\dim(l_1)^{\perp}_Q = \dim(l_2)^{\perp}_Q = 1$. On the other hand, $l_1' \subset (l_1)^{\perp}_Q, l_2' \subset (l_2)^{\perp}_Q$. Thus we have the equalities. But now $l_1 \subset K^{\perp}_Q = \mathbb{P}^3$, hence $K \subset (l_1)^{\perp}_Q = l_1'$ and similarly, $K \subset l_2'$. Since l_1', l_2' are skew, we get a contradiction.

Assume now that $\dim K = 1$. Since K cannot intersect all six lines l_i (otherwise it is contained in S and there are no such lines in S), we may assume that K does not intersect l_1. Then, as above, $l_1' = (l_1)^{\perp}_Q$ and $K = l_1'$. Now, K does not intersect l_2'. Repeating the argument, we obtain that $K = l_2$. Thus $l_1' = l_2$, which is a contradiction.

Finally assume that $\dim K = 2$. Then K intersects all lines. Then $(l_i)^{\perp}_Q$ are all of dimension ≥ 2 and contain K. Since K may contain at most two lines from the double-six, we may assume that $(l_1)^{\perp}_Q = (l_2)^{\perp}_Q = K$. Since $l_1' \subset (l_1)^{\perp}_Q = K, l_2' \subset (l_2)^{\perp}_Q = K$, we see that the lines l_1', l_2' are coplanar and hence intersect. This is a contradiction. □

Definition 9.1. *The quadric Q is called the* Schur quadric *with respect to a given double-six.*

Consider the intersection curve C of the Schur quadric Q with the cubic surface S. Obviously, it belongs to the linear system $|-2K_S|$. Let $\pi : S \to \mathbb{P}^2, \pi' : S \to \mathbb{P}^2$ be the birational morphisms defined by the double-sixer $(l_1, \ldots, l_6), (l_1', \ldots, l_6')$ corresponding to Q. The image of C under π (or π') is a curve of degree 6 with double points at the points $p_i = \pi(l_i)$. We call this curve the *Schur sextic*. It is defined as soon as we choose 6 points on $\mathbb{P}^2$ such that S is isomorphic to the blow-up of these points.

Proposition 9.1.11. *The six double points of the Schur sextic are bi-flexes, i.e. the tangent line to each branch is tangent to the branch with multiplicity ≥ 3.*

Proof. Let Q be the Schur quadric corresponding a Schur sextic. $l_i \cap Q = \{a, b\}$ and $l_i' \cap Q = \{a', b'\}$. We know that

$$P_a(Q) \cap Q = \{x \in Q : a \in PT(Q)_x\}.$$

Since $l_i' = (l_i)^{\perp}_Q$, we have

$$l_i' \cap Q = (P_a(Q) \cap P_b(Q)) \cap Q = \{a', b'\}.$$

This implies that $a', b' \in PT(Q)_a$ and hence the lines $\overline{a, a'}, \overline{a, b'}$ span the tangent space of Q at the point a. The tangent plane $PT(Q)_a$ contains the line l_i' and hence intersects the cubic surface S along l_i' and a conic K_a. We have

$$PT(K_a) = PT(S)_a \cap PT(Q)_a = PT(Q \cap S)_a.$$

Thus the conic K_a and the curve $\mathcal{C} = Q \cap S$ are tangent at the point a. Since the line l_i' is equal to the proper inverse transform of the conic C_i in $\mathbb{P}^2$ passing through the points $p_j, j \neq i$, the conic K_a is the proper inverse transform of some line ℓ in the plane passing through p_i. The point a corresponds to the tangent direction at p_i defined by a branch of the Schur sextic at p_i. The fact that K_a is tangent to $\mathcal{C}$ at a means that the line ℓ is tangent to the branch with multiplicity ≥ 3. Since similar is true, when we replace a with b, we obtain that p_i is a bi-flex of the Schur sextic. $\square$

Remark 9.1.6. A bi-flex is locally given by an equation whose Taylor expansion looks like $xy + xy(ax+by) + f_4(x,y) + \ldots$. This shows that one has impose 5 conditions to get a bi-flex. To get 6 bi-flexes for a curve of degree 6 one has to satisfy 30 linear equations. The space of homogeneous polynomials of degree 6 in 3 variables has dimension 28. So, the fact that such sextics exist is very surprising.

Also observe that the set of quadrics Q such that $l_Q^\perp = l'$ for a fixed pair of skew lines (l, l') is a linear (projective) subspace of codimension 4 of the 9-dimensional space of quadrics. So the existence of the Schur quadric is quite unexpected!

I do not know whether for a given set of 6 points on $\mathbb{P}^2$ defining a nonsingular cubic surface, there exists a unique sextic with bi-flexes at these points.

Example 9.1.1. Let S be the *Clebsch diagonal surface* given by two equations in $\mathbb{P}^4$:

$$\sum_{i=1}^{5} t_i = \sum_{i=1}^{5} t_i^3 = 0. \tag{9.14}$$

It exhibits an obvious symmetry defined by permutations of the coordinates. Let $a = \frac{1}{2}(1+\sqrt{5}), a' = \frac{1}{2}(1-\sqrt{5})$ be two roots of the equation $x^2 - x - 1 = 0$. One checks that the skew lines

$$l : t_1 + t_3 + at_2 = at_3 + t_2 + t_4 = at_2 + at_3 - t_5 = 0$$

and

$$l' : t_1 + t_2 + a't_4 = t_3 + a't_1 + t_4 = a't_1 + a't_4 - t_5 = 0$$

lie on S. Applying to each line even permutations we obtain a double-six. The Schur quadric is $\sum t_i^2 = \sum t_i = 0$.

Let $\pi_1 : S \to \mathbb{P}^2, \pi_2 : S \to \mathbb{P}^2$ be two birational maps defined by blowing down two sixers forming a double-six. We will see later in section 9.3.2 that there exists a 3×3-matrix $A = (a_{ij}(t))$ with entries in linear forms such that $S = V(|A|)$. The two maps are given by taking the left (resp. the right) nullspaces of A. The adjugate matrix $\mathrm{adj}(A)$ of cofactors is a matrix of rank 1 when restricted to S. The coordinates of its proportional columns (resp. rows) are quadrics spanning the linear system $\pi_1^*(\mathcal{O}_{\mathbb{P}^2}(1)$ (resp. $\pi_1^*(\mathcal{O}_{\mathbb{P}^2}(1))$. Let $x = [x_0, \ldots, x_3]$ be any point in $\mathbb{P}^3$. The polar quadric of S with center at x is given by the equation

$$\begin{vmatrix} Da_{11} & Da_{12} & Da_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ Da_{21} & Da_{22} & Da_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ Da_{31} & Da_{32} & Da_{33} \end{vmatrix} = 0,$$

where D is the linear differential operator $\sum x_i \frac{\partial}{\partial t_i}$. Since $|A| = |{}^tA|$, we see that the polar quadric belongs to the image of the map (9.13). By definition of the Schur quadric, its dual quadric is apolar to the polar. This proves the following.

Proposition 9.1.12. *The duals of 36 Schur quadrics belong to the 5-dimensional projective space of quadrics apolar to the 3-dimensional linear system of polar quadrics of S.*

This result was first mentioned by H. Baker in [14], its proof appears in his book [15], Vol. 3, p. 187. In the notation of Theorem 9.1.2, let Q_α is the Schur quadric corresponding to the double-six defined by the root α (see Theorem 9.1.2). Any three of type $Q_{\alpha_{\max}}, Q_{\alpha_{123}}, Q_{\alpha_{456}}$ are linearly dependent. Among $Q_{\alpha_{ij}}$'s at most 5 are linearly independent ([339]).

9.1.4 Eckardt points

A point of intersection of three lines in a tritangent plane is called an *Eckardt point*. As we will see later the locus of nonsingular cubic surfaces with an Eckardt point is of codimension 1 in the moduli space of cubic surfaces.

Proposition 9.1.13. *There is a bijective correspondence between Eckardt points on a nonsingular cubic surface S and automorphisms of order 2 with one isolated fixed point.*

Proof. Let $p \in S$ be an Eckardt point and let $\pi : S' \to S$ be the blow-up of p. This is a Del Pezzo surface of degree 2. The preimage of the linear system $|-K_S - p|$ is the linear system $|-K_{S'}|$. It defines a degree 2 regular map $f : S' \to \mathbb{P}^2$ whose restriction to $S \setminus \pi^{-1}(p) \cong S \setminus \{p\}$ is the linear projection of S with center at p. Let R_1, R_2, R_3 be the proper inverse transforms of the lines in S from the tritangent plane defined by p. These are (-2)-curves on S'. Their image in $\mathbb{P}^2$ is a singular point of the branch curve B of degree 4. The image of $E = \pi^{-1}(p)$ is a line passing through 3 singular points of a quartic curve. It must be an irreducible component of B. Thus B is the union of a line ℓ and a cubic curve C which intersect at three distinct points x_1, x_2, x_3. Let X be the double cover of the blow-up of $\mathbb{P}^2$ at the points x_1, x_2, x_3 ramified along the proper transform of the curve B. We have a birational map $f : S' - \to X$ which is a regular map outside the union of curves R_i. It is easy to see that it extends to the whole S' by mapping the curves R_i isomorphically to the preimages of the points x_i under the map $X \to \mathbb{P}^2$.

Thus $f : S' - \to X$ is a finite map of degree 2, and hence is a Galois cover of degree 2. The corresponding automorphism of S' leaves the curve E pointwisely invariant, and hence descends to an automorphism σ of the cubic surface S. Since it must leave $|-K_S|$ invariant, it is induced by a linear projective transformation $\bar{g}$ of $\mathbb{P}^3$. Its set of fixed points in $\mathbb{P}^3$ is the point p and a plane which intersects S along a curve C'. The linear projection from p maps C' isomorphically to the plane cubic C. Thus σ has one isolated fixed point on S.

Conversely, assume S admits an automorphism σ of order 2 with one isolated fixed point p. As above we see that σ is induced by a projective transformation $\bar{\sigma}$. Diagonalizing the corresponding linear map of $\mathbb{C}^4$, we see that $\bar{g}$ has one eigenspace of dimension 1 and one eigensubspace of dimension 3. Thus in $\mathbb{P}^3$ it fixes a point and a plane Π. The fixed locus of σ is the point p and a plane section C' not passing through p. Let P be the tangent plane of S at p. It is obviously invariant and its intersection with S is a cubic plane curve Z with a singular point at p. Its intersection with C' gives 3 fixed nonsingular points on Z. If Z is irreducible, its normalization is isomorphic to $\mathbb{P}^1$ which has only two fixed points of any non-trivial automorphism of order 2. Thus Z is reducible. If it consists of a line and a conic, then one of the components has 3 fixed points including the point p. Again this is impossible. So we conclude that Z consists of three concurrent lines and hence a tritangent plane. It is clear that the automorphism associated to this tritangent plane coincides with σ. $\square$

Example 9.1.2. Consider a cubic surface given by equation

$$f_3(t_0, t_1, t_2) + t_3^3 = 0,$$

where $C = V(f_3)$ is a nonsingulat plane cubic. Let ℓ be a flex tangent of C. It is easy to see that the preimage of C under the projection $[x_0, x_1, x_2, x_3] \mapsto [x_0, x_1, x_2]$ splits in the union of three lines passing through a common point (the preimage of the inflection point). Thus the surface contains 9 Eckardt points. Note that the corresponding 9 tritangent planes contain all 27 lines.

Example 9.1.3. Consider a cubic surface given by equation

$$\sum_{i=0}^{4} a_i t_i^3 = \sum_{i=0}^{4} t_i = 0,$$

where $a_i \neq 0$. We will see later that a general cubic surface is projectively equivalent to such surface. Assume $a_0 = a_1$. Then the point $p = [1, -1, 0, 0, 0]$ is an Eckardt point. In fact, the tangent plane at this point is $t_0 + t_1 = t_2 + t_3 + t_4 = 0$. It cuts out the surface along the union of three lines intersecting at the point p. Similarly, we have an Eckardt point whenever $a_i = a_j$ for some $i \neq j$. Thus we may have 1,2,3,4, 6 or 10 Eckardt points dependent on whether we have two coefficients are equal, or two pairs of the coefficients are equal, or three coefficients are equal, or two and three coefficients are equal, or four or five coefficients are equal. The other possibilities for the number of Eckardt points are 9 as in the previous example or 18 in the case when the surface is isomorphic to a Fermat cubic surface.

For the future need let us prove the following.

Proposition 9.1.14. *Let p_1 and p_2 be two Eckardt points on S such that the line $\ell = \overline{p_1, p_2}$ is not contained in S. Then ℓ intersects S in a third Eckardt point.*

Proof. Let σ be an automorphism of S defined by the projection from the point p_1. Then l intersects S at the point $p_3 = g(p_2)$. Note the projection of any line ℓ passing through p_2 must contain a singular point of the branch locus since otherwise ℓ intersects the line component of the branch locus at a nonsingular point and hence passes through p_1. Thus ℓ intersects one of the lines passing through p_1 and the plane spanned by these two lines cuts out S in an additional line passing through p_3. In this way we find three lines through p_3. $\square$

Proposition 9.1.15. *No more than two Eckardt points lie on a line contained in the surface.*

Proof. Consider the linear projection from one of the Eckardt points p_1. Its branch curve is the union of a plane cubic C and a line intersecting at three points. The second Eckardt point p_2 is projected to one of the intersection points, say q. The plane spanned by the lines $\langle p_1, p_2\rangle$ and one of the other 2 lines passing through p_2 is a tritangent plane with Eckardt point p_2. Since it is invariant with respect to the involution σ defined by p_1, the point p_2 is a fixed point and hence lies on the curve of fixed points of σ. The projection of the tritangent plane is a line which intersects C only at the point q. Hence q is an inflection point. Clearly this shows that if there is a third Eckardt point p_3, it must coincide with p_2. $\square$

9.1.5 27 lines and 28 bitangents

Let S be a nonsingular cubic surface and $x_0 \in S$ be a point on it. Projecting from the point, we obtain a map of degree 2 $\phi : S' \to \mathbb{P}^2$, where S' is the blow-up of x_0. If x_0 is not an Eckardt point, the ramification divisor of this map is equal to the proper transform of the intersection W of S with the polar quadric $P_{x_0}(S)$. The curve W is a curve of degree 6 which has a double point at x_0 (because the second polar is the tangent plane of S at x_0). If x_0 is an Eckardt point, the curve W is the union of three lines passing through x_0 and an irreducible curve of degree 3. The ramification locus is the union of the proper transform of W and the exceptional curve E_0 blown up from x_0. The branch divisor of ϕ is a plane curve C of degree 4. If x_0 is an Eckardt point, C is the union of a cubic curve and a line. The three intersection points are the images of the three lines passing through x_0.

The surface S' is a weak Del Pezzo surface of degree 2. If x_0 does not lie on any line, then S' is a Del Pezzo surface. Assume the latter. Then $\phi : S' \to \mathbb{P}^2$ is a finite map of degree 2 given by the anti-canonical linear system $|-K_{S'}|$. The curve C is a nonsingular curve of degree 4. Let ℓ be a line on S. Then its projection is a line $\bar{\ell}$ in the plane. The pre-image of $\bar{\ell}$ under the projection is the plane section of S spanned by ℓ and x_0. It consists of the line ℓ and a conic γ. The conic passes through x_0. The intersection points ℓ and γ belong to the ramification locus. Thus the line $\bar{\ell}$ intersects C at two points, hence it is a bitangent. In this way we obtain 27 bitangents of C. The last one is of course the image $\bar{\lambda}_0$ of E_0 in the plane. Its pre-image is equal to the union of E_0 and the plane section of S by the tangent plane at x_0. Thus we see that the 28 bitangens of C correspond to 28 pairs of (-1)-curves on S'. The two (-1)-lines in each pair are exchanged by the Bertini involution.

Let $(\ell_1, \ldots, \ell_6)$ be a sixer on S. We continue to assume that x_0 does not lie on any line in S. Let $\sigma : S \to \mathbb{P}^2$ be the blowing-down of the sixer to points $p_1, \ldots, p_6$. Then the surface S' is obtained by blowing up $p_1, \ldots, p_6$ and $p_7 = \sigma(x_0)$. This shows that the 7 bitangents $\bar{\ell}_1, \ldots, \bar{\ell}_6, \bar{\lambda}_0$ form an Aronhold set of bitangents. Thus we see that we have 72 Aronhold sets containing a common bitangent.

Next we assume that x_0 lies on a line. Then its proper transform on S' is blown down to a singular point of the quartic curve C. Since x_0 lies on at most 3 lines, we see that C has at most 3 nodes. If the number of nodes is equal to 3, then x_0 is an Eckardt point, and C acquires a line component. In other cases C is an irreducible curve with at most 2 singular points.

Conversely, let C be a plane quartic. We know that a minimal resolution of the double cover of the plane ramifies over C is isomorphic is a weak Del Pezzo surface S' of degree 2. This surface S' is obtained by blowing up a point on a nonsingular cubic surface S if and only if there exists a (-1)-curve E_0 on S' which intersects all (-2)-curves on S'. Since the map $S' \to \mathbb{P}^2$ is given by $|-K_{S'}|$, the image of this (-1)-curve in the plane is a line passing through singular points of C. This easily implies that C has at most three nodes, and it has three nodes, then C is the union of a line an an irreducible cubic. Otherwise, it is an irreducible quartic curve.

9.2 Singularities

9.2.1 Non-normal cubic surfaces

Let X be an irreducible cubic surface in $\mathbb{P}^3$. Assume that X is not normal. Then its singular locus contains a one-dimensional part C. Let $C_1, \ldots, C_k$ be irreducible components of C and m_i be the multiplicity of a general point η_i of C_i as a point on X. A general section of X is a plane cubic curve H. Its intersection points with C_i are singular points of multiplicity m_i. Their number is equal to $d_i = \deg(C_i)$. By Bertini's theorem, H is irreducible. Since an irreducible plane cubic curve has only one singular point of multiplicity 2, we obtain that C is irreducible and of degree 1.

Let us choose coordinates in such a way that C is given by the equations $t_0 = t_1 = 0$. Then the equation of X must look like

$$l_0 t_0^2 + 2 l_1 t_0 t_1 + l_2 t_1^2 = 0,$$

where $l_i, i = 0, 1, 2,$ are linear forms in t_0, t_1, t_2. This shows that the left-hand side contains t_2 and t_3 only in degree 1. Thus we can rewrite the equation in the form

$$t_2 f + t_3 g + h = 0, \tag{9.15}$$

where f, g, h are binary forms in t_0, t_1, the first two of degree 2, and the third one of degree 3.

Suppose f, g have no common zeros. Then the map $\mathbb{P}^1 \to \mathbb{P}^1$ defined by (f, g) is of degree 2, and hence has two ramification points. This implies that $f = al^2 + bm^2, g = a'l^2 + b'm^2$ for some linear polynomials l, m. After linear change of variables we may assume that $l = t_0, m = t_1$. Thus every monomial in the left-hand side of the equation (9.15) is divisible either by t_0^2 or by t_1^2. Thus we can rewrite it in the form $pt_0^2 + qt_1^2$, where p, q are linear forms in t_0, t_1, t_2, t_3. Without loss of generality, we may assume that p has a non-zero coefficient at t_3. After a linear change of variables we may assume that $p = t_3$. If q has zero coefficient at t_2, our surface is a cone over a singular plane cubic. If the coefficient is non-zero, after a linear change of variables we may assume that $q = t_3$ and the equation becomes

$$t_2 t_0^2 + t_3 t_1^2 = 0.$$

Suppose f, g has one common non-multiple zero. After a linear change of variables t_0, t_1, we may assume that $f = t_0t_1, g = t_0(t_0 + t_1)$ and the equation becomes

$$t_2t_0t_1 + t_3t_0t_1 + t_3t_0^2 + t_0t_1(at_0 + bt_1) + ct_0^3 + dt_1^3 = 0.$$

After the linear change of variables

$$t_2 \mapsto t_2 + t_3 + at_0 + bt_1, \quad t_3 \mapsto t_3 + ct_0,$$

we reduce the equation to the form

$$t_2t_0t_1 + t_3t_0^2 + dt_1^3 = 0.$$

Obviously, $d \neq 0$. Multiplying by d^2 and changing $t_0 \mapsto dt_0,\ t_1 \to dt_1,\ t_2 \mapsto d^{-1}t_2$, we may assume that $d = 1$.

Finally, if g is proportional to f, say $g = \lambda f$, replacing t_2 with $t_2 + \lambda t_3$, we reduce the equation 9.15 to the form $t_2f + h = 0$. In this case X is again a cone.

Summarizing we get

Theorem 9.2.1. *Let X be an irreducible non-normal cubic surface. Then, either X is a cone over an irreducible singular plane cubic, or it is projectively equivalent to one of the following cubic surfaces singular along a line:*

(i) $t_0^2t_2 + t_1^2t_3 = 0$;

(ii) $t_2t_0t_1 + t_3t_0^2 + t_1^3 = 0$.

The two surfaces are not projectively isomorphic.

The last assertion follows from considering the normalization $\bar{X}$ of the surface X. In both cases it is a nonsingular surface, however in (i), the preimage of the singular line is irreducible, but in the second case it is reducible.

9.2.2 Normal cubic surfaces

A normal cubic surface S has only isolated singularities. Let p be a singular point. Choose projective coordinates such that $p = [1, 0, 0, 0]$. Then the equation of the surface can be written in the form

$$t_0f_2(t_1, t_2, t_3) + f_3(t_1, t_2, t_3) = 0,$$

where f_2 and f_3 are homogeneous polynomials of degree given by the subscripts. If $f_2 = 0$, the surface is a cone over a nonsingular plane cubic curve. If $f_2 \neq 0$, then p is a singular point of multiplicity 2. Projecting from p we see that S is birationally isomorphic to $\mathbb{P}^2$. Let $\pi : S' \to S$ be a minimal resolution of singularities of S. The sheaf $R^1\pi_*\mathcal{O}_{S'}$ has support only at singular points of S. Since S is normal, $\pi_*\mathcal{O}_{S'} = \mathcal{O}_S$. Applying the Leray spectral sequence we obtain an exact sequence

$$0 \to H^1(S, \mathcal{O}_S) \to H^1(S', \mathcal{O}_{S'}) \to H^0(S, R^1\pi_*\mathcal{O}_{S'}) \to H^2(S, \mathcal{O}_S).$$

Since S' is a nonsingular rational surface, we have $H^1(S', \mathcal{O}_{S'}) = 0$. The canonical sheaf of S is $\mathcal{O}_S(-1)$, hence, by Serre's duality, $H^2(S, \mathcal{O}_S) = H^0(S, \omega_S) = 0$. Thus we obtain that $H^0(S, R^1\pi_*\mathcal{O}_{S'}) = 0$. This shows that, for any singular points $s \in S$, we have $(R^1\pi_*\mathcal{O}_{S'})_s = 0$. As is known (see [328]) this characterizes canonical singularities (or RDP) of a surface.

This gives

Theorem 9.2.2. *Let S be a normal cubic surface in $\mathbb{P}^3$. Then S is either a cone over a nonsingular plane cubic curve or an anticanonical model of a weak Del Pezzo surface of degree 3.*

9.2.3 Canonical singularities

From now on we assume that X is a cubic surface with canonical singularities, i.e. X is an anticanonical model of a weak Del Pezzo surface S of degree 3.

All possible Dynkin curves on S can be easily found from the list of root bases in $\mathbf{E}_7$. These are all root bases in $\mathbf{E}_7$ of rank ≤ 6 except of types $D_6, D_5 + A_1$ and $2A_1 + D_4$. These do not occur since the discriminants of the corresponding sublattices of $\mathbf{E}_6$ do not satisfy the assertion of Lemma 8.2.1. Let us list the remaining types of root bases:

$$
\begin{array}{ll}
(r=6) & E_6,\ A_6,\ D_4+A_2, \sum_{k=1}^{s} A_{i_k}, i_1+\cdots+i_s=6 \\
(r=5) & D_5,\ D_4+A_1,\ \sum_{k=1}^{s} A_{i_k}, i_1+\cdots+i_s=5 \\
(r=4) & D_4,\ \sum_{k=1}^{s} A_{i_k}, i_1+\cdots+i_s=4 \\
(r=3) & A_3,\ A_2+A_1,\ 3A_1 \\
(r=2) & A_2,\ A_1+A_1 \\
(r=1) & A_1.
\end{array}
$$

Lemma 9.2.3. *Let $p_0 = (1, 0, 0, 0)$ be a singular point of $V(f_3)$. Write*

$$f_3 = t_0 g_2(t_1, t_2, t_3) + g_3(t_1, t_2, t_3),$$

where g_2, g_3 are homogeneous polynomials of degree 2,3. Let $p = (t_0, t_1, t_2, t_3) \in V(f_3)$. If the line $\overline{p_0, p}$ is contained in $V(f_3)$, then the point $q = (t_1, t_2, t_3)$ is a common point of the conic $V(g_2)$ and the cubic $V(g_3)$. If, moreover, p is a singular point of $V(f_3)$, then the conic and the cubic intersect at q with multiplicity > 1.

Proof. This is easy to verify and is left to the reader. □

Corollary 9.2.4. *$V(f_3)$ has at most 4 singular points. Moreover, if $V(f_3)$ has 4 singular points, then each point is of type A_1.*

Proof. Let p_0 be a singular point. Choose coordinates such that $p_0 = (1,0,0,0)$ and apply Lemma 9.2.3. Suppose we have more than 4 singular points. The conic and the cubic will intersect at at least four singular points with multiplicity > 1. Since they do not share an irreducible component (otherwise f_3 is reducible), this contradicts Bézout's Theorem. Suppose we have 4 singular points and p_0 is not of type A_1. Since p_0 is not an ordinary double point, the conic $V(g_2)$ is reducible. Then the cubic $V(g_3)$ intersects it at 3 points with multiplicity > 1 at each point. It is easy to see that this also contradicts Bézout's Theorem. □

Lemma 9.2.5. *The cases, $A_{i_1}+\cdots+A_{i_k}, i_1+\cdots+i_k = 6$, except the cases $3A_2, A_5+A_1$ do not occur.*

Proof. Assume $M = A_{i_1}+\cdots+A_{i_k}, i_1+\cdots+i_k = 6$. Then $d_M = (i_1+1)\cdots(i_k+1)$. Since $3|d_M$, one of the numbers, say i_1+1, is equal either to 3 or 6. If $i_1+1 = 6$, then $M = A_5+A_1$. If $i_1+1 = 3$, then $(i_2+1)\ldots(i_k+1)$ must be a square, and $i_2+\cdots+i_k = 4$. It is easy to see that the only possibility are $i_2 = i_3 = 2$ and $i_2 = i_3 = i_4 = i_5 = 1$. The last possibility is excluded by applying Corollary 9.2.4. □

Lemma 9.2.6. *The cases D_4+A_1 and D_4+A_2 do not occur.*

Proof. Let p_0 be a singular point of S of type D_4. Again, we assume that $p_0 = (1,0,0,0)$ and apply Lemma 9.2.3. As we have already noted, the singularity of type D_4 is analytically (or formally) isomorphic to the singularity $z^2+xy(x+y) = 0$. This shows that the conic $V(g_2)$ is a double line ℓ. The plane $z = 0$ cuts out a germ of a curve with 3 different branches. Thus there exists a plane section of $S = V(f_3)$ passing through p_0 which is a plane cubic with 3 different braches at P. Obviously, it must be a union of 3 lines with a common point at p_0. Now the cubic $V(g_3)$ intersects the line ℓ at 3 points corresponding to the lines through p_0. Thus S cannot have more singular points. □

Let us show that all remaining cases are realized. We will exhibit the corresponding Del Pezzo surface as the blow-up of 6 bubble points $p_1,\ldots,p_6$ in $\mathbb{P}^2$.

A_1: 6 points in $\mathbb{P}^2$ on an irreducible conic;

A_2: $p_3 \succ_1 p_1$;

$4A_1$: $p_2 \succ_1 p_1, p_4 \succ_1 p_2$;

A_3: $p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1$;

A_2+A_1: $p_3 \succ_1 p_2 \succ_1 p_1, p_5 \succ_1 p_4$;

A_4: $p_5 \succ_1 p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1$;

$3A_1$: $p_2 \succ_1 p_1, p_4 \succ_1 p_3, p_6 \succ_1 p_5$;

$2A_2$: $p_3 \succ_1 p_2 \succ_1 p_1, p_6 \succ_1 p_5 \succ_1 p_4$;

A_3+A_1: $p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1, p_6 \succ_1 p_5$;

A_5: $p_6 \succ_1 p_5 \succ_1 p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1$;

D_4: $p_2 \succ_1 p_1, p_4 \succ_1 p_3, p_6 \succ_1 p_5$ and p_1, p_3, p_5 are collinear;

A_2+2A_1: $p_3 \succ_1 p_2 \succ_1 p_1, p_5 \succ_1 p_4$, and $|\ell - p_1 - p_2 - p_3| \neq \emptyset$;

A_4+A_1: $p_5 \succ_1 p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1$ and $|2\ell - p_1 - \ldots - p_6| \neq \emptyset$;

D_5: $p_5 \succ_1 p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1$ and $|\ell - p_1 - p_2 - p_6| \neq \emptyset$;

$4A_1$: $p_1, \ldots, p_6$ are the intersection points of 4 lines in a general linear position;

$2A_2+A_1$: $p_3 \succ_1 p_2 \succ_1 p_1, p_6 \succ_1 p_5 \succ_1 p_4$ and $|\ell - p_1 - p_2 - p_3| \neq \emptyset$;

A_3+2A_1: $p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1, p_6 \succ_1 p_5$ and $|\ell - p_1 - p_2 - p_3| \neq \emptyset$;

A_5+A_1: $p_6 \succ_1 p_5 \succ_1 p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1$ and $|2\ell - p_1 - \ldots - p_6| \neq \emptyset$;

E_6: $p_6 \succ_1 p_5 \succ_1 p_4 \succ_1 p_3 \succ_1 p_2 \succ_1 p_1$ and $|\ell - p_1 - p_2 - p_3| \neq \emptyset$;

$3A_2$: $p_3 \succ_1 p_2 \succ_1 p_1, p_6 \succ_1 p_5 \succ_1 p_4$, $|\ell - p_1 - p_2 - p_3| \neq \emptyset$, $|\ell - p_4 - p_5 - p_6| \neq \emptyset$;

Projecting from a singular point and applying Lemma 9.2.3 we see that each singular cubic surface can be given by the following equation.

A_1: $V(t_0g_2(t_1,t_2,t_3)+g_3(t_1,t_2,t_3))$, where $V(g_2)$ is a nonsingular conic which intersects $V(g_3)$ transversally;

A_2: $V(t_0t_1t_2+g_3(t_1,t_2,t_3))$,where $V(t_1t_2)$ intersects $V(g_3)$ transversally;

$2A_1$: $V(t_0g_2(t_1,t_2,t_3)+g_3(t_1,t_2,t_3))$, where $V(g_2)$ is a nonsingular conic which is simply tangent to $V(g_3)$ at one point;

A_3: $V(t_0t_1t_2+g_3(t_1,t_2,t_3))$, where $V(t_1t_2)$ intersects $V(g_3)$ at the point $[0,0,1]$ and at other 4 distinct points;

A_2+A_1: $V(t_0t_1t_2+g_3(t_1,t_2,t_3))$, where $V(g_3)$ is tangent to $V(t_2)$ at $[1,0,0]$;

A_4: $V(t_0t_1t_2+g_3(t_1,t_2,t_3))$, where $V(g_3)$ is tangent to $V(t_1)$ at $[0,0,1]$;

$3A_1$: $V(t_0g_2(t_1,t_2,t_3)+g_3(t_1,t_2,t_3))$, where $V(g_2)$ is nonsingular and is tangent to $V(g_3)$ at 2 points;

$2A_2$: $V(t_0t_1t_2+g_3(t_1,t_2,t_3))$, where $V(t_1)$ intersects $V(g_3)$ transverally and $V(t_2)$ is a flex tangent to $V(g_3)$ at $[1,0,0]$;

A_3+A_1: $V(t_0t_1t_2+g_3(t_1,t_2,t_3))$, where $V(g_3)$ passes through $[0,0,1]$ and $V(t_1)$ is tangent to $V(g_3)$ at a point $[1,0,0]$;

A_5: $V(t_0t_1t_2+g_3(t_1,t_2,t_3))$, where $V(t_1)$ is a flex tangent of $V(g_3)$ at the point $[0,0,1]$;

D_4: $V(t_0t_1^2+g_3(t_1,t_2,t_3))$, where $V(t_1)$ intersects transversally $V(g_3)$;

A_2+2A_1: $V(t_0t_1t_2+g_3(t_1,t_2,t_3))$, where $V(g_3)$ is tangent $V(t_1t_2)$ at two points not equal to $[0,0,1]$;

A_4+A_1: $V(t_0t_1t_2+g_3(t_1,t_2,t_3))$, where $V(g_3)$ is tangent to $V(t_1)$ at $[0,0,1]$ and is tangent to $V(t_2)$ at $[1,0,0]$;

D_5: $V(t_0t_1^2+g_3(t_1,t_2,t_3))$, where $V(t_1)$ is tangent to $V(g_3)$ at $[0,0,1]$;

$4A_1$: $V(t_0g_2(t_1,t_2,t_3)+g_3(t_1,t_2,t_3))$, where $V(g_2)$ is nonsingular and is tangent to $V(g_3)$ at 3 points;

$2A_2+A_1$: $V(t_0g_2(t_1,t_2,t_3)+g_3(t_1,t_2,t_3))$, where $V(g_2)$ is tangent to $V(g_3)$ at 2 points $[1,0,0]$ with multiplicity 3;

A_3+2A_1: $V(t_0t_1t_2+g_3(t_1,t_2,t_3))$, where $V(g_3)$ passes through $[0,0,1]$ and is tangent to $V(g_1)$ and to $V(g_2)$ at one point not equal to $[0,0,1]$;

A_5+A_1: $V(t_0t_1t_2+g_3(t_1,t_2,t_3))$, where $V(t_1)$ is a flex tangent of $V(g_3)$ at the point $[0,0,1]$ and $V(t_2)$ is tangent to $V(g_3)$;

E_6: $V(t_0t_1^2+g_3(t_1,t_2,t_3))$, where $V(t_1)$ is a flex tangent of $V(g_3)$.

$3A_2$: $V(t_0t_1t_2+g_3(t_1,t_2,t_3))$, where $V(t_1),V(t_2)$ are flex tangents of $V(g_3)$ at points different from $[0,0,1]$.

Remark 9.2.1. Applying a linear change of variables, one can simplify the equations. For example, in the case XXI, we may assume that the inflection points are $[1,0,0]$ and $[0,1,0]$. Then $g_3 = t_3^3+t_1t_2L(t_1,t_2,t_3)$. Replacing t_0 with $t_0' = t_0+L(t_1,t_2,t_3)$, we reduce the equation to the form

$$t_0t_1t_2+t_3^3=0. \tag{9.16}$$

Another example is the E_6-singularity (case XX). We may assume that the flex point is $[0,0,1]$. Then $g_3 = t_2^3+t_1g_2(t_1,t_2,t_3)$. The coefficient at t_3^2 is not equal to zero, otherwise the equation is reducible. After a linear change of variables we may assume that $g_2 = t_3^2+at_1^2+bt_1t_2+ct_2^2$. Replacing t_0 with $t_0+at_1+bt_2$, we may assume that $a=b=0$. After scaling the unknowns, we get

$$t_0t_1^2+t_1t_2^2+t_2^3=0. \tag{9.17}$$

The following table gives the classification of possible canonical singularities of a cubic surface, the number of lines and the class of the surface (i.e., the degree of the dual surface).

Type	Singularity	Lines	Class	Type	Singularity	Lines	Class
I	$\emptyset$	27	12	XII	D_4	6	6
II	A_1	21	10	XIII	A_2+2A_1	8	5
III	A_2	15	9	XIV	A_4+A_1	4	5
IV	$2A_1$	16	8	XV	D_5	3	5
V	A_3	10	8	XVI	$4A_1$	9	4
VI	A_2+A_1	11	7	XVII	$2A_2+A_1$	5	4
VII	A_4	6	7	XVIII	A_3+2A_1	5	4
VIII	$3A_1$	12	6	XIX	A_5+A_1	2	4
IX	$2A_2$	7	6	XX	E_6	1	4
X	A_3+A_1	7	6	XXI	$3A_2$	3	3
XI	A_5	3	6				

Table 9.1: Singularities of cubic surfaces

Note that the number of lines can be checked directly by using the equations. The map from $\mathbb{P}^2$ to S is given by the linear system of cubics generated by $V(g_3),V(t_1g_2)$, $V(t_2g_2),V(t_3g_2)$. The lines are images of lines or conics which has intersection 1 with a general member of the linear system. We omit the computation of the class of the surface.

9.3 Determinantal equations

9.3.1 Cayley-Salmon equation

Let l_1, l_2, l_3 be three skew lines in $\mathbb{P}^3$. Let $\mathcal{P}_i$ be the pencil of planes through the line l_i. Let us identify $\mathcal{P}_i$ with $\mathbb{P}^1$ and consider the rational map

$$f : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 - \to \mathbb{P}^3$$

which assigns to the triple of planes (Π_1, Π_2, Π_3) the intersection point $\Pi_1 \cap \Pi_2 \cap \Pi_3$. This map is undefined at a triple (Π_1, Π_2, Π_3) such that the line $l_{ij} = \Pi_i \cap \Pi_j$ is contained in Π_k, where $\{i, j, k\} = \{1, 2, 3\}$. The line l_{ij} obviously intersects all three lines. The union of such lines is the nonsingular quadric Q containing l_1, l_2, l_3 (count parameters to convince yourself that any 3 skew lines are contained in a unique nonsingular quadric). A plane from $\mathcal{P}_i$ intersects Q along l_i and a line m_i on Q from another ruling. The triple belongs to the indeterminacy locus I of f if and only if $m_1 = m_2 = m_3$. Consider the map

$$\phi : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to (\mathbb{P}^1)^3, \quad (\Pi_1, \Pi_2, \Pi_3) \mapsto (m_1, m_2, m_3).$$

We see that $I = \phi^{-1}(\Delta)$, where Δ is the small diagonal. Obviously ϕ is an isomorphism, so I is a smooth rational curve. Let Δ_{ij} be one of the three diagonals (the locus of points with equal i-th and j-th coordinates). Its preimage $D_i = \phi^{-1}(\Delta_{ij})$ is blown down under f to the line l_k. In fact, if $(\Pi_1, \Pi_2, \Pi_3) \in D_{12}$, then $m_1 = m_2$ and $\Pi_1 \cap \Pi_2 \cap \Pi_3 = m_1 \cap l_3$. Clearly, D_{12}, D_{13}, D_{23} are divisors on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of degree $(1, 1, 0), (1, 0, 1), (0, 1, 1)$, respectively. The map f can now be resolved by blowing up the curve I, followed by blowing down the proper inverse transforms of the divisors D_{ij} to the lines l_k. One should compare it with the standard birational map from the quadric $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^2$ defined by the projection from a point.

Note that in coordinates, $f(\Pi_1, \Pi_2, \Pi_3)$ is the line of solutions of a system of 3 linear equations, thus depends linearly in coefficients of each equation. This shows that the rational map is given by a linear system of divisors of degree $(1, 1, 1)$. Let S be a cubic surface containing the lines l_1, l_2, l_3. Its full preimage under f is a divisor of degree $(3, 3, 3)$. It contains the divisors D_{12}, D_{13}, D_{23} whose sum is the divisor of degree $(2, 2, 2)$. Let R be the residual divisor of degree $(1, 1, 1)$. It is equal to the proper inverse transform of S. Let

$$R = V\big(\sum_{i,j,k=0,1} a_{i,j,k} \lambda_i \mu_j \gamma_k\big),$$

where $(\lambda_0, \lambda_1), (\mu_0, \mu_1), (\gamma_0, \gamma_1)$ are coordinates in the pencils $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$.

Thus we obtain

Theorem 9.3.1. *(F. August). Any cubic surface containing 3 skew lines l_1, l_2, l_3 can be generated by 3 pencils of planes in the following sense. There exists a correspondence R of degree $(1, 1, 1)$ on $\mathcal{P}_1 \times \mathcal{P}_2 \times \mathcal{P}_3$ such that*

$$S = \{x \in \mathbb{P}^3 : x \in \Pi_1 \cap \Pi_2 \cap \Pi_3 \textit{ for some } (\Pi_1, \Pi_2, \Pi_3) \in R\}.$$

Let us rewrite R in the form

$$R = V(\lambda_0 A_0(\mu_0, \mu_1, \gamma_0, \gamma_1) + \lambda_1 A_1(\mu_0, \mu_1, \gamma_0, \gamma_1)),$$

where A_0, A_1 are bihomogeneous forms in (λ_0, λ_1) and (γ_0, γ_1). Suppose that S contains two distinct lines l, m which intersect l_2, l_3 but do not intersect l_1. Let $\Pi_2 = \overline{l, l_2}, \Pi_3 = \overline{l, l_3}$. Since any plane Π in $\mathcal{P}_1$ intersects ℓ, the point (Π, Π_2, Π_3) is mapped to S but not contained in any divisor D_{12}, D_{13}, D_{23}. Thus it belongs to R. Since Π is arbitrary, the point (Π_2, Π_3) is the intersection point of the curves $V(A_0), V(A_1)$ in $\mathcal{P}_2 \times \mathcal{P}_3$. This shows that the curves $V(A_0), V(A_1)$ of bidegree $(1,1)$ intersect at two distinct points. Change coordinates μ, γ to assume that these points are $([0,1],[1,0])$ and $([1,0],[0,1])$. Plugging in the equations of $A_0 = 0, A_1 = 0$, we see that the curves $V(A_0), V(A_1)$ belong to the pencil spanned by the curves $V(\mu_0\gamma_0), V(\mu_1\gamma_1)$. Changing the coordinates (λ_0, λ_1) we may assume that

$$R = V(\lambda_0\mu_0\gamma_0 + \lambda_1\mu_1\gamma_1). \tag{9.18}$$

The surface S is the set of solutions (t_0, t_1, t_2, t_3) of the system of equations

$$\begin{aligned} \lambda_0 l_1(t_0, t_1, t_2, t_3) &= \lambda_1 m_1(t_0, t_1, t_2, t_3), \\ \mu_0 l_2(t_0, t_1, t_2, t_3) &= \mu_1 m_2(t_0, t_1, t_2, t_3), \\ \gamma_0 l_3(t_0, t_1, t_2, t_3) &= \gamma_1 m_3 i(t_0, t_1, t_2, t_3). \end{aligned}$$

where l_i, m_i are linear forms and

$$\lambda_0\mu_0\gamma_0 + \lambda_1\mu_1\gamma_1 = 0.$$

Multiplying the left-hand sides and the right-hand sides, we get

$$S = V(l_1 l_2 l_3 + m_1 m_2 m_3). \tag{9.19}$$

Corollary 9.3.2. *Assume additionally that S contains 2 distinct lines which intersect two of the lines l_1, l_2, l_3 but not the third one. Then S can be given by the equation*

$$l_1 l_2 l_3 - m_1 m_2 m_3 = 0. \tag{9.20}$$

An equation of cubic surface of this type is called a *Cayley-Salmon equation*.

Observe that S contains the lines $\ell_{ij} = V(l_{ij})$. Obviously, $\ell_{ii} = l_i$ and ℓ_{23}, ℓ_{32} are the two lines which intersect ℓ_2, ℓ_3 but not ℓ_1. The lines ℓ_{12}, ℓ_{21} intersect ℓ_1, ℓ_2 but not l_3 (since otherwise $V(A_0), V(A_1)$ in above have more than two intersection points). Similarly, we see that ℓ_{13}, ℓ_{31} intersect ℓ_1, ℓ_3 but not ℓ_2. Thus we have 9 different lines. As is easy to see they form a pair of two conjugate triads of tritangent planes (which can be defined as in the nonsingular case)

$$\begin{matrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23}. \\ l_{31} & l_{32} & l_{33} \end{matrix} \tag{9.21}$$

Thus the condition on S imposed in Corollary 2.2.9 implies that S contains a pair of conjugate triples of tritangent planes. Conversely, such a set of 9 lines gives a Cayley-Salmon equation of S. In fact, each triple of the tritangent planes defines S along the same set of 9 lines. Thus S is contained in the pencil spanned by surfaces $V(l_1l_2l_3), V(m_1m_2m_3)$. It is clear that two Cayley-Salmon equations defining the same set of 9 lines can be transformed to one another by a linear change of variables. Thus the number of essentially different Cayley-Salmon equations is equal to the number of pairs of conjugate triads of tritangent planes.

Theorem 9.3.3. *Let S be a normal cubic surface. The number of different Cayley-Salmon equations for S is equal to 120 (type I), 10 (type II), 1 (type III,IV, VIII), and zero otherwise.*

Proof. We know that the number of conjugate pairs of triads of tritangent trios of exceptional vectors is equal to 120. Thus the number of conjugate triads of triples of tritangent planes on a nonsingular cubic surface is equal to 120. It follows from the proof of Corollary 2.2.9 that a pair of conjugate triples of tritangent planes on a singular surface exists only if we can find 3 skew lines and 2 lines which intersect two of them but not the third. Also we know that the number of lines on S must be at least 9. So we have to check only types $II - VI$ and $VIII$. We leave to the reader to verify the assertion in these cases. □

Corollary 9.3.4. *Let S be a nonsingular cubic surface. Then S is projectively equivalent to a surface*

$$V(t_0t_1t_2 + t_3(t_0 + t_1 + t_2 + t_3)l(t_0, \ldots, t_3)).$$

A general S can be written in this form in exactly 120 ways (up to projective equivalence).

Proof. Consider a Cayley-Salmon equation $l_1l_2l_3 + m_1m_2m_3 = 0$ of S. Let (9.21) be the corresponding 9 lines on S. If l_1, l_2, l_3, m_j are linearly independent, we choose a coordinate system such that $l_1 = t_0, l_2 = t_1, l_3 = t_2, m_j = t_3$. If not, the lines $\ell_{1j}, \ell_{2j}, l_{3j}$ intersect at one point $p_j = l_1 \cap l_2 \cap l_3$. Assume that this is not the case for all j so that S is projectively equivalent to $V(t_0t_1t_2 + t_3m_2m_3)$. let $m_2 = \sum a_it_i$. If one of the a_i's is equal to zero, say $a_3 = 0$, the linear form m_2 is a linear combination of coordinates t_0, t_1, t_2. We have assumed that this does not happen. Thus, after scaling, the coordinates we may assume that $m_2 = \sum t_i$. This gives the promised equation. Since we can start with any conjugate pair of triads of tritangent planes, the previous assumption is not satisfied only if any such pair consists of tritangent planes containing three concurrent lines. We will see later that the number of such tritangent planes on a nonsingular surface is at most 18. So we can always start with a conjugate triad of tritangent planes for which each plain does not contain concurrent lines. □

Corollary 9.3.5. *let S be a normal surface of type $I - IV$ or $VIII$. Then there exists a 3×3 matrix $A(t)$ with linear forms in $t_0, \ldots, t_3$ such that*

$$S = V\big(\det(A(t))\big).$$

Proof. Observe that

$$l_1l_2l_3 + m_1m_2m_3 = \det\begin{pmatrix} l_1 & m_1 & 0 \\ 0 & l_2 & m_2 \\ m_3 & 0 & l_3 \end{pmatrix}.$$

□

9.3.2 Hilbert-Burch Theorem

By other methods we will see that Corollary 9.3.5 can be generalized to any normal cubic surface of type different from XX. We will begin with the approach using the following well-known result from Commutative Algebra (see [156]).

Theorem 9.3.6. *(Hilbert-Burch). Let I be an ideal in polynomial ring R such that* $\mathrm{depth}(I) = \mathrm{codim} I = 2$ *(thus R/I is a Cohen-Macaulay ring). Then there exists a projective resolution*

$$0 \longrightarrow R^{n-1} \xrightarrow{\phi_2} R^n \xrightarrow{\phi_1} R \longrightarrow R/I \longrightarrow 0.$$

The i-th entry of the vector $(a_1, \ldots, a_n)$ defining ϕ_1 is equal to $(-1)^i c_i$, where c_i is the complementary minor obtained from the matrix A defining ϕ_2 by deleting its i-th row.

We apply this theorem to the case when $R = \mathbb{C}[X_0, X_1, X_2]$ and I is the homogeneous ideal of a closed 0-dimensional subscheme Z of $\mathbb{P}^2 = \mathrm{Proj}(R)$ generated by four linearly independent homogeneous polynomials of degree 3. Let $\mathcal{I}_Z$ be the ideal sheaf of Z. Then $(I_Z)_m = H^0(\mathbb{P}^2, \mathcal{I}_Z(m))$. By assumption

$$H^0(\mathbb{P}^2, \mathcal{I}_Z(2)) = 0. \tag{9.22}$$

Applying the Hilbert-Burch Theorem, we find a resolution of the graded ring R/I

$$0 \longrightarrow R(-4)^3 \xrightarrow{\phi_2} R(-3)^4 \xrightarrow{\phi_1} R \longrightarrow R/I \to 0,$$

where ϕ_2 is given by a 3×4 matrix $A(X)$ whose entries are linear forms in X_0, X_1, X_2. Passing to the projective spectrum, we get an exact sequence of sheaves

$$0 \longrightarrow W_2 \otimes \mathcal{O}_{\mathbb{P}^2}(-4) \xrightarrow{\phi_2} W_1 \otimes \mathcal{O}_{\mathbb{P}^2}(-3) \xrightarrow{\phi_1} \mathcal{I}_Z \longrightarrow 0,$$

where W_2, W_1 are vector spaces of dimension 3 and 4. Twisting by $\mathcal{O}_{\mathbb{P}^2}(3)$, we get the exact sequence

$$0 \longrightarrow W_2 \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{\tilde{\phi}_2} W_1 \otimes \mathcal{O}_{\mathbb{P}^2} \xrightarrow{\tilde{\phi}_1} \mathcal{I}_Z(3) \longrightarrow 0. \tag{9.23}$$

Taking global sections, we obtain $W_1 = H^0(\mathbb{P}^2, \mathcal{I}_Z(3))$. Twisting by $\mathcal{O}_{\mathbb{P}^2}(-2)$, and using a canonical isomorphism $H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3)) \cong \mathbb{C}$, we obtain that $W_2 = H^1(\mathbb{P}^2, \mathcal{I}_Z(1))$. The exact sequence

$$0 \to \mathcal{I}_Z(1) \to \mathcal{O}_{\mathbb{P}^2}(1) \to \mathcal{O}_Z \to 0$$

shows that

$$W_2 \cong \mathrm{Coker}(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \to H^0(\mathcal{O}_Z)) \cong \mathrm{Coker}\big(\mathbb{C}^3 \to \mathbb{C}^{h^0(\mathcal{O}_Z)}\big).$$

Since $\dim W_2 = 3$, we obtain that $h^0(\mathcal{O}_Z) = 6$. Thus Z is a 0-cycle of length 6.

Now we see that the homomorphism $\tilde{\phi}_2$ of vector bundles is defined by a linear map

$$t : E \to \mathrm{Hom}(W_2, W_1), \tag{9.24}$$

where $\mathbb{P}^2 = |E|$. We can identify the linear map t with the tensor $E^\vee \otimes W_2^* \otimes W_1$. Let us now view this tensor as a linear map

$$u : W_1^* \to \mathrm{Hom}(E, W_2^*). \tag{9.25}$$

In plain language, if t is viewed as a system of 3 linear equations with unknowns t_0, t_1, t_2, t_3 whose coefficients are linear forms in variables x_0, x_1, x_2, then u is the same system rewritten as a system of 3 equations with unknowns x_0, x_1, x_2 whose coefficients are linear forms in variables t_0, t_1, t_2, t_3.

The linear map (9.24) defines a rational map

$$f : |E| \to |W_1^*| = |\mathcal{I}_Z(3)|^*, \quad [v] \mapsto |t(v)(W_2)^\perp|.$$

This is the map given by the linear system $|\mathcal{I}_Z(3)|$. In coordinates, it is given by maximal minors of the matrix $A(X)$ defining ϕ_2. For any $\alpha \in t(v)(W_2)^\perp$, we have $u(\alpha)(v) = 0$. This shows that $\mathrm{rank}(u(\alpha)) < 3$. Thus S is contained in the locus of $[\alpha]$ such that α belongs to the preimage of the determinantal locus in $\mathrm{Hom}(E, W_2^*)$, i.e. the locus of linear maps of rank < 3. It is a cubic hypersurface in the space $\mathrm{Hom}(E, W_2^*)$. Thus the image S' of f is contained in a determinantal cubic surface S. Since the intersection scheme of two general members C_1, C_2 of the linear system $|\mathcal{I}_Z(3)|$ is equal to the 0-cycle Z of degree 6, the image of f is a cubic surface. This gives a determinantal representation of S.

Theorem 9.3.7. *Assume S is a normal cubic surface which does not have a singular point of type E_6. Then S admits a determinantal representation $S = V(\det(A))$, where A is a matrix whose entries are linear forms. A surface with a singular point of type E_6 does not admit such a representation.*

Proof. Assume S has no singular point of type E_6 and let X be a minimal resolution of S. Then the set of (-2)-curves is a proper subset of the set of roots of the lattice $K_X^\perp$ and span a proper sublattice M of $\mathrm{Pic}(X)$. Let α be a root in $K_X^\perp$ which does not belong to M. Since the Weyl group $W(X)$ acts transitively on the set of roots, we can choose a marking $\phi : I^{1,6} \to \mathrm{Pic}(X)$ such that $\alpha = 2e_0 - e_1 - \ldots - e_6$. Let $w \in W(X)^n$ be a an element of the Weyl group generated by reflections with respect to (-2)-curves such that $w \circ \phi$ is a geometric marking defining a geometric basis $e_0', e_1', \ldots, e_6'$. Since w preserves M, $w(\alpha) = 2e_0' - e_1' - \ldots - e_6'$ is not a linear combination of (-2)-curves. However, any effective root x is a linear combination of (-2)-curves (use that $x \cdot K_X) = 0$ and for any irreducible component E of x with $E^2 \neq -2$ we have $E \cdot K_X < 0$). Now X is obtained by blowing up a set of 6 points (maybe infinitely near) not lying on a conic. It is easy that this blow-up is isomorphic to the blow-up of a 0-dimensional cycle Z of length 6. Blowing up a sequence of k infinitely near points $p_k \succ_1 \ldots \succ_1 p_1$ is the same as to blow up the ideal (x, y^k). The linear system $|\mathcal{I}_Z(3)|$ is equal to the linear system of cubics through the points $p_1, \ldots, p_6$. The ideal I_Z is generated by a basis of the 3-dimensional linear system $|\mathcal{I}_Z(3)|$ defining a rational map $\mathbb{P}^2 - \to S \subset \mathbb{P}^3$. Thus we can apply the Hilbert-Burch Theorem to obtain a determinant representation of S.

Assume S has a singular point of type E_6 and $A(t) = (A_{ij})_{1 \leq i,j \leq 3}$. Consider the system of linear equations

$$\sum_{j=1}^{3} l_{ij}(t_0, \ldots, t_3)x_j = 0, \quad i = 1, 2, 3. \tag{9.26}$$

For any $x = [x_0, x_1, x_2] \in \mathbb{P}^2$ the set of points $p = [t_0, t_1, t_2, t_3]$ such that $A(t)x = 0$ is a linear space. Consider the rational map $\pi : S \to \mathbb{P}^2$ which assigns to $t \in S$ the solution x of $A(t)x = 0$. Since π is not bijective, there exists a line ℓ on S which is blown down to a point $[a_1, a_2, a_3]$. This means that the equations (9.26) with $[x_0, x_1, x_2]$ substituted with $[a_1, a_2, a_3]$ define three planes intersecting along a line. Thus the three planes are linearly dependent, hence we can write

$$\alpha(a_j \sum_{j=1}^{3} l_{1j}) + \beta(a_j \sum_{j=1}^{3} l_{2j}) + \gamma(\sum_{j=1}^{3} l_{3j})$$

$$= \sum_{j=1}^{3} a_j(\alpha l_{1j} + \beta l_{2j} + \gamma l_{3j}) = 0,$$

for some α, β, γ not all zeros. Choose coordinates for x such that $[a_1, a_2, a_3] = [1, 0, 0]$. Then we obtain that the entries in the first column of $A(t)$ are linearly dependent. This allows us to assume that $l_{11} = 0$ in the matrix $A(t)$. The equations $l_{21} = l_{31} = 0$ define the line ℓ. The equations $l_{12} = l_{13} = 0$ define a line m. Obviously, $l \neq m$ since otherwise S has equation

$$-l_{12}l_{21}l_{33} + l_{12}l_{31}l_{23} + l_{13}l_{21}l_{32} - l_{13}l_{31}l_{23} = 0,$$

which shows that the line $l = m$ is the double line of S. So, we see that S has at least two lines, but a surface of type XX has only one line. $\square$

We have already seen that each time S is represented as the image of S under a rational map given by the linear system $|\mathcal{I}_Z(3)|$, where Z is a 0-cycle of length 6 satisfying condition (9.22), we can write S by a determinantal equation. A minimal resolution of indeterminacy points defines a blowing down morphism $\pi : X \to \mathbb{P}^2$ of the weak Del Pezzo surface X isomorphic to a minimal resolution of S. The inverse map is given by assigning to $t \in S$ the nullspace $N(A(t))$. Changing Z to a projectively equivalent set replaces the matrix $A(t)$ by a matrix $A(t)C$, where C is an invertible scalar matrix. This does not change the equation of S. Thus the number of essentially different determinantal representations is equal to the number of linear systems $|e_0|$ on X which define a blowing down morphism $X \to \mathbb{P}^2$ such that the corresponding geometric markings $(e_0, e_1, \ldots, e_6)$ of X satisfy $|2e_0 - e_1 - \ldots - e_6| = \emptyset$. This gives

Theorem 9.3.8. *The number of essentially different determinantal representations of S is equal to the number of unordered geometric markings $(e_0, e_1, \ldots, e_6)$ of $\mathrm{Pic}(X)$ such that $|2e_0 - e_1 - \ldots - e_6| = \emptyset$. It is equal to 72 if S is nonsingular.*

Consider again (9.24) as a tensor $t \in W_1 \otimes V^\vee \otimes W_2^*$ which defines a linear map $t : W_1^* \to \mathrm{Hom}(W_2, V^\vee)$. We have the corresponding rational map

$$g : S \to \mathbb{P}(W_2) \cong \mathbb{P}^2, \quad [w] \mapsto \mathrm{Ker}(t^*).$$

Let $\mathbb{P}^3 = \mathbb{P}(W_1^*)$. Consider the projective resolution

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \otimes W_2 \xrightarrow{\phi} \mathcal{O}_{\mathbb{P}^3} \otimes V^\vee \longrightarrow \mathcal{F} \to 0,$$

where ϕ is defined by the linear map viewed as a 3×3 matrix with entries in $W_1 = (W_1^*)^*$. Since the determinant of the matrix is equal to the equation of S, the sheaf $\mathcal{F}$ is locally isomorphic to $\mathcal{O}_S$. We can write it as $\mathcal{F} = \mathcal{O}_S(D)$ for some divisor class D. Taking global sections, we obtain $H^0(S, \mathcal{O}_S(D)) \cong V^\vee$. Thus the linear system $|D|$ on S defines our rational map $\pi : S \to |E|$. Twisting by $\mathcal{O}_{\mathbb{P}^3}(-3)$ and taking cohomology, we obtain an isomorphism $W_2 \cong H^2(S, \mathcal{O}_S(-3H + D))$, where H is a hyperplane section. Since $\mathcal{O}_S(H) \cong \mathcal{O}_S(-K_S)$, we obtain an isomorphism

$$W_2 \cong H^2(S, \mathcal{O}_S(3K_S + D)) \cong H^0(S, \mathcal{O}_S(-2K_S - D))^*.$$

Let $\pi' : X \to \mathbb{P}(W_2)$ be the rational map of a minimal resolution defined by g. The preimage of D on X is equal to the class e_0 and $-K_S = 3e_0 - e_1 - \ldots - e_6$, where $(e_0, e_1, \ldots, e_6)$ is a geometric marking of X defined by the blowing down morphism π'. Thus π' is given by the linear system $|5e_0 - 2e_1 - \ldots - 2e_6|$. If S is nonsingular, π blows down 6 skew lines $l_1, \ldots, l_6$ to 6 points $p_1, \ldots, p_6$ on $|E|$ and the map $g = \pi'$ blows down the proper inverse transforms $m_1, \ldots, m_6$ of conics C_j through the points $p_i, i \neq j$ to some points $q_1, \ldots, q_6$ in $\mathbb{P}(W_2)$. The lines $(l_1, \ldots, l_6; m_1, \ldots, m_6)$ form a double-sixer on S. It follows from above, that the lines $(m_1, \ldots, m_6)$ define a determinantal representation of S corresponding to the transpose of the matrix $A(t)$.

Remark 9.3.1. We can also deduce Theorem 9.3.8 from the theory of determinantal equation from Chapter 4. Applying this theory we obtain that S admits a determinantal equation with entries linear forms if it contains a projectively normal curve C such that

$$H^0(S, \mathcal{O}_S(C)(-1)) = H^2(S, \mathcal{O}_S(C)(-2)) = 0. \tag{9.27}$$

Moreover, the set of non-equivalent determinantal representations is equal to the set of divisor classes of such curves. Let $f : X \to S$ be a minimal resolution and $C' = f^*(C)$. Since $f^*\mathcal{O}_S(-1) = \mathcal{O}_X(K_X)$, the conditions (9.27) are equivalent to

$$H^0(X, \mathcal{O}_X(C' + K_X)) = H^2(X, \mathcal{O}_X(C' + 2K_X)) = 0. \tag{9.28}$$

Since C' is nef, $H^1(X, \mathcal{O}_X(C' + K_X)) = 0$. Also $H^2(X, \mathcal{O}_X(C' + K_X)) = H^0(X, \mathcal{O}_X(-C')) = 0$. By Riemann-Roch,

$$0 = \chi(X, \mathcal{O}_X(C' + K_X)) = \tfrac{1}{2}((C' + K_X)^2 - (C' + K_X) \cdot K_X) + 1$$
$$= \tfrac{1}{2}(C'^2 + C' \cdot K_X) + 1.$$

Thus C' is a smooth rational curve, hence C is a smooth rational curve. It is known that a projectively normal rational curve in $\mathbb{P}^n$ must be of degree n. Thus $-K_X \cdot C' = 3$, hence $C'^2 = 1$. The linear system $|C'|$ defines a birational map $\pi : X - \to \mathbb{P}^2$. Let $e_0 = [C'], e_1, \ldots, e_6$ be the corresponding geometric basis of Pic(X). We have $K_X = -3e_0 + e_1 + \cdots + e_6$ and the condition

$$0 = H^2(X, \mathcal{O}_X(C' + 2K_X)) = H^0(X, \mathcal{O}_X(-C' - K_X)) = 0$$

is equivalent to

$$|2e_0 - e_1 - \ldots - e_6| = \emptyset. \tag{9.29}$$

9.3.3 The cubo-cubic Cremona transformation

Consider a system of linear equations in variables t_0, t_1, t_2, t_3:

$$\sum_{j=1}^{3} l_{ij}(z_0, z_1, z_2, z_3)t_j = 0, i = 1, 2, 3, 4, \tag{9.30}$$

where l_{ij} are linear forms in variables z_0, z_1, z_2, z_3. It defines a rational map $\Phi : \mathbb{P}^3 - \to \mathbb{P}^3$ by assigning to $[a_0, \ldots, a_3] \in \mathbb{P}^3$ the space of solutions of the system (9.30) with z_i substituted with a_i. In coordinate-free way, we can consider the system as a linear map $\tau : W \to \mathrm{Hom}(W_1, W_2)$, where $\dim W = 4, \dim W_1 = 4, \dim W_2 = 3$. The map $\Phi : \mathbb{P}(W) \to \mathbb{P}(W_1)$ is defined by sending $w \in W$ to the linear space $\mathrm{Ker}(\tau(v)) \subset W_1$. The inverse map Φ^{-1} is defined by rewriting the system as a system with unknowns z_i. Or, in corrdinate-free language, by viewing the tensor $\tau \in W^\vee \otimes W_1^* \otimes W_2$ as a linear map $W_1 \to \mathrm{Hom}(W, W_2)$. Let D be the set of linear maps (considered up to proportionality) $W_1 \to W_2$ of rank ≤ 2. The map Φ is not defined at the preimage of D in $|W|$. It is given by the common zeros of the four maximal minors Δ_i of the matrix $(l_{ij}(z))$. The map Φ is given by

$$[z_0, \ldots, z_3] \mapsto [\Delta_1, -\Delta_2, \Delta_3, -\Delta_4].$$

Lemma 9.3.9. *The scheme-theoretical locus Z of common zeros of the cubic polynomials Δ_i is a connected curve of degree 6 and arithmetic genus 3.*

Proof. We apply the Hilbert-Burch Theorem to the ring $R = \mathbb{C}[z_0, z_1, z_2, z_3]$ and the homogeneous ideal I of Z. We get a resolution

$$0 \longrightarrow R(-4)^3 \xrightarrow{\phi_2} R(-3)^4 \xrightarrow{\phi_1} I \longrightarrow 0. \tag{9.31}$$

Twisting by n and computing the Euler-Poincaré characteristic we obtain the Hilbert polynomial of the scheme Z

$$P(Z;n) = \chi(\mathbb{P}^3, \mathcal{O}_Z(n)) = 6n - 2.$$

This shows that Z is one-dimensional, and comparing with Riemann-Roch, we see that $\deg(Z) = 6$ and $\chi(\mathcal{O}_Z) = -2$. The exact sequence

$$0 \to \mathcal{I}_Z \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_Z \to 0$$

gives $\dim H^0(\mathcal{O}_Z) = 1$ if and only if $H^1(\mathbb{P}^3, \mathcal{I}_Z) = 0$. The latter equality follows from considering the resolution of $\mathcal{I}_Z$. Thus Z is connected and $p_a(Z) = 1 - \chi(\mathcal{O}_Z) = 3$. □

The inverse map is also given by cubic polynomials. This explains the classical name for the transformation T, the *cubo-cubic transformation*. The preimages of planes under Φ are cubic surfaces in $\mathbb{P}(W)$ containing the curve Z. The images of planes under Φ are cubic surfaces in $\mathbb{P}(W_1)$ containing the curve Z', defined similarly to Z for the inverse map Φ^{-1}.

Now let V be a 3-dimensional subspace of W. Then restricting the linear map $t : W \to \mathrm{Hom}(W_1, W_2)$ to V we obtain a determinantal representation of the cubic surface $S = \Phi(|E|) \subset \mathbb{P}(W_1)$. The map $\Phi : |E| - \to \mathbb{P}(W_1)$ is not defined at the set $|E| \cap Z$. Tensoring (9.31) with $\mathcal{O}_{|E|}$ we obtain a projective resolution for the ideal sheaf of $Z \cap |E|$ in $|E|$ (use that $\mathrm{Tor}_1(R/J, I) = 0$, where J is the ideal generated by the hyperplane in W defining V see [156], Exercise A3.16). If $|E|$ intersects Z transversally at 6 points, we see that S is a nonsingular cubic surface.

9.3.4 Cubic symmetroids

A *cubic symmetroid* is a hypersurface in $\mathbb{P}^n$ admitting a representation as a symmetric (3×3)-determinant whose entries are linear forms in $n+1$ variables. Here we will be interested in cubic symmetroid surfaces. An example of a cubic symmetroid is a 4-nodal cubic surface

$$t_0t_1t_2 + t_0t_1t_3 + t_0t_2t_3 + t_1t_2t_3 = \det \begin{pmatrix} t_0 & 0 & t_2 \\ 0 & t_1 & -t_3 \\ -t_3 & t_3 & t_2 + t_3 \end{pmatrix}.$$

It is called the *Cayley cubic surface*. By choosing the singular points to be the reference points $[1,0,0,0]$, $[0,1,0,0]$, $[0,0,1,0]$, $[0,0,0,1]$, it is easy to see that cubic surfaces with 4 singularities of type $A_1$are projectively isomorphic. Note that the condition for a determinantal representation of a cubic surface with canonical singularities to be a symmetric determinantal representation is the existence of an isomorphism

$$\mathcal{O}_S(C) \cong \mathcal{O}_S(C)(2).$$

This is obviously impossible for a nonsingular cubic surface.

Lemma 9.3.10. *Let $L \subset |\mathcal{O}_{\mathbb{P}^2}(2)|$ be a pencil of conics. Then it is projectively isomorphic to one of the following pencils:*

(i) $\lambda(t_0t_1 - t_0t_2) + \mu(t_1t_2 - t_0t_2) = 0$;

(ii) $\lambda(t_0t_1 + t_0t_2) + \mu t_1t_2 = 0$;

(iii) $\lambda t_2^2 + \mu(t_0t_1 + t_0t_2 + t_1t_2) = 0$;

(iv) $\lambda t_2(t_2 - t_0) + \mu t_1(t_0 + t_2) = 0$;

(v) $\lambda t_0^2 + \mu(t_0t_2 + t_1^2) = 0$;

(vi) $\lambda t_0^2 + \mu t_1^2 = 0$;

(vii) $\lambda t_0t_1 + \mu t_0t_2 = 0$;

(viii) $\lambda t_0t_1 + \mu t_0^2 = 0$.

Proof. Let $C_1 = V(f_1), C_2 = V(f_2)$ be two generators of a pencil. If C_1 and C_2 have a common irreducible component we easily reduce it, by a projective transformation to cases (vii) or (viii). Assume now that C_1 do not have a common component. Let $k = \#C_1 \cap C_2$.

Assume $k = 4$. Then, no three of the intersection points lie on a line since otherwise the line is contained in both conics. By a linear transformation we may assume that the intersection points are $[1,0,0], [0,1,0], [0,0,1], [1,1,1]$. The linear system of conics passing through these points is given in (i).

Assume $k = 3$. After a linear change of variables we may assume that C_1, C_2 are tangent at $[1,0,0]$ with tangency direction $t_1 + t_2 = 0$ and intersect transversally at $[0,1,0]$ and $[0,0,1]$. The linear system of conics passing through the three points is $\lambda t_0t_1 + \mu t_0t_2 + \gamma t_1t_2 = 0$. The tangency condition gives $\lambda = \mu$. This gives case (ii).

Assume $k = 2$. Let $[1,0,0]$ and $[0,1,0]$ be the base points. First we assume that C_1 and C_2 are tangent at both points. Obviously, one of the conics from the pencil is the double line $t_2^2 = 0$. We can also fix the tangency directions to be $t_1 + t_2 = 0$ at $[1,0,0]$ and $t_0 + t_2 = 0$ at $[0,1,0]$. The other conic could be $t_0t_1 + t_0t_2 + t_1t_2 = 0$. This gives case (iii).

Now we assume that C_1 and C_2 intersect transversally at $[1,0,0]$ and with multiplicity 3 at $[0,1,0]$ with tangency direction $t_0 + t_2 = 0$. A conic passing through $[1,0,0]$ and tangent to the line $t_0 + t_2 = 0$ at the point $[0,1,0]$ has equation $at_2^2 + b(t_0t_1 + t_1t_2) + dt_0t_2 = 0$. It is easy to check that the condition of triple tangency is $a + d = 0$. This gives case (iv).

Finally assume that $k = 1$. Obviously, the pencil is spanned by a conic and its tangent line taken with multiplicity 2. By a projective transformation it is reduced to form given in case (v) if the conic is irreducible and case (vi) if the conic is a double line. □

Theorem 9.3.11. *Let S be an irreducible cubic symmetroid. Assume that S has only canonical singularities. Then S is projectively isomorphic to one of the following determinantal surfaces:*

(i) $\mathcal{C}_3 = V(t_0t_1t_2 + t_0t_1t_3 + t_0t_2t_3 + t_1t_2t_3)$ *with four RDP of type* A_1*;*

(ii) $\mathcal{C}_3' = V(t_0t_1t_2 + t_1t_3^2 - t_2t_3^2)$ *with two RDP of type* A_1 *and one RDP of type* A_3*;*

(iii) $\mathcal{C}_3'' = V(t_0t_1t_2 - t_3^2(t_0 + t_2) - t_1t_2^2)$ *with one RDP of type* A_1 *and one RDP of type* A_5.

Proof. Let $A = (l_{ij})$ be a symmetric 3×3 matrix with linear entries $l_{ij}(t_0, t_1, t_2, t_3)$ defining the equation of S. It can be written in the form $A(t) = t_0A_0 + t_1A_1 + t_2A_2 + t_3A_3$, where $A_i, i = 1, 2, 3, 4$, are symmetric 3×3 matrices. Let W be a linear system of conics spanned by the conics

$$C_i = [t_0, t_1, t_2] \cdot A \cdot \begin{pmatrix} t_0 \\ t_1 \\ t_2 \end{pmatrix} = 0.$$

The matrices A_i are linearly independent since otherwise $S = V(\det A(t))$ is a cone with vertex $[c_0, c_1, c_2, c_3]$, where $\sum c_iA_i = 0$. Thus $\mathcal{W}$ is a web of conics. Let $\mathbb{P}^2 = |E|$ so that $\mathcal{W} = \mathbb{P}(W)$ for a 4-dimensional linear subspace W of $S^2E^\vee$. Consider the polarity $S^2E \cong (S^2E^\vee)^\vee$. Then the projectivization of the dual of W is a pencil L of apolar conics in dual projective space $\mathbb{P}(E^\vee)$. Since the apolarity is equivariant with respect to the representation of $\mathrm{SL}(3)$ in S^2E and in $S^2E^\vee$, we see that we may assume that L is given in one of the cases from the previous lemma. Here we have to replace the unknowns t_i with the differential operators ∂_i. We list the corresponding dual 4-dimensional spaces of quadratic forms.

(i) $t_0t_0^2 + t_1t_1^2 + t_2t_2^2 + 2t_3(t_0t_2 + t_1t_2 + t_1t_2) = 0;$

(ii) $t_0t_0^2 + t_1t_1^2 + t_2t_2^2 + 2t_3(t_0t_1 - t_0t_2) = 0;$

(iii) $t_0t_0^2 + t_1t_1^2 + 2t_2(t_0t_1 - t_1t_2) + 2t_3(t_0t_2 - t_1t_2) = 0;$

(iv) $t_0t_0^2 + t_1t_1^2 + 2t_2(t_2^2 + t_0t_2) + 2t_3(t_1t_2 - t_0t_1) = 0;$

(v) $t_0(2t_0t_2 - t_1^2) + t_1t_2^2 + 2t_2t_0t_1 + 2t_3t_1t_2 = 0;$

(vi) $t_0t_2^2 + 2t_1t_0t_1 + 2t_2t_1t_2 + 2t_3t_0t_2 = 0;$

(vii) $t_0t_0^2 + t_1t_1^2 + t_2t_2^2 + 2t_3t_0t_1 = 0;$

(viii) $t_0t_1^2 + t_1t_2^2 + 2t_2t_0t_2 + 2t_3t_1t_2 = 0.$

The corresponding determinantal varieties are the following.

(i)
$$\det \begin{pmatrix} t_0 & t_3 & t_3 \\ t_3 & t_1 & t_3 \\ t_3 & t_3 & t_2 \end{pmatrix} = t_0t_1t_2 + t_3^2(-t_0 - t_2 - t_1 + t_3) = 0.$$

It has 4 singular points $[1, 0, 0, 0]$, $(0, 1, 0, 0]$, $[0, 0, 1, 0]$, and $[1, 1, 1, 1]$. The surface is isomorphic to the 4-nodal cubic surface $\mathcal{C}_3$.

(ii)

$$\det\begin{pmatrix} t_0 & t_3 & -t_3 \\ t_3 & t_1 & 0 \\ -t_3 & 0 & t_2 \end{pmatrix} = t_0t_1t_2 + t_1t_3^2 - t_2t_3^2 = 0.$$

It has 2 ordinary nodes $[0,1,0,0]$, $[0,0,1,0]$ and a RDP $[1,0,0,0]$ of type A_3.

(iii)

$$\det\begin{pmatrix} t_0 & t_2 & t_3 \\ t_2 & t_1 & -2(t_2+t_3) \\ t_3 & -2(t_2+t_3) & 0 \end{pmatrix} = 4(t_2+t_3)^2t_0 + (t_2+t_3)t_2t_3 + t_1t_3^2 = 0.$$

The surface has a double line given by $t_3 = t_2 + t_3 = 0$. This case is excluded.

(iv)

$$\det\begin{pmatrix} t_0 & -t_3 & t_2 \\ -t_3 & t_1 & t_3 \\ t_2 & t_3 & 2t_2 \end{pmatrix} = 2t_0t_1t_2 - t_3^2(t_0 + 4t_2) - t_1t_2^2 = 0.$$

The point $[1,0,0,0]$ is of type A_1 and the point $[0,1,0,0]$ is of type A_5.

(v)

$$\det\begin{pmatrix} 0 & t_2 & t_0 \\ t_2 & -t_0 & t_3 \\ t_0 & t_3 & t_1 \end{pmatrix} = t_1t_2^2 + 2t_0t_2t_3 + t_0^3 = 0.$$

The surface has a double line $t_0 = t_2 = 0$. This case has to be excluded.

(vi)

$$\det\begin{pmatrix} 0 & t_1 & t_3 \\ t_1 & 0 & t_2 \\ t_3 & t_2 & t_0 \end{pmatrix} = -t_1(t_0t_1 + 2t_2t_3) = 0.$$

This surface is reducible, the union of a plane and a nonsingular quadric.

(vii)

$$\det\begin{pmatrix} t_0 & 0 & 0 \\ 0 & t_1 & t_3 \\ 0 & t_3 & t_2 \end{pmatrix} = t_0(t_1t_2 - t_3^2) = 0.$$

The surface is the union of a plane and a quadratic cone.

(viii)

$$\det\begin{pmatrix} 0 & 0 & t_2 \\ 0 & t_0 & t_3 \\ t_2 & t_3 & t_1 \end{pmatrix} = t_0t_2^2 = 0.$$

The surface is a cone.

□

Remark 9.3.2. If S is a cone over a plane cubic curve C. Then S admits a symmetric determinantal representation if and only if C admits such a representation. We refer to Chapter ? for determinantal representations of plane cubics.

If S is irreducible non-normal surface, then S admits a symmetric determinantal representation. This corresponds to cases (iii) and (v) from the proof of the previous theorem. Case (iii) (resp. (v)) gives a surface isomorphic to the surface from case (i) (resp. (ii)) of Theorem 9.3.11. We also see that a reducible cubic surface which is not a cone admits a symmetric determinantal representation only if it is the union of an irreducible quadric and a plane which intersects the quadric transversally.

The 4-nodal cubic surface exhibits an obvious symmetry defined by the permutation group $\mathfrak{S}_4$. It also admits a double cover ramified only over its singular points. In fact, all three determinantal normal cubic surfaces with singularities of types $4A_1$, $2A_1+A_3$ and A_1+A_5 admit such a cover. It is defined by the family $\{(\ell,Q)\in(\check{\mathbb{P}}^2)\times\mathcal{W}:\ell\subset Q\}$, where $\mathcal{W}$ is the web of conics whose Hessian surface is the cubic surface. Note that the three cubic surfaces can be obtained as the projections of quartic surfaces with singularities of types $4A_1$ and $2A_1+A_3$ which have the similar covering property.

This cover can be seen in many different ways. We give only one, the others can be found in Exercises.

We consider only the Cayley cubic surface S. First, note that S can be obtained as the blow-up of the vertices of a complete quadrangle. Let X' be a weak Del Pezzo surface of degree 2 obtained as a minimal resolution of the double cover of $\mathbb{P}^2$ branched along the union of the four sides of the complete quadrangle. The double cover extends to a double cover $f: X'\to X$ of a minimal resolution X of S branched over the exceptional curves of the singularities. The ramification locus of f consists of the union of 4 disjoint (-1)-curves. Blowing them down we get a weak Del Pezzo surface Y of degree 6. The cover descends to a double cover of S ramified over the nodes.

9.4 Representations as sums of cubes

9.4.1 Sylvester's pentahedron

Counting constants we see that it is possible that a general homogeneous cubic form in 4 variables can be written as a sum of 5 cubes of linear forms in finitely many ways. Since there are no cubic surfaces singular at 5 general points, the theory of apolarity tells us that the count of constants gives a correct answer. The following result of J. Sylvester gives more:

Theorem 9.4.1. *A general homogeneous cubic form in 4 variables can be written uniquely as a sum*

$$f=l_1^3+l_2^3+l_3^3+l_4^3+l_5^3,$$

where l_i are linear forms in 4 variables.

Proof. Suppose

$$f=\sum_{i=1}^{5}l_i^3=\sum_{i=1}^{5}m_i^3.$$

Let x_i, y_i be the points in $\check{\mathbb{P}}^3$ corresponding to the hyperplanes $V(l_i), V(m_i)$. Consider the linear system of quadrics in $\check{\mathbb{P}}^3$ which pass through the points $x_5, y_1, \ldots, y_5$. If x_5 is not equal to any y_j, this is a linear projective subspace of dimension 3. Applying the corresponding differential operators to f we find 4 linearly independent relations between the linear forms l_1, l_2, l_3, l_4. This shows that the points x_1, x_2, x_3, x_4 are coplanar. It does not happen for general f. Thus we may assume that $x_5 = y_5$, so that we can write $m_5 = \lambda_5 l_5$ for some λ_5. After subtraction, we get

$$\sum_{i=1}^{4} l_i^3 + (1-\lambda_5^3)l_5^3 = \sum_{i=1}^{4} m_i^3.$$

Now we consider quadrics through y_1, y_2, y_3, y_4. They span a 6-dimensional linear space. Its elements define linear relations between the forms $l_1, \ldots, l_5$. Since the dimension of the linear span of these forms is equal to 4 (the genericity assumption), we obtain that there exists a 4-dimensional linear system of quadrics in $\check{\mathbb{P}}^3$ vanishing at $x_1, \ldots, x_5, y_1, \ldots, y_4$. Assume that all of these points are distinct. Two different quadrics from the linear system intersect along a curve B of degree 4. If this curve is irreducible, a third quadric intersects it at ≤ 8 points. So, the curve B must be reducible. Recall that the linear system of quadrics through an irreducible curve of degree 3 is of dimension 2. Thus, one of the quadrics in our linear system must contain a curve of degree ≤ 2 and so the base locus contains a curve of degree ≤ 2. The dimension of a linear system of quadrics containing an irreducible conic is of dimension 4 and its base locus is equal to the conic. Since the points x_i's are not coplanar, we see that this case does not occur. Assume that the base locus contains a line. Then 3 linearly independent quadrics intersect along a line ℓ and a cubic curve R. A forth quadric will intersect R at ≤ 4 points outside ℓ. Thus we have at most 4 points among $x_1, \ldots, x_5, y_1, \ldots, y_4$ which do not lie on ℓ. This implies that 5 points lie on the line ℓ. Since no three points among the x_i's and y_j's can lie on a line, we obtain a contradiction.

Thus one of the x_j's coincides with some y_i. We may now assume that $m_4 = \lambda_4 l_4$ and get

$$\sum_{i=1}^{3} l_i^3 + (1-\lambda_4^3)l_4^3 + (1-\lambda_5^3)l_5^3 = \sum_{i=1}^{3} m_i^3.$$

Take a plane through y_1, y_2, y_3 and get a linear dependence

$$a_1 l_1^2 + a_2 l_2^3 + a_3 l_3^2 + a_4(1-\lambda_4^3)l_4^2 + a_5(1-\lambda_5^2)l_5^2.$$

Here $a_4, a_5 \neq 0$, since otherwise the points y_1, y_2, y_3 and $x_4 = y_4$ or $x_4 = y_5$ are coplanar. A linear dependence between squares of linear forms means that the corresponding points in the dual space do not impose independent conditions on quadrics. The subvariety of $(\mathbb{P}^3)^5$ of such 5-tuples is a proper closed subset. By generality assumption, we may assume that our set of 5 points $x_1, \ldots, x_5$ is not in this variety. Now we get $\lambda_4^3 = \lambda_5^3 = 1$ and

$$\sum_{i=1}^{3} l_i^3 = \sum_{i=1}^{3} m_i^3.$$

A plane through y_1, y_2, y_3 gives a linear relation between l_1^2, l_2^2, l_3^2 which as we saw before must be trivial. Thus the points $x_1, x_2, x_3, y_1, y_2, y_3$ lie in the same plane Π. A linear system $\mathcal{Q}$ of quadrics in $\mathbb{P}^3$ through 3 non-collinear points does not have unassigned base points (i.e. its base locus consists of the three points). Since linear relations between l_1, l_2, l_3 form a one-dimensional linear space, $\mathcal{Q}$ contains a hyperplane of quadrics vanishing at additional 3 points x_1, x_2, x_3. Thus the linear system of quadrics through a set of 4 points x_1, x_2, x_3, y_i or y_1, y_2, y_3, x_i has 2 unassigned base points lying in the same plane. By restricting the linear system to the plane, we see that this is impossible unless all 6 points are collinear. This is excluded by the generality assumption. This final contradiction shows that $m_i = \lambda_i l_i$ and $\sum_{i=1}^3 (1-\lambda_i^3) l_i^3 = 0$. Since a general form cannot be written as a sum of 4 cubes, we get $\lambda_i^3 = 1$ and $l_i^3 = m_i^3$ for all $i = 1, \ldots, 6$. □

Corollary 9.4.2. *A general cubic surface is projectively isomorphic to a surface in $\mathbb{P}^4$ given by the equations*

$$\sum_{i=0}^{4} a_i t_i^3 = \sum_{i=0}^{4} t_i = 0. \tag{9.32}$$

The coefficients $(a_0, \ldots, a_4)$ are determined uniquely up to permutation and a common scaling.

Proof. Assume that the linear forms $l_1, \ldots, l_5$ in the Sylvester presentation span the linear space of linear form. Let $b_1 l_1 + \ldots + b_5 l_5 = 0$ be a unique, up to proportionality, linear relation. Consider the embedding of $\mathbb{P}^3$ into $\mathbb{P}^4$ given by the formula

$$[x_0, \ldots, x_4] \mapsto [y_0, \ldots, y_5] = [l_0(x), \ldots, l_5(x)].$$

Then the surface is isomorphic to the intersection of the cubic hypersurface $V(\sum y_i^3)$ with the hyperplane $V(\sum b_i y_i)$. Now change the coordinates by $t_i = b_i y_i$. In the new coordinates we get equation (9.32), where $a_i = b_i^3$. The Sylvester presentation is unique, up to permutation of the linear functions l_i, multiplication l_i by third roots of 1, and a common scaling. It is clear that the coefficients $(a_0, \ldots, a_4)$ are determined uniquely up to permutation and common scaling. □

We refer to equations (9.32) as a *Sylvester equation* of a general cubic surface.

Suppose a cubic form f can be written as the sum of powers of distinct non-proportional linear forms $l_1, \ldots, l_5$. It is obvious that $V(f)$ is a cone if and only if the forms l_i are linearly independent. Assume that $V(f)$ is not a cone. If no four of the forms l_i are linearly dependent we use the proof of Corollary 9.4.2 to reduce the equation of the cubic surface to the Sylvester form. We say in this case that the cubic surface is *Sylvester non-degenerate*. If four of the forms are linearly dependent, after a linear change of variables, we may assume that $l_1 = x_0, l_2 = x_1, l_3 = x_2, l_4 = x_3, l_5 = ax_0 + bx_1 + cx_2$. The equation becomes

$$f = x_3^3 + g(x_0, x_1, x_2), \tag{9.33}$$

where g_3 is a ternary cubic form. A cubic surface given by such equation is called *cyclic*. Conversely, any f as above such that $V(g)$ admits a polar quadrangle can be written as a sum of 5 powers of linear independent linear forms with four of them linearly dependent. If three of the forms are linearly dependent, the equation becomes

$$x_0^3 + x_1^3 + (ax_0 + bx_1)^3 + x_2^3 + x_3^3 = 0.$$

If the binary form $x_0^3 + x_1^3 + (ax_0 + bx_1)^3$ has no multiple zeros, we can further reduce it to the sum of cubes and obtain that the surface is isomorphic to the Fermat cubic.

9.4.2 The Hessian surface

The Sylvester Theorem gives the equation of the Hessian surface of a Sylvester non-degenerate cubic surface.

Definition 9.2. *Let $S = V(f)$ be a Sylvester non-degenersate cubic surface and $f = \sum_{i=1}^5 l_i^3$ be its equation. The set of 5 planes $V(l_i)$ is called the* Sylvester pentahedron. *This is the polar pentahedron of f. The points $V(l_i, l_j, l_k), 1 \leq i < j < k \leq 5$, are called the* vertices*, and the lines $V(l_i, l_j), 1 \leq i < j \leq 5$, are called the* edges.

Theorem 9.4.3. *Let $S = V(f)$ be a general cubic surface and $\mathrm{He}(S)$ be the Hessian surface of S. Assume $f = \sum_{i=1}^5 l_i^3$. Then $\mathrm{He}(S)$ contains the edges of the Sylvester pentahedron, and the vertices are its ordinary double points. The equation of $\mathrm{He}(S)$ can be written in the form*

$$l_1 l_2 l_3 l_4 l_5 \sum_{i=1}^5 \frac{a_i^2}{l_i} = 0,$$

where $\sum_{i=1}^5 a_i l_i = 0$.

Proof. Recall that

$$\mathrm{He}(S) = \{x \in \mathbb{P}^3 : P_x(S) \text{ is singular }\}.$$

For any point $x \in V(l_i, l_j)$ we have $D_x(f) = \sum_{k \neq i,j} \lambda_k l_k^2$. This is a quadric of rank ≤ 3. Thus each edge is contained in $\mathrm{He}(S)$. Since each vertex lies in 3 non-coplanar edges it must be a singular point. Observe that any edge contains 3 vertices. Any quartic containing 10 vertices and two general points on each edge contains the 10 edges. Thus the linear system of quartics containing 10 edges is of dimension $34 - 30 = 4$. Obviously, any quartic with equation $\sum_{i=1}^5 \frac{\lambda_i}{l_i} = 0$ contains the edges. Thus the equation of $\mathrm{He}(S)$ can be written in this form. We derive the same conclusion in another way which will also allow us to compute the coefficients λ_i.

Consider the isomorphism from S to a surface in $\mathbb{P}^4$ given by two equations

$$\sum_{i=1}^5 z_i^3 = \sum_{i=1}^5 a_i z_i = 0.$$

This isomorphism is given by the map $\mathbb{P}^3 \to \mathbb{P}^4$ defines by the linear forms l_i. The polar quadric $V(P_x(f))$ is given by two equations in $\mathbb{P}^4$

$$\sum l_i(a) z_i^2 = \sum_{i=1}^5 a_i z_i = 0.$$

It is singular if and only if the matrix

$$\begin{pmatrix} l_1(x)z_1 & l_2(x)z_2 & l_3(x)z_3 & l_4(x)z_4 & l_5(x)z_5 \\ a_1 & a_2 & a_3 & a_4 & a_5 \end{pmatrix}$$

is of rank 1 at some point $z = (z_1, \ldots, z_5) = (l_1(t), \ldots, l_5(t))$. This can be expressed by the equalities $l_i(x) = \lambda a_i/l_i(t), i = 1, \ldots, 5$. Since $\sum_{i=1}^5 a_i l_i(x) = 0$, we obtain

$$0 = \sum_{i=1}^{5} a_i l_i(x) = \sum_{i=1}^{5} a_i^2/l_i(t).$$

This gives the asserted equation of $\mathrm{He}(S)$. □

We can also obtain the equation of the Hessian of a general cubic surface in terms of its Sylvester equation (9.32),

$$a_0 \cdots a_4 t_0 \cdots t_4 \sum_{i=0}^{4} \frac{1}{a_i t_i} = 0 \tag{9.34}$$

Remark 9.4.1. Recall that the Hessian of any cubic hypersurface admits a birational automorphism σ which assigns to the polar quadric of corank 1 its singular point. Let X be a minimal nonsingular model of $\mathrm{He}(S)$. It is a K3 surface. The birational automorphism σ extends to a biregular automorphism of X. It exchanges the proper inverse transforms of the edges with the exceptional curves of the resolution. One can show that for a general S, the automorphism of X has no fixed points, and hence the quotient is an Enriques surface.

Suppose now that $V(f)$ is not Sylvester non-degenerate but f admits a representation as a sum of cubes of 5 linearly independent linear forms. As we observed in the previous section, the equation of the surface can be brought to the form

$$x_0^3 + x_1 + x_2^3 + (ax_0 + bx_1 + cx_2)^3 + x_3^3 = 0.$$

The Hessian surface is the union of the plane $V(x_3)$ and the cone over the Hessian of the cubic curve $V(x_0^3 + x_1 + x_2^3 + (ax_0 + bx_1 + cx_2)^3)$. It has the equation

$$x_3[x_0x_1x_2 + (ax_0 + bx_1 + cx_2)(a^2x_1x_2 + b^2x_0x_2 + c^2x_0x_1)] = 0.$$

A cubic form may not admit a polar pentahedral, so its equation may not be written as a sum of powers of liner forms. For example, consider a cubic surface given by the equation

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 + 3x_3^2(ax_0 + bx_1 + cx_2) = 0.$$

For a general choice of the coefficients, the surface is nonsingular and non-cyclic. Its Hessian has the equation

$$x_0x_1x_2x_3 + x_0x_1x_2(ax_0 + bx_1 + cx_2) - x_3^2(a^2x_1x_2 + b^2x_0x_2 + c^2x_0x_1) = 0.$$

It is an irreducible surface and its singular points $[0,0,1,0],[0,1,0,0],[1,0,0,0]$ are singular points of type A_3. The point $[0,0,0,1]$ is a singular point of type A_1. So we see that the surface cannot be Sylvester non-degenerate. The surface does not admit a polar pentahedral, it admits a generalized polar pentahedral in which two of the planes coincide. We refer to [338] and [112] for more examples of Sylvester-degenerate cubic surfaces and their Hessians.

Proposition 9.4.4. *Let a cubic surface be given by Sylvester equation (9.32). Then it is nonsingular if and only if, for all choices of signs,*

$$\sum_{i=1}^{5} \pm \frac{1}{\sqrt{a_i}} \neq 0. \tag{9.35}$$

Proof. The surface is singular if and only if

$$\operatorname{rank}\begin{pmatrix} a_0t_0^2 & a_1t_1^2 & a_2t_2^2 & a_3t_3^2 \\ 1 & 1 & 1 & 1 \end{pmatrix} = 1.$$

This gives $a_it_i^2 = c, i = 0,\ldots,3.$ for some $c \neq 0$. Thus $t_i = \pm c/\sqrt{a_i}$ for some choice of signs, and we get

$$\sum_{i=0}^{3} a_it_i^3 = c\sum_{i=0}^{3} t_i = c\sum_{i=0}^{3} \pm\frac{c}{\sqrt{a_i}} = 0.$$

Conversely, if (9.35) holds for some choice of signs, then the point $[\pm\frac{1}{\sqrt{a_0}},\ldots,\pm\frac{1}{\sqrt{a_3}}]$ satisfies $\sum t_i = 0$ and $\sum a_it_i^3 = 0$. It also satisfies the equations $a_it_i^2 = a_jt_j^2$. Thus it is a singular point. □

9.4.3 Cremona's hexahedral equations

The Sylvester Theorem has the deficiency that it cannot be applied to any nonsingular cubic surface. The Cremona's hexahedral equations which we consider here work for any nonsingular cubic surface. As we will see later in this chapter allows one to define a regular map of degree 36 from an open Zariski subset U of $\mathbb{P}^4$ to $\mathcal{M}_{\text{nscub}}$. Its fibres can be viewed as a choice of a double-sixer on the surface.

Theorem 9.4.5. *(L. Cremona). Assume that a cubic surface S is not a cone and admits a Cayley-Salmon equation (e.g. S is a nonsingular surface). Then S is isomorphic to a cubic surface in $\mathbb{P}^5$ given by the equations*

$$\sum_{i=1}^{6} t_i^3 = \sum_{i=1}^{6} t_i = \sum_{i=1}^{6} a_it_i = 0. \tag{9.36}$$

Proof. Let $F = V(l_1l_2l_3 + m_1m_2m_3)$ be a Cayley-Salmon equation of S. Let us try to find some constants such that, after scaling, the linear forms they add up to zero. Write

$$l_i' = \lambda_il_i, \quad m_i' = \mu_im_i,\ i = 1,2,3.$$

Since S is not a cone, four of the linear forms are linearly independent. After reordering the linear forms, we may assume that the linear forms l_1, l_2, l_3, m_1 are linearly independent. Let

$$m_2 = al_1 + bl_2 + cl_3 + dm_1,\ m_3 = a'l_1 + b'l_2 + c'l_3 + d'l_4.$$

The constants λ_i, μ_i must satisfy the following system of equations

$$\begin{aligned}
\lambda_1 + a\mu_2 + a'\mu_3 &= 0,\\
\lambda_2 + b\mu_2 + b'\mu_3 &= 0,\\
\lambda_3 + c\mu_2 + c'\mu_3 &= 0,\\
\mu_1 + d\mu_2 + d'\mu_3 &= 0,\\
\lambda_1\lambda_2\lambda_3 + \mu_1\mu_2\mu_3 &= 0.
\end{aligned}$$

The first four linear equations allow us to express linearly all unknowns in terms of μ_2, μ_3. Plugging in the last equation, we get a cubic equation in μ_2/μ_3. Solving it, we get a solution. Now set

$$z_1 = l_2' + l_3' - l_1', \quad z_2 = l_3' + l_1' - l_2', \quad z_3 = l_1' + l_2' - l_3',$$

$$z_4 = \mu_2' + \mu_3' - \mu_1', \quad z_5 = \mu_3' + \mu_1' - \mu_2', \quad z_6 = \mu_1' + \mu_2' - \mu_3'.$$

One checks that these six linear forms satisfy the equations from the assertion of the theorem. □

Corollary 9.4.6. *(T. Reye) A general homogeneous cubic form f in 4 variables can be written as a sum of 6 cubes in ∞^4 different ways. In other words,*

$$\dim \mathrm{VSP}(f;6)^o = 4.$$

Proof. This follows from the proof of the previous theorem. Consider the map

$$(\mathbb{C}^4)^6 \to \mathbb{C}^{20}, \quad (l_1, \ldots, l_6) \mapsto l_1^3 + \cdots + l_6^3.$$

It is enough to show that it is dominant. We show that the image contains the open subset of nonsingular cubic surfaces. In fact, we can use a Clebsch-Salmon equation $l_1l_2l_3 + m_1m_2m_3$ for $S = V(f)$ and apply the proof of the Theorem to obtain that, up to a constant factor,

$$f = z_1^3 + z_2^3 + z_3^3 + z_4^3 + z_5^3 + z_6^3.$$

□

Now let us see in how many ways one can write a surface by a Cremona hexahedral equation.

Suppose a nonsingular S is given by equations (9.36) which one calls *Cremona hexahedral equations*. They allow us to locate 15 lines on S such that the remaining lines form a double-six. The equations of these lines in $\mathbb{P}^5$ are

$$z_i + z_j = 0,\ z_k + z_l = 0,\ z_m + z_n = 0,\ \sum_{i=1}^6 a_i z_i = 0,$$

where $\{i,j,k,l,m,n\} = \{1,2,3,4,5,6\}$. Let us denote the line given by the above equations by $l_{ij,kl,mn}$.

Let us identify a pair a,b of distinct elements in $\{1,2,3,4,5,6\}$ with a transposition (ab) in $\mathfrak{S}_6$. We have the product $(ij)(kl)(mn)$ of three commuting transpositions corresponding to each line $l_{ij,kl,mn}$. The group $\mathfrak{S}_6$ admits a unique (up to a composition with a conjugation) outer automorphism which sends each transposition to the product of three commuting transpositions. In this way we can match lines $l_{ij,kl,mn}$ with exceptional vectors c_{ab} of the $\mathbf{E}_6$-lattice. To do it explicitly, one groups together 5 products of three commuting transpositions in such a way that they do not contain a common transposition. Such a set is called a *total* and the triples (ij,kl,mn) are called *synthemes*. Here is the set of 6 totals

$$\begin{aligned}
T_1 &= (12)(36)(45),\ (13)(24)(56),\ (14)(26)(35),\ (15)(23)(46),\ (16)(25)(34),\\
T_2 &= (12)(36)(45),\ (13)(25)(46),\ (14)(23)(56),\ (15)(26)(34),\ (16)(24)(35),\\
T_3 &= (12)(35)(46),\ (13)(24)(56),\ (14)(25)(36),\ (15)(26)(34),\ (16)(23)(45),\\
T_4 &= (12)(34)(56),\ (13)(25)(46),\ (14)(26)(35),\ (15)(24)(36),\ (16)(23)(45),\\
T_5 &= (12)(34)(56),\ (13)(26)(45),\ (14)(25)(36),\ (15)(23)(46),\ (15)(24)(35),\\
T_6 &= (12)(35)(46),\ (13)(26)(45),\ (14)(23)(56),\ (15)(24)(36),\ (16)(25)(34).
\end{aligned}$$

Two different totals T_a, T_b contain one common product $(ij)(kl)(mn)$. The correspondence $(a,b) \mapsto (ij)(kl)(mn)$ defines the outer automorphism

$$\alpha : \mathfrak{S}_6 \to \mathfrak{S}_6. \tag{9.37}$$

For example, $\alpha((12)) = (12)(34)(56)$ and $\alpha((23) = (13)(45)(56)$.

After we matched the lines $l_{ij,kl,mn}$ with exceptional vectors c_{ab}, we check that this matching defines an isomorphism of the incidence subgraph of the lines with the subgraph of the incidence graph of 27 lines on a cubic surface whose vertices correspond to exceptional vectors c_{ab} .

Theorem 9.4.7. *Each Cremona hexahedral equations of a nonsingular cubic surface S defines an ordered double-sixer of lines. Conversely, a choice of an ordered double-sixer defines uniquely Cremona hexahedral equations of S.*

Proof. We have seen already the first assertion of the theorem. If two surfaces given by hexahedral equations define the same double-six, then they have common 15 lines. Obviously, this is impossible. Thus the number of different hexahedral equations of S is less or equal than 36. Now consider the identity

$$\begin{gathered}
(z_1+\cdots+z_6)\big((z_1+z_2+z_3)^2+(z_4+z_5+z_6)^2-(z_1+z_2+z_3)(z_4+z_5+z_6)\big)\\
=(z_1+z_2+z_3)^3+(z_4+z_5+z_6)^3 = z_1^3+\cdots+z_6^3\\
+3(z_2+z_3)(z_1+z_3)(z_1+z_2)+3(z_4+z_5)(z_5+z_6)(z_4+z_6).
\end{gathered}$$

It shows that Cremona hexahedral equations define a Cayley-Salmon equation

$$(z_2+z_3)(z_1+z_3)(z_1+z_2)+(z_4+z_5)(z_5+z_6)(z_4+z_6)=0,$$

where we have to eliminate one unknown with help of the equation $\sum a_i z_i = 0$. Applying permutations of $z_1, \ldots, z_6$, we get 10 Cayley-Salmon equations of S. Each 9 lines formed by the corresponding conjugate pair of triads of tritangent planes are among the 15 lines determined by the hexahedral equation. It follows from the classification of the conjugate pairs that we have 10 such pairs composed of lines c_{ij}'s (type II). Thus a choice of Cremona hexahedral equations defines exactly 10 Cayley-Salmon equations of S. Conversely, it follows from the proof of Theorem 9.4.5 that each Cayley-Salmon equation gives three Cremona hexahedral equations (unless the cubic equation has a multiple root). Since we have 120 Cayley-Salmon equations for S we get $36 = 360/10$ hexahedral equations for S. They match with 36 double-sixers. $\square$

9.4.4 The Segre cubic primal

Let $p_1, \ldots, p_m$ be a set of points in $\mathbb{P}^n$, where $m > n+1$. For any ordered subset $(p_{i_1}, \ldots, p_{i_{n+1}})$ of $n+1$ points we denote by $(i_1 \ldots i_{n+1})$ the determinant of the matrix whose rows are projective coordinates of the points $(p_{i_1}, \ldots, p_{i_{n+1}})$ in this order. We consider $(i_1 \ldots i_{n+1})$ as a section of the invertible sheaf $\otimes_{j=1}^{n+1} p_{i_j}^* \mathcal{O}_{\mathbb{P}^n}(1)$ on $(\mathbb{P}^n)^m$. It is called a *bracket-function*. A monomial in bracket-functions such that each index $i \in \{1, \ldots, m\}$ occurs exactly d times defines a section of the invertible sheaf

$$\mathcal{L}_d = \bigotimes_{i=1}^{n} p_i^* \mathcal{O}_{\mathbb{P}^n}(d).$$

According to the Fundamental Theorem of Invariant Theory (see [136]) the subspace $(R_n^m)(d)$ of $H^0((\mathbb{P}^n)^m, \mathcal{L}_d)$ generated by such monomials is equal to the space of invariants $H^0((\mathbb{P}^n)^m, \mathcal{L}_d)^{\mathrm{SL}(n+1)}$, where the group $\mathrm{SL}(n+1)$ acts linearly on the space of sections via its diagonal action on $(\mathbb{P}^n)^m$. The graded ring

$$R_n^m = \bigoplus_{d=0}^{\infty} (R_n^m)(d) \tag{9.38}$$

is a finitely generated algebra. Its projective spectrum is isomorphic to the GIT-quotient

$$P_n^m := (\mathbb{P}^n)^m // \mathrm{SL}(n+1)$$

of $(\mathbb{P}^n)^m$ by $\mathrm{SL}(n+1)$. Let $r_1, \ldots, r_N$ be homogeneous generators of R_n^m. The complement of it set of common zeros U^{ss} admits a regular map $U^{\mathrm{ss}} \to P_n^m$. The set U^{ss} does not depend on the choice of generators. Its points are called *semi-stable*. Let U^{s} be the largest open subset such that the fibres of the restriction map $U^{\mathrm{s}} \to P_n^m$ are orbits. Its points are called *stable*.

It follows from the Hilbert-Mumford numerical stability criterion that a points set $(p_1, \ldots, p_m)$ in $\mathbb{P}^1$ is semi-stable (resp. stable) if and only if at most $\frac{1}{2}m$ (resp. $< \frac{1}{2}m$) points coincide. We have already seen the definition of the bracket-functions in the case $m = 4$. They define the cross ratio of 4 points

$$[p_1, p_2, p_3, p_4] = \frac{(12)(34)}{(13)(24)}.$$

The cross ratio defines the rational map $(\mathbb{P}^1)^4 - \to \mathbb{P}^1$. It is defined on the open set U^{s} of points where no more that 2 coincide and it is an orbit space over the complement of three points $0, 1, \infty$.

In the case of points in $\mathbb{P}^2$ the condition of stability (semi-stability) is that at most $\frac{1}{3}m$ (resp. $< \frac{1}{3}m$) coincide and at most $\frac{2}{3}m$ (resp. $< \frac{2}{3}m$) points are on a line.

Proposition 9.4.8. *Let $(q_1, \cdots, q_6)$ be an ordered set of distinct points in $\mathbb{P}^1$. The following conditions are equivalent.*

(i) There exists an involution of $\mathbb{P}^1$ such that the pairs $(q_1, q_2), (q_3, q_4), (q_5, q_6)$ are orbits of the involution.

(ii) The binary forms $g_i, i = 1, 2, 3$ with zeros $(q_1, q_2), (q_3, q_4)$ and $q_5, q_6)$ are linearly dependent.

(iii) Let p_i be the image of q_i under a Veronese map $\mathbb{P}^1 \to \mathbb{P}^2$. Then the lines $\overline{p_1, p_2}, \overline{p_3, p_4}, \overline{p_5, p_6}$ are concurrent.

(iv) The bracket-function $(14)(36)(25) - (16)(23)(54)$ vanishes on $(q_1, \ldots, q_6)$.

Proof. (i) ⇔ (ii) Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be the degree 2 map defined by the involution. Let f be given by $[t_0, t_1] \mapsto [g_1(t_0, t_1), g_2(t_0, t_1)]$, where g_1, g_2 are binary forms of degree 2. By choosing coordinates in the target space we may assume that $f(q_1) = f(q_2) = 0, f(q_3) = f(q_4) = 1, f(q_5) = f(q_6) = \infty$, i.e. $g_1(q_1) = g_1(q_2) = 0, g_2(q_3) = g_2(q_4) = 0, (g_1 - g_2)(q_5) = (g_1 - g_2)(q_6) = 0$. Obviously, the binary forms $g_1, g_2, g_3 = g_1 - g_2$ are linearly dependent. Conversely, suppose g_1, g_2, g_3 are linearly dependent. By scaling we may assume that $g_3 = g_1 - g_2$. We define the involution by $[t_0, t_1] \mapsto [g_1(t_0, t_1), g_2(t_0, t_1)]$.

(ii) ⇔ (iii) Without loss of generality, we may assume that $q_i = [1, a_i]$ and $g_1 = t_1^2 - (a_1 + a_2)t_0t_1 + a_1a_2t_0^2, g_2 = t_1^2 - (a_3 + a_4)t_0t_1 + a_3a_4t_0^2, g_3 = t_1^2 - (a_5 + a_6)t_0t_1 + a_5a_6t_0^2$. The condition that the binary forms are linearly dependent is

$$\det \begin{pmatrix} 1 & a_1 + a_2 & a_1a_2 \\ 1 & a_3 + a_4 & a_3a_4 \\ 1 & a_5 + a_6 & a_5a_6 \end{pmatrix} = 0. \tag{9.39}$$

The image of q_i under the Veronese map $[t_0, t_1] \mapsto [t_0^2, t_0t_1, t_2^2]$ is the point $p_i = [1, a_i, a_i^2]$. The line $\overline{p_i, p_j}$ has the equation

$$\det \begin{pmatrix} x_0 & x_1 & x_2 \\ 1 & a_i & a_i^2 \\ 1 & a_j & a_j^2 \end{pmatrix} = (a_j - a_i)(a_ia_jx_0 - (a_i + a_j)x_1 + x_2) = 0.$$

Obviously, the lines are concurrent if and only if (9.39) is satisfied.

(iii) ⇔ (iv) We have

$$\begin{pmatrix} 1 & a_1 + a_2 & a_1a_2 \\ 1 & a_3 + a_4 & a_3a_4 \\ 1 & a_5 + a_6 & a_5a_6 \end{pmatrix} \cdot \begin{pmatrix} a^2 & b^2 & c^2 \\ -a & -b & -c \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} (a-a_1)(a-a_2) & (b-a_1)(b-a_2) & (c-a_1)(c-a_2) \\ (a-a_3)(a-a_4) & (b-a_3)(b-a_4) & (c-a_3)(c-a_4) \\ (a-a_5)(a-a_6) & (b-a_5)(b-a_6) & (c-a_5)(c-a_6) \end{pmatrix}.$$

Substituting $a = a_1, b = a_3, c = a_5$ and computing the determinant we obtain that it is equal to

$$(a_3-a_5)(a_5-a_1)(a_1-a_3)\det\begin{pmatrix} 0 & a_2-a_3 & a_5-a_2 \\ a_1-a_4 & 0 & a_4-a_5 \\ a_6-a_1 & a_3-a_6 & 0 \end{pmatrix}$$

$$= (a_3-a_5)(a_5-a_1)(a_1-a_3)[(a_1-a_4)(a_3-a_6)(a_5-a_2)+(a_6-a_1)(a_2-a_3)(a_4-a_5)].$$

Since the points are distinct the vanishing of determinant (9.39) is equivalent to vanishing of $(a_1-a_4)(a_3-a_6)(a_5-a_2)+(a_6-a_1)(a_2-a_3)(a_4-a_5)$, i.e. the vanishing of the bracket-function $(14)(36)(25)-(16)(23)(54)$ on our point set.

$\square$

We let

$$[ij, kl, mn] := (il)(kn)(jm) - (jk)(lm)(ni). \tag{9.40}$$

For example, $[12, 34, 56] = (14)(36)(25)-(16)(23)(45)$. The expressions $[ij, kl, mn]$ are elements of the linear space $(R_1^6)(1)$. Note as multi-linear functions on V they belong to the 5-dimensional irreducible component of the representation of $\mathfrak{S}_6$ in $(V^\vee)^{\otimes 6}$ corresponding to the partition $6 = 3+3$. In other words, the expression is invariant under permuting elements in the same pair, it is invariant under permuting the pairs by an even permutation, and changes the sign under permuting the pairs by an odd permutation. It is known (and is easy to check) that the linear representation of type $3+3$ is isomorphic to the composition of α with the tensor product of the sign-representation with the standard irreducible representation of $\mathfrak{S}_6$ in the space

$$V_{\text{st}} = \{(a_1, \ldots, a_6) \in \mathbb{C}^6 : a_1 + \cdots + a_6 = 0\}.$$

Let us identify the set $(1, 2, 3, 4, 5, 6)$ with the set of points $(\infty, 0, 1, 2, 3, 4, 5)$ of the projective line $\mathbb{P}^1(\mathbb{F}_5)$. The group $\mathrm{PSL}(2, \mathbb{F}_5) \cong \mathfrak{A}_5$ identified with the group of Moebius transformations $z \mapsto \frac{az+b}{cz+d}$ acts naturally on this set. Let $u_0 = [\infty 0, 14, 23]$ and let $u_i, i = 1, \ldots, 4$, be obtained from u_0 via the action of the transformation $z \mapsto z+i$. Let

$$U_1 := u_0 + u_1 + u_2 + u_3 + u_4$$

$$= \big([\infty 0, 14, 23] + [\infty 1, 20, 34] + [\infty 2, 31, 40] + [\infty 3, 42, 01] + [\infty 4, 03, 12]\big).$$

Obviously, U_1 is invariant under the subgroup of order 5 generated by the transformation $z \mapsto z+1$. It is also invariant under the transformation $\tau : z \mapsto -1/z$. It is well-known that $\mathfrak{A}_5$ is generated by these two transformations. The orbit of U_∞ under the group $\mathfrak{A}_6$ acting by permutations of $\infty, 0, \ldots, 4$ consists of 6 functions $U_1, U_2, U_3, U_4, U_5, U_6$. We will rewrite them now returning to our old notation of indices by the set $(1,2,3,4,5,6)$.

$$\begin{pmatrix}U_1\\U_2\\U_3\\U_4\\U_5\\U_6\end{pmatrix}=\begin{pmatrix}0&[12,36,45]&[13,24,56]&[14,35,26]&[15,46,23]&[16,25,34]\\&0&[15,26,34]&[13,46,52]&[16,35,24]&[14,56,23]\\&&0&[16,32,45]&[14,25,36]&[12,35,46]\\&&&0&[12,34,56]&[15,36,24]\\&&&&0&[13,45,26]\\&&&&&0\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\\1\\1\\1\end{pmatrix},\tag{9.41}$$

where the matrix is skew-symmetric. We immediately observe that

$$U_1+U_2+U_3+U_4+U_5+U_6=0.\tag{9.42}$$

Next observe that the triples of pairs $[ij,kl,mn]$ in each row of the matrix constitute a total from (9.37). The permutation group $\mathfrak{S}_6$ acts in the $R_6^1(1)$ as an irreducible 5-dimensional linear representation defined by the Young tableau of type $(2,2,2)$. The composition of this representation with the outer automorphism (9.37) is isomorphic to the standard representation V_{st} of $\mathfrak{S}_6$ tensored with the sign representation. For example, (12) acts as $U_1\mapsto -U_2, U_2\mapsto -U_1, U_i\mapsto U_i,\ i>2$. Under the outer automorphism (9.37) of $\mathfrak{S}_6$ we have

$$(12)\mapsto(12)(36)(45),\ (3456)\to(1524)$$

One checks that $(U_1,\ldots,U_6)$ are transformed under $(12)(3456)$ to $(-U_2,-U_1,-U_4,$ $-U_5,-U_6,-U_3)$. This implies that the space of invariant functions is one-dimensional and is spanned by the function U_1-U_2. On the other hand, we check immediately that the function $[12,36,54]$ is invariant under σ. This gives $U_1-U_2=c[12,36,54]$ for some scalar c. Evaluating these functions on a point set $(p_1,\ldots,p_6)$ with $p_1=p_2,p_3=p_6,p_4=p_5$ we find that $c=6$. Now applying permutations we obtain:

$$\begin{aligned}U_1-U_2&=6[12,36,54],\ U_1-U_3=6[13,42,65],\ U_1-U_4=6[14,53,26],\\U_1-U_5&=6[15,64,32],\ U_1-U_6=6[16,52,34],\ U_2-U_3=6[15,26,34],\\U_2-U_4&=6[13,46,52],\ U_2-U_5=6[16,35,24],\ U_2-U_6=6[14,23,56],\\U_3-U_4&=6[16,45,32],\ U_3-U_5=6[14,52,63],\ U_3-U_6=6[12,46,53],\\U_4-U_5&=6[12,43,56],\ U_4-U_6=6[15,36,24],\ U_5-U_6=6[13,54,62].\end{aligned}\tag{9.43}$$

Similarly, we find that U_1+U_2 is the only anti-invariant function under σ and hence coincides with $c(12)(36)(45)$. After evaluation the functions at a point set $(p_1,\ldots,p_6)$ with $p_1=p_3,p_2=p_4,p_5=p_6$ we find that $c=4$. In this way we get the relations:

$$\begin{aligned}U_1+U_2&=4(12)(36)(45),\ U_1+U_3=4(13)(42)(56),\ U_1+U_4=4(41)(53)(26),\\U_1+U_5&=4(15)(46)(32),\ U_1+U_6=4(16)(25)(34),\ U_2+U_3=4(15)(26)(43),\\U_2+U_4&=4(13)(46)(25),\ U_2+U_5=4(16)(35)(42),\ U_2+U_6=4(14)(23)(56),\\U_3+U_4&=4(16)(54)(32),\ U_3+U_5=4(14)(25)(63),\ U_3+U_6=4(12)(46)(53),\\U_4+U_5&=4(12)(34)(56),\ U_4+U_6=4(15)(36)(24),\ U_5+U_6=4(13)(45)(62).\end{aligned}\tag{9.44}$$

Using (9.42), we obtain

$$\begin{aligned}U_1&=(12)(36)(45)+(13)(42)(56)+(14)(35)(26)+(15)(46)(32)+(16)(25)(34),\\U_2&=(12)(36)(45)+(13)(46)(25)+(14)(56)(23)+(15)(26)(43)+(16)(24)(53),\\U_3&=(12)(53)(46)+(13)(42)(56)+(14)(52)(36)+(15)(26)(43)+(16)(23)(45),\\U_4&=(12)(34)(56)+(13)(46)(25)+(14)(35)(26)+(15)(24)(36)+(16)(23)(45),\\U_5&=(12)(34)(56)+(13)(54)(26)+(14)(52)(36)+(15)(46)(32)+(16)(24)(53),\\U_6&=(12)(53)(46)+(13)(54)(26)+(14)(56)(23)+(15)(36)(24)+(16)(25)(34).\end{aligned}\tag{9.45}$$

We see that our functions are in bijective correspondence with 6 totals from above. We call the functions $U_1,\ldots,U_6$ the *Joubert functions*.

It is easy to see that the functions U_i do not vanish simultaneously on semi-stable point sets. Thus they define a morphism

$$J : P_1^6 \to \mathbb{P}^5.$$

Theorem 9.4.9. *The morphism J defined by the Joubert functions is an isomorphism onto the subvariety $\mathcal{S}_3$ of $\mathbb{P}^5$ given by the equations*

$$\sum_{i=0}^{5} z_i = \sum_{i=0}^{5} z_i^3 = 0. \tag{9.46}$$

Proof. It is known that the graded ring R_1^6 is generated by the following bracket-functions (*standard tableaux*)

$$(12)(34)(56),\ (12)(35)(46),\ (13)(24)(56),\ (13)(25)(46),\ (14)(25)(36)$$

(see [130]). The subspace of $R_1^6(1)$ generated by the Joubert functions is invariant with respect to $\mathfrak{S}_6$. Since $R_1^6(1)$ is an irreducible representation, this implies that the relation $\sum U_i = 0$ spans the linear relations between the Joubert functions. Consider the sum $\Sigma = \sum U_i^3$. Obviously, it is invariant with respect to $\mathfrak{A}_6$. One immediately checks that an odd permutation in $\mathfrak{S}_6$ transforms each sum Σ to $-\Sigma$. This implies that $\Sigma = 0$ whenever two points p_i and p_j coincide. Hence Σ must be divisible by the product of 15 functions (ij). This product is of degree 5 in coordinates of each point but Σ is of degree 3. This implies that $\Sigma = 0$. Since the functions U_i generate the graded ring R_1^6, by definition of the space P_1^6, we obtain an isomorphism from P_1^6 to a closed subvariety of $\mathcal{S}_3$. Since the latter is irreducible and of dimension equal to the dimension of P_1^6, we obtain the assertion of the theorem. □

The cubic threefold $\mathcal{S}_3$ is called the *Segre cubic primal*. We will often consider it as a hypersurface in $\mathbb{P}^4$.

It follows immediately by differentiating that the cubic hypersurface $\mathcal{S}_3$ has 10 double points. They are the points $p = [1,1,1,-1,-1,-1]$ and others obtained by permuting the coordinates. We will see in a later chapter that this is maximal possible for a cubic hypersurface of dimension 3 with isolated singularities. A point p is given by the equations $z_i + z_j = 0, 1 \ge i \le 3, 4 \ge j \le 6$. Using (9.42) this implies that p is the image of a point set with $p_1 = p_4 = p_6$ or $p_2 = p_3 = p_5$. Thus the singular points of the Segre cubic primal are the images of semi-stable but not stable point sets.

Also $\mathcal{S}_3$ has 15 planes with equations $z_i + z_j = z_k + z_l = z_l + z_m = 0$. Let us see that they are the images of point sets with two points coincide. Without loss of generality, we may assume that $z_1 + z_2 = z_3 + z_4 = z_5 + z_6 = 0$. Again from (9.42), we obtain that $(12)(36)(45), (16)(23)(45)$ and $(13)(26)(45)$ vanish. This happens if and only if $p_4 = p_5$.

We know that the locus of point sets $(q_1,\ldots,q_6)$ such that the pairs $(q_i,q_j),(q_k,q_l)$, and (q_m,q_n) are orbits of an involution are defined by the equation $[ij,kl,mn] = 0$.

By (9.43), we obtain that they are mapped to a hyperplane section of $\mathcal{S}_3$ defined by the equation $z_a - z_b = 0$, where $\alpha((ab)) = (ij)(kl)(mn)$.

It follows from Cremona hexahedral equations that a nonsingular cubic surface is isomorphic to a hyperplane section of the Segre cubic. In a theorem below we will make it more precise. But first we need some lemmas.

Lemma 9.4.10. *Let $p_1, \ldots, p_6$ be six points in $\mathbb{P}^2$. Let $\{1, \ldots, 6\} = \{i, j\} \cup \{k, l\} \cup \{m, n\}$. The condition that the lines $\overline{p_i, p_j}, \overline{p_k, p_l}, \overline{p_m, p_n}$ are concurrent is*

$$(ij, kl, mn) := (kli)(mnj) - (mni)(klj) = 0. \tag{9.47}$$

Proof. The expression $(kli)(mnx)-(mni)(klx)$ can be considered as a linear function defining a line on $\mathbb{P}^2$. Plugging in $x = p_i$ we see that it passes through the point p_i. Also if x is the intersection point q of the lines $\overline{p_k, p_l}$ and $\overline{p_m, p_n}$, then, writing the coordinates of x as a linear combination of the coordinates of p_k, p_l, and of p_m, p_n, we see that the line passes through the point q. Now equation (9.47) expresses the condition that the point p_j lies on the line passing through p_i and the intersection point of the lines $\overline{p_k, p_l}$ and $\overline{p_m, p_n}$. This proves the assertion. □

We have already noted that $[ij, kl, mn] \in R_1^6(1)$ are transformed by $\mathfrak{S}_6$ in the same way as $(ij)(kl)(mn)$ up to the sign representation. Also the functions $(ij, kl, mn) \in R_2^6(1)$ are transformed by $\mathfrak{S}_6$ in the same way as the functions $(ijk)(lmn)$ up to the sign representation. However, the corresponding irreducible components in $(V^\vee)^{\otimes 6}$ exchange the type. The functions (ij, kl, mn) belong to the component of type $(2, 2, 2)$ and the functions $(ijk)(mnl)$ belong to the component of type $(3, 3)$.

Let

$$\begin{pmatrix} \bar{U}_1 \\ \bar{U}_2 \\ \bar{U}_3 \\ \bar{U}_4 \\ \bar{U}_5 \\ \bar{U}_6 \end{pmatrix} = \begin{pmatrix} 0 & (12,36,45) & (13,42,65) & (14,53,26) & (15,46,32) & (16,52,34) \\ & 0 & (15,26,43) & (13,46,25) & (16,24,53) & (14,56,32) \\ & & 0 & (16,32,45) & (14,52,63) & (12,53,46) \\ & & & 0 & (12,43,56) & (15,36,24) \\ & & & & 0 & (13,54,62) \\ & & & & & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Observe that the matrix used here is obtained from the matrix in (9.41) defining the functions U_i by replacing $[ij, kl, mn]$ with (ij, kl, mn). Equations (9.43) extend to the functions $\bar{U}_i$.

Lemma 9.4.11. *We have the relation*

$$\bar{U}_1 + \bar{U}_2 + \bar{U}_3 = -6(146)(253) \tag{9.48}$$

and similar relations obtained from this one by permuting the numbers $(1, \ldots, 6)$.

Proof. Adding up, we get

$$\bar{U}_1 + \bar{U}_2 + \bar{U}_3 = \big((14,53,26) + (14,52,63) + (14,56,23)\big) + \big((16,52,34)$$
$$+(16,24,53) + (16,32,45)\big) + \big((46,31,52) + (46,15,32) + (46,12,53)\big).$$

Next we obtain

$$
\begin{gathered}
(14,53,26)+(14,52,63)+(14,56,23)=(142)(536)-(146)(532)+(146)(523)\\
-(143)(526)+(142)(563)-(143)(562)=-2(146)(253),\\
(16,52,34)+(16,24,53)+(16,32,45)=(163)(524)-(164)(523)+(165)(243)\\
-(163)(245)+(164)(325)-(165)(324)=-2(146)(253),\\
(46,31,52)+(46,15,32)+(46,12,53)=(465)(312)-(462)(315)+(463)(152)\\
-(461)(153)+(465)(123)-(463)(125)=2\big((465)(312)-(462)(315)+(463)(152)\big).
\end{gathered}
$$

Now we use the Plücker relation (10.1)

$$(ijk)(lmn)-(ijl)(kmn)+(ijm)(kln)-(ijn)(klm)=0. \tag{9.49}$$

It gives

$$(465)(312)-(462)(315)+(463)(152)=-(146)(253).$$

Collecting all of this together, we get the assertion. □

Let $(p_1,\dots,p_6)$ be a fixed ordered set of 6 points in $\mathbb{P}^2$. Consider the following homogeneous cubic polynomials in coordinates $x=(x_0,x_1,x_2)$ of a point in $\mathbb{P}^2$.

$$
\begin{aligned}
F_1 &= (12x)(36x)(54x)+(13x)(42x)(65x)+(14x)(53x)(26x)+(15x)(64x)(32x)+(16x)(25x)(43x),\\
F_2 &= (12x)(36x)(54x)+(13x)(46x)(52x)+(14x)(65x)(23x)+(15x)(26x)(34x)+(16x)(24x)(35x),\\
F_3 &= (12x)(35x)(46x)+(13x)(42x)(65x)+(14x)(52x)(63x)+(15x)(26x)(34x)+(16x)(32x)(45x),\\
F_4 &= (12x)(43x)(56x)+(13x)(46x)(52x)+(14x)(53x)(26x)+(15x)(63x)(24x)+(16x)(32x)(45x),\\
F_5 &= (12x)(43x)(56x)+(13x)(54x)(62x)+(14x)(52x)(63x)+(15x)(64x)(32x)+(15x)(24x)(35x),\\
F_6 &= (12x)(35x)(46x)+(13x)(54x)(62x)+(14x)(65x)(23x)+(15x)(63x)(24x)+(16x)(25x)(43x).
\end{aligned}
$$

Theorem 9.4.12. *The rational map $\mathbb{P}^2\to\mathbb{P}^5$ given by the polynomials $F_1,\dots,F_6$ has the image given by the equations*

$$z_1^3+z_2^3+z_3^3+z_4^3+z_5^3+z_6^3=0, \tag{9.50}$$

$$z_1+z_2+z_3+z_4+z_5+z_6=0,$$

$$a_1z_1+a_2z_2+a_3z_3+a_4z_4+a_5z_5+a_6z_6=0,$$

where $(a_1,\dots,a_6)$ are the values of $(\bar{U}_1,\dots,\bar{U}_6)$ at the point set $(p_1,\dots,p_6)$.

Proof. Take $x=(1,0,0)$, then each determinant (ijx) is equal to the determinant (ij) for the projection of $p_1,\dots,p_6$ to $\mathbb{P}^1$. Since all the bracket-functions are invariant with respect to $\mathrm{SL}(3)$ we see that any (ijx) is the bracket function for the projection of the points to $\mathbb{P}^1$ with center at x. This shows that the relations for the functions U_i imply the similar relations for the polynomials F_i. This is what classics called the *Clebsch transference principle*. Let us find the additional relation of the form $\sum_{i=0}^5 a_iz_i=0$. Consider the cubic curve

$$C=a_1F_1(x)+\cdots+a_6F_6(x)=0,$$

where $a_1, \ldots, a_6$ are as in the assertion of the theorem. We have already noted that (ij, kl, mn) are transformed by $\mathfrak{S}_6$ in the same way as $(ij)(kl)(mn)$ up to the sign representation. Thus the expression $\sum_i a_i F_i(x)$ is transformed to itself under an even permutation and transformed to $-\sum_i a_i F_i(x)$ under an odd permutation. Thus the equation of the cubic curve is invariant with respect to the order of the points $p_1, \ldots, p_6$). Obviously, C vanishes at the points p_i. Suppose we prove that C vanishes at the intersection point of the lines $\overline{p_1, p_2}$ and $\overline{p_3, p_4}$, then by symmetry it vanishes at the intersection points of all possible pairs of lines, and hence contains 5 points on each line. Since C is of degree 3 this implies that C vanishes on 15 lines, hence C is identical zero and we are done.

So, let us prove that the polynomial C vanishes at $p = \overline{p_1, p_2} \cap \overline{p_3, p_4}$. Recall from analytic geometry (or multi-linear algebra) that p can be represented by the vector $(v_1 \times v_2) \times (v_3 \times v_4) = (v_1 \wedge v_2 \wedge v_3)v_4 - (v_1 \wedge v_2 \wedge v_4)v_3 = (123)v_4 - (124)v_3$. Thus the value of (ijx) at p is equal to

$$(ijp) = (123)(ij4) - (124)(ij3) = (12)(ij)(34). \tag{9.51}$$

Applying the transference principle to (9.44), we obtain

$$F_1(x) + F_2(x) = 4(12x)(36x)(45x),\ F_4(x) + F_5(x) = 4(12x)(34x)(56x),$$

$$F_1(x) + F_6(x) = 4(16x)(25x)(34x),\ F_3(x) + F_6(x) = 4(12x)(53x)(46x),$$

$$F_2(x) + F_3(x) = (15x)(26x)(43x).$$

This implies that $F_1 + F_2,\ F_4 + F_5,\ F_1 + F_6,\ F_3 + F_6,\ F_2 + F_3$ all vanish at p. Thus the value of C at p is equal to

$$(a_4 - a_5)F_4(p) + (a_2 + a_6 - a_1 - a_3)F_6(p)$$

$$= (a_4 - a_5)(F_4(p) + F_6(p)) + (a_2 + a_6 + a_5 - a_1 - a_3 - a_4)F_6(p)$$

$$= (a_4 - a_5)(F_4(p) + F_6(p)) + (a_2 + a_5 + a_6)(F_1(p) + F_3(p).$$

Here we used that $a_1 + \cdots + a_6 = 0$ and $F_1(p) + F_3(p) + 2F_6(p) = 0$. Using Lemma 9.4.11, we find

$$a_4 - a_5 = (a_4 + a_1 + a_2) - (a_5 + a_1 + a_2) = 6(125)(436) - 6(126)(435) = 6(12, 43, 56).$$

$$a_2 + a_5 + a_6 = 6(346)(125).$$

Using (9.44) and (9.51), we get

$$F_4(p) + F_6(p) = (51p)(42p)(36p) = (42p)(12, 34, 15)(12, 36, 34),$$

$$F_1(p) + F_3(p) = (13p)(42p)(56p) = (42p)(12, 56, 34)(12, 13, 34).$$

Collecting this together we obtain that the value of $\frac{1}{6}C$ at p is equal to

$$(12, 43, 56)(42p)[(12, 34, 15)(12, 36, 34) + (125)(436)(12, 13, 34)).$$

It remains to check that

$$(12,34,15)(12,36,34)+(125)(436)(12,13,34)$$

$$=(125)(314)(123)(364)+(125)(463)(123)(134)=0.$$

$\square$

Recall that the Segre cubic has 15 planes defined by the equations

$$\Pi_{ij,kl.mn}: z_i+z_j=z_k+z_l=z_m+z_n=0,$$

where $\{i,j\}\cup\{k,l\}\cup\{m,n\}=[1,6]$. The intersection of this plane with the hyperplane $H:\sum a_iz_i=0$ is the union of three lines on the cubic surface. In this way we see 15 lines. Each hyperplane $H_{ij}: z_i=z_j=0$ cuts out the Segre cubic $\mathcal{S}_3$ along the union of three planes $\Pi_{ij,kl,mn},$, where the union of $\{k,l\}$ and $\{m,n\}$ is equal to $[1,6]\setminus\{i,j\}$. The hyperplane H intersects $H_{ij}\cap\mathcal{S}_3$ along the union of three lines. Thus we see 15 tritangent planes and 15 lines forming a configuration (15_3). This is a subconfiguration of the configuration $(27_5,45_3)$ of 27 lines and 45 tritangent planes on a nonsingular cubic surface. The dual of the hyperplanes H_{ij} define 15 points in the dual $\mathbb{P}^4$. The duals of the planes $\Pi_{ij,kl,mn}$ are 15 lines. The 15 lines and 15 points form a configuration (15_3) in the dual space. The Igusa-Richmond quartic dual of the Segre cubic is singular along the 15 lines.

9.4.5 Moduli spaces of cubic surfaces

The methods of the Geometric Invariant Theory (GIT) allows one to construct the moduli space of nonsingular cubic surfaces $\mathcal{M}_{\text{cub}}$ as an open subset of the GIT-quotient

$$\mathbb{P}(S^3(\mathbb{C}^4)^\vee)//\mathrm{SL}(4)=\mathrm{Proj}\bigoplus_{d=0}^{\infty}S^d(S^3(\mathbb{C}^4)^\vee)^\vee)^{\mathrm{SL}(4)}. \tag{9.52}$$

The analysis of stability shows that, except one point, the points of this variety represent the orbits of cubic surfaces with ordinary double points. The exceptional point corresponds to the isomorphism class of a unique surface with three A_2-singularities. So, the GIT-quotient can be taken as a natural compactification $\bar{\mathcal{M}}_{\text{cub}}$ of the moduli space $\mathcal{M}_{\text{cub}}$. The computations from the classical invariant theory due to G. Salmon [353], [358] and A. Clebsch [74] (see a modern exposition in [226]) show that the invariant graded ring in (9.52) is generated by elements I_d of degrees $d=8,16,24,32,40$, and 100 (a modern proof of completeness can be found in [26]). The first four basic invariants are invariants with respect to the group $G$of invertible matrices with the determinant equal to ± 1. This explains why their degrees are divisible by 8 (see [136]). The last invariant is what the classics called a skew invariant, it is not an invariant of G but an invariant of $\mathrm{SL}(4)$. There is one basic relation expressing I_{100}^2 as a polynomial in the remaining invariants. The graded subalgebra generated by elements of degree divisible by 8 is freely generated by the first 5 invariants. Since the projective spectrum of this subalgebra is isomorphic to the projective spectrum of the whole algebra, we obtain an isomorphism

$$\bar{\mathcal{M}}_{\text{cub}} \cong \mathbb{P}(8, 16, 24, 32, 40) \cong \mathbb{P}(1, 2, 3, 4, 5). \tag{9.53}$$

The discriminant Δ of a homogeneous cubic form in four variables is expressed in terms of the basic invariants by the formula

$$\Delta = (I_8^2 - 64 I_{16})^2 - 2^{14}(I_{32} + 2^{-3} I_8 I_{24}) \tag{9.54}$$

(the exponent -3 is missing in Salmon's formula, it has been corrected in [112]).

We may restrict the invariants to the open Zariski subset of Sylvester nondegenerate cubic surfaces, It allows one to identify the first four basic invariants with symmetric functions of the coefficients of the Sylvester equations. Salmon's computations give

$$I_8 = \sigma_4^2 - 4\sigma_3\sigma_5, \quad I_{16} = \sigma_1\sigma_5^3, \quad I_{24} = \sigma_4\sigma_5^4, \quad I_{32} = \sigma_2\sigma_5^6, \quad I_{40} = \sigma_5^8, \tag{9.55}$$

where σ_i are elementary symmetric polynomials. Evaluating Δ from above, we obtain a symmetric polynomial of degree 8 obtained from (9.35) by eliminating the irrationality.

The invariant I_{40} restricts to $(a_0a_1a_2a_3a_4)^8$. It does not vanish on the set of Sylvester non-degenerate cubic surfaces. Its locus of zeros is the closure of locus of Sylvester-degenerat nonsingular cubic surfaces.

The skew invariant I_{100} is given by the equation

$$I_{100} = (a_0a_1a_2a_3a_4)^{19} \det \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ a_0^{-1} & a_1^{-1} & a_2^{-1} & a_3^{-1} & a_4^{-1} \\ a_0^2 & a_1^2 & a_2^2 & a_3^2 & a_4^2 \\ a_0^3 & a_1^3 & a_2^3 & a_3^3 & a_4^3 \end{pmatrix}.$$

It vanishes on the closure of the locus of nonsingular surfaces with an Eckardt point . Observe that it vanishes if $a_i = a_j$ and that agrees with Example 9.1.3.

Following [112] we can interpret (9.55) as a rational map

$$\mathbb{P}(\mathbb{C}^4)/\mathfrak{S}_5 \cong \mathbb{P}(1, 2, 3, 4, 5) - \to \bar{\mathcal{M}}_{\text{cub}} \cong \mathbb{P}(1, 2, 3, 4, 5).$$

We have

$$\sigma_1 = \frac{I_{16}}{\sigma_5^3}, \quad \sigma_2 = \frac{I_{32}}{\sigma_5^6}, \quad \sigma_3 = \frac{I_{24}^2 - I_8 I_{40}}{\sigma_5^9}, \quad \sigma_4 = \frac{I_{24} I_{40}}{\sigma_5^{12}}, \quad \sigma_5 = \frac{I_{40}^2}{\sigma_5^{15}}.$$

This gives the inverse rational map

$$\bar{\mathcal{M}}_{\text{cub}} - \to \mathbb{P}(\mathbb{C}^4)/\mathfrak{S}_5.$$

The map is not defined at the set of points where all the invariants I_{8d} vanish except I_8. It is shown in [112], Theorem 6.1 that the set of such points is the closure of the orbit of a Fermat cubic surface.

One can also make explicit the GIT-space P_2^6 of ordered sets of 6 points in the plane. It can be viewed as a compactification of the the moduli space $\mathcal{M}_{\text{cub}}^{\text{m}}$ of *marked* *nonsingular cubic surfaces*, i.e. nonsingular cubic surfaces together with a choice of a geometric marking (or, equivalently, a choice of an order of its 27 lines) Since $\bar{U}_i \in R_2^6(1)$, we obtain a regular map

$$\Phi : P_2^6 \to \mathbb{P}^4$$

defined by $(\bar{U}_1, \ldots, \bar{U}_6)$. It can be shown that this map is of degree 2 and factors through the *association involution* (see [130]). Let $(p_1, \ldots, p_6)$ be a general point set and S be a cubic surface isomorphic to its blow-up. The point set defines six skew lines $\ell_1, \ldots, \ell_6$ on S. Let $\ell_1', \ldots, \ell_6'$ be the six skew lines such that $(\ell_1, \ldots, \ell_6)$ and $(\ell_1', \ldots, \ell_6')$ form the double-six. The blowing down of $\ell_1', \ldots, \ell_6'$ defines an ordered set of points $(p_1', \ldots, p_6')$ whose orbit is the value of the association involution on the orbit of $(p_1, \ldots, p_6)$. This agrees with Theorem 9.4.7. Note that the graded ring R_2^6 is generated by the functions $\bar{U}_i$ (or $(ijk)(lmn)$) and a function Υ from $R_2^6(2)$. The association involution is the identity on (R_2^6) and sends Υ to $-\Upsilon$.

The morphism Φ is $\mathfrak{S}_6$-equivariant, where $\mathfrak{S}_6$ acts on P_2^6 via permuting the factors in $(\mathbb{P}^2)^6$ and acts in $\mathbb{P}^4$ via the composition of the standard irreducible 5-dimensional representation and the outer automorphism α. This representation corresponds to the partition $6 = 2 + 2 + 2$ and differs from the representation of type $3 + 3$ by the sign-representation. Passing to invariants we obtain a map of degree 2:

$$P_2^6/\mathfrak{S}_6 \to \mathbb{P}^4/\mathfrak{S}_6 \cong \mathbb{P}(2, 3, 4, 5, 6). \tag{9.56}$$

This defined a birational isomorphism between the moduli space of nonsingular cubic surfaces together with a choice of a double-sixer and the weighted projective space $\mathbb{P}(2, 3, 4, 5, 6)$ (or, equivalently, the moduli space of sets unordered six points in the plane modulo the association involution).

The set of fixed points of the association involution τ on P_2^6 is represented by ordered point sets $(p_1, \ldots, p_6)$ lying on a conic.The involution commutes with the action of $\mathfrak{S}_6$. Its descent $\bar{\tau}$ to the quotient $P_2^6/\mathfrak{S}_6$ acquires more fixed points represented by point sets $(p_1, \ldots, p_6)$ such that the associated set is projectively equivalent to the same set but in different order. For example, assume that the lines $\overline{p_1, p_2}, \overline{p_3, p_4}, \overline{p_5, p_6}$ have a common point. By Lemma 9.4.10, this happens if and only the function $(12, 34, 56)$ vanishes at this set. By (9.43) (with U_i replaced by $\bar{U}_i$), this is equivalent to vanishing of $\bar{U}_4 - \bar{U}_5$ which is equal to $a_4 - a_5$ in the Cremona hexahedral equation. The intersection point of the three lines defines an Eckardt point on the cubic surface S corresponding to the set $(p_1, \ldots, p_6)$. By Proposition 9.1.13, the projection from this point defines an automorphism g of S such that in the geometric basis $(e_1, \ldots, e_6)$ of Pic(S) defined by $(p_1, \ldots, p_6)$ it g^* acts as the permutation $(12)(34)(56)$. This shows that the set $(p_2, p_1, p_4, p_3, p_6, p_5)$ is projectively equivalent to the set $(p_1, \ldots, p_6)$. Conversely, one can show that a point set $(p_1, \ldots, p_6)$ corresponding to a nonsingular cubic surface defines is a fixed point of the involution $\bar{\tau}$ lies in the locus of zeros of some function $\bar{U}_i - \bar{U}_j, i \neq j$ [85], Part I (16).

It follows from the above discussion that the ramification locus of the double cover (9.56) is equal to the locus of zeros of the product of two functions: the function Υ defining the locus of point sets on a conic and the function Δ equal to the product of the differences $\bar{U}_i - \bar{U}_j$. The branch locus in $\mathbb{P}^4/\mathfrak{S}_6$ is equal to the locus of the product of two functions: one is the symmetric function $\sigma_2^2 - 4\sigma_4$ defining the Igusa-Richmond quartic in $\mathbb{P}^4$ and the function D^2, where D equal to the discriminant of the polynomial $(X - a_1)\cdots(X - a_6)$. The first part of the branch locus is isomorphic to $\mathbb{P}(2,3,5,6)$ and hence is irreducible. The second part is isomorphic to the discriminant hypersurface, also known to be irreducible.

Remark 9.4.2. It is easy to see that under the Veronese map $\nu : \mathbb{P}^1 \to \mathbb{P}^2$, we have $\nu^*((ij)(kl)(mn)) = [ij][kl][mn](ij)(kl)(mn) \in R_1^6$. Under the isomorphism $P_1^6 \cong \mathcal{S}_3$ these functions define the quadrics through the singular points of the Segre cubic $\mathcal{S}_3$. This can be interpreted by asserting that the composition of rational maps

$$P_1^6 \overset{\nu_2}{\dashrightarrow} P_2^6 \xrightarrow{\Phi} \mathbb{P}^4$$

is given by polar quadrics of the Segre cubic. One can show that its image is a hypersurface of degree 4 isomorphic to the dual hypersurface of $\mathcal{S}_3$ (see[130]). This hypersurface is isomorphic to the Igusa-Richmond quartic.

A cubic surface in $\mathbb{P}^3$ can be given as a hyperplane section of a cubic threefold in $\mathbb{P}^4 = |W|$. In this way the theory of projective invariant of cubic surfaces becomes equivalent to the theory of projective invariants of $\mathrm{PGL}(5)$ in the space $S^3W^\vee \times W^\vee$. The Cremona hexahedral equations of a cubic surface represents a subvariety of this representation isomorphic to $\mathbb{C}^6$. The Clebsch transference principle (see for a modern explanation [226]) allows one to express projective invariants of $\mathrm{GL}(4)$ as polynomial functions on $\mathbb{C}^6$. The degree of an invariant polynomial of degree m equal to their weights $3m/4$. In particular, the basic polynomials $I_8, \ldots, I_{100}$ become polynomials $J_6, J_{12}, J_{18}, J_{24}, J_{30}, J_{75}$ in $(a_1, \ldots, a_6)$ of degrees indicated in the subscript. The first five polynomials are symmetric polynomials in $a_1, \ldots, a_6$, the last one is a skew-symmetric polynomial. For example,

$$J_6 = 24(4\sigma_2^3 - 3\sigma_3^2 - 16\sigma_2\sigma_4 + 12\sigma_6)$$

(see [85] Part III, p. 336, and [390]).

The skew-invariant J_{75} defining the locus of cubic surfaces with an Eckardt points is reducible. It contains as a factor of degree 15 the discriminant $\prod_{i<j}(a_i - a_j)$ of the polynomial $(X - a_1)\cdots(X - a_6)$. The remaining factor of degree 60 is equal to the product of 30 polynomials of the form

$$T_{1256;3} = (126)(356)(134)(253) - (136)(256)(123)(354), \tag{9.57}$$

where we use Lemma 9.4.11 to express the product of two brackets as a function $a_i + a_j + a_k$. The vanishing of $T_{1256;3}$ expresses the condition that the conic through the points p_1, p_2, p_3, p_5, p_6 is touched at p_3 by the line $\overline{p_3, p_4}$ (equivalently, the tritangent plane defined by the lines $e_3, 2e_0 - e_1 - e_2 - e_3 - e_5 - e_6, e_0 - e_3 - e_4$ has an Eckard point).

We can also find the expression of the discriminant invariant Δ (9.54) in terms of the coefficients $a_0, \ldots, a_5$.

The cubic surface given by $(a_0, \ldots, a_5)$ is singular if and only if $p_1, \ldots, p_6$ lie on a conic or there are three collinear point among them. Applying Lemma 9.4.11, we find that the latter condition is expressed by vanishing $a_i + a_k + a_k$ for some distinct i, j, k. Since the coefficients add up to 0, we see that we have 10 linear equations of this sort. The former condition is given by the equation

$$d_2 = (341)(561)(532)(462) - (342)(562)(531)(461) \tag{9.58}$$

similar to (9.57). If we replace in this expression 6 with x, and let x vary we get an equation of a conic. When $x = 1, 2, 3, 4, 5$, the expression vanishes. Thus this conic passes through the points $p_1, \ldots, p_5$. Putting back $x = 6$, we wee (9.58) vanishes if and only if the six points are on a conic.

The expression d_2 is an invariant of degree 2 of 6 points from the ring R_2^6 which coincides with the function Υ. The square d_2^2 is a polynomial of degree 4 in a spanning set of $R_2^6(1)$. It is a symmetric expression in $(1, \ldots, 6)$ equal to $\sigma_2^2 - 4\sigma_4$. Thus we see that the discriminant invariant in $(a_0, \ldots, a_5)$, being of of degree 24, must be a scalar multiple of the product of powers of $(\sigma_2 - 4\sigma_4)$ and powers of $(a_i + a_j + a_k),\ 1 \le i < j < k \le 5$. The only way to make a symmetric polynomial of degree 24 in this way is to take all factors in the first power. We also use that σ_1 vanishes on $(a_0, \ldots, a_5)$. The computer computation gives the following expression in terms of the elementary symmetric polynomials.

$$\Delta = (\sigma_2^2 - 4\sigma_4)(\sigma_3^4\sigma_4^2 - 2\sigma_2\sigma_3^3\sigma_4\sigma_6 + \sigma_2^2\sigma_3^2\sigma_5^2 + 2\sigma_3^2\sigma_4\sigma_5^2 - 2\sigma_2\sigma_3\sigma_5^4 + 2\sigma_2\sigma_3\sigma_4\sigma_6 -$$

$$8\sigma_3^2\sigma_4^2\sigma_6 - 2\sigma_2^3\sigma_3\sigma_5\sigma_6 + 8\sigma_2\sigma_3\sigma_4\sigma_5\sigma_5\sigma_6 + 2\sigma_2^2\sigma_5^2\sigma_6 + \sigma_2^4\sigma_6^2 - 8\sigma_2^2\sigma_4\sigma_6^2 + 16\sigma_4^2\sigma_6^2)$$

Remark 9.4.3. The story goes on. The group $W(\mathbf{E}_6)$ acts birationally on the space P_2^6 by changing the markings and Coble describes in [85], Part III, rational invariants of this action. He also defines a linear system of degree 10 of elements of degree 3 in R_2^6 which gives a $W(\mathbf{E}_6)$-equivariant embedding of a certain blow-up of P_2^6 in $\mathbb{P}^9$ corresponding to some irreducible 10-dimensional linear representation of the Weyl group. We refer for a modern treatment of this construction to [88], [178]. Other $W(\mathbf{E}_6)$-equivariant birational models of R_2^6 were given in [298] and [199]. We also refer to a recent construction of the GIT-moduli space of cubic surfaces as a quotient of a complex 4-dimensional ball by a reflection group [5],[139]. The embedding of the moduli spaces in $\mathbb{P}^9$ by means of automorphic forms on the 4-dimensional complex ball is discussed in [168], [6].

9.5 Automorphisms of cubic surfaces

9.5.1 Elements of finite order in Weyl groups

Let W be the Weyl group of a simple root system of type A, D, E. The conjugacy classes of elements of finite order can be classified. We will follow the classification due to R. Carter [43].

We know that each $w \in W$ is equal to the product of reflections with respect to roots α. Let $l(w)$ be the smallest number k of roots such that w can be written as such a product. This number is equal to the number of eigenvalues of w in $Q_{\mathbb{C}}$ different from 1. If $w = r_{\boldsymbol{\alpha}_1} \ldots r_{\boldsymbol{\alpha}_{l(w)}}$, then the corresponding roots are linearly independent.

Each element w can be written as the product $w = w_1 w_2$, where $w_1^2 = w_2^2 = 1$ and $l(w) = l(w_1) + l(w_2)$. Moreover, if $w_1 = r_{\boldsymbol{\alpha}_1} \dots r_{\boldsymbol{\alpha}_{l(w_1)}}$, where the roots $\boldsymbol{\alpha}_i$ are orthogonal to each other. The same is true for $w_2 = r_{\beta_1} \cdots r_{\beta_{l(w_2)}}$. Each such decomposition defines the *Carter graph* $\Gamma(w)$ of w. Its vertices correspond to each root in the decomposition of

$$w = r_{\boldsymbol{\alpha}_1} \dots r_{\boldsymbol{\alpha}_{l(w_1)}} r_{\beta_1} \cdots r_{\beta_{l(w_2)}}.$$

Two vertices corresponding to $\boldsymbol{\alpha}_i$ and β_j are joined by an edge if $(\boldsymbol{\alpha}_i, \beta_j) \neq 0$.

Example 9.5.1. Suppose Q is the root lattice of type A_n. Its Weyl group is the symmetric group $\mathfrak{S}_{n+1}$. Let $\sigma = (i_1, \dots, i_{2k})$ be a $2k$-cycle. It is the product of transpositions $(i_1 i_2) \dots (i_{2k-1} i_{2k})$. Write

$$\sigma = (i_1 i_2)(i_3 i_4)(i_3 i_4)(i_2 i_3)(i_3 i_4) \dots (i_{2k-1} i_{2k}) = (i_1 i_2)(i_3 i_4)(i_2 i_4) \dots (i_{2k-1} i_{2k}).$$

Continuing in this way we will be able to write σ as the product of two involutions

$$\sigma = [(i_1 i_2)(i_3 i_4) \dots (i_{2k-1} i_{2k})][(i_2 i_3)(i_4 i_5) \dots (i_{2k-2} i_{2k-1})].$$

Each transposition (ij) is the reflection with respect to the root $\mathbf{e}_i - \mathbf{e}_j$, where we consider the lattice Q as the sublattice of $\mathbb{Z}^{n+1}$ formed by the vectors perpendicular to $\mathbf{e}_1 + \dots + \mathbf{e}_{n+1}$. Now it is easy to see that the Carter graph of σ is the Dynkin diagram of type A_{2k}. The same conclusion can be derived in the case when σ is cycle of odd length. Since any permutation is the product of commuting cycles, we obtain that the Carter graph of a permutation is equal to the disconnected sum of graphs of type A_k.

Carter proves that two elements in the Weyl group $W(Q)$ are conjugate if and only if their graphs coincide. Anytime we embed the root lattice Q in a root lattice Q', we get an embedding of the Weyl groups $W(Q) \hookrightarrow W(Q')$. Since the conjugacy class in $W(Q)$ is determind by the Carter graph, we see that two nonconjugate elements in the subgroup stay nonconjugate in the group.

Each Carter graph Γ has the following properties:

- Its vertices correspond to linearly independent roots.
- Each subgraph of Γ which is a cycle contains even number of vertices.
- Each graph without cycles is a Dynkin diagram of some root lattice Q'. The corresponding sublattice of Q defines the embedding of $W(Q')$ in $W(Q)$ and the conjugacy class corresponding to Γ can be represented by the Coxeter element in $W(Q)$.

Recall that the *Coxeter element* is a product of reflections corresponding to simple roots (forming a root basis of Q). Its order is the *Coxeter number* of the Weyl group.

The graphs corresponding to subroot lattices can be described by the Borel-De Siebenthal-Dynkin.

Let $w \in W(Q)$, its characteristic polynomial in its linear action on $Q_{\mathbb{C}}$ can be read off from the diagram.

The following table contains the list of connected components of Carter's graphs, the orders of the corresponding elements and the characteristic polynomial.

Graph	Order	Characteristic polynomial
A_k	$k+1$	$t^k + t^{k-1} + \cdots + 1$
D_k	$2k-2$	$(t^{k-1}+1)(t+1)$
$D_k(a_1)$	l.c.m$(2k-4,4)$	$(t^{k-2}+1)(t^2+1)$
$D_k(a_2)$	l.c.m$(2k-6,6)$	$(t^{k-3}+1)(t^3+1)$
$\vdots$	$\vdots$	$\vdots$
$D_k(a_{\frac{k}{2}-1})$	even k	$(t^{\frac{k}{2}}+1)^2$
E_6	12	$(t^4-t^2+1)(t^2+t+1)$
$E_6(a_1)$	9	t^6+t^3+1
$E_6(a_2)$	6	$(t^2-t+1)^2(t^2+t+1)$
E_7	18	$(t^6-t^3+1)(t+1)$
$E_7(a_1)$	14	t^7+1
$E_7(a_2)$	12	$(t^4-t^2+1)(t^3+1)$
$E_7(a_3)$	30	$(t^5+1)(t^2-t+1)$
$E_7(a_4)$	6	$(t^2-t+1)^3(t+1)$
E_8	30	$t^8+t^7-t^5-t^4-t^3+t+1$
$E_8(a_1)$	24	t^8-t^4+1
$E_8(a_2)$	20	$t^8-t^6+t^4-t^2+1$
$E_8(a_3)$	12	$(t^4-t^2+1)^2$
$E_8(a_4)$	18	$(t^6-t^3+1)(t^2-t+1)$
$E_8(a_5)$	15	$t^8-t^7+t^5-t^4+t^3-t+1$
$E_8(a_6)$	10	$(t^4-t^3+t^2-t+1)^2$
$E_8(a_7)$	12	$(t^4-t^2+1)(t^2-t+1)^2$
$E_8(a_8)$	6	$(t^2-t+1)^4$

Table 9.2: Carter graphs and characteristic polynomials

9.5.2 Subgroups of $W(\mathbf{E}_6)$

We will need some known information about the structure of the Weyl group $W(\mathbf{E}_6)$.

Lemma 9.5.1. *Let H be a maximal subgroup of $W(\mathbf{E}_6)$. Then one of the following cases occurs:*

(i) $H \cong 2^4 : \mathfrak{S}_5$ *of order* $2^4 \cdot 5!$ *and index 27;*

(ii) $H \cong \mathfrak{S}_6 \times 2$ *of order* $2 \cdot 6!$ *and index 36;*

(iii) $H \cong 3_+^{1+2} : 2\mathfrak{S}_4$ *of order* 1296 *and index 40;*

(iv) $H \cong 3^3 : (\mathfrak{S}_4 \times 2)$ *of order* 1296 *and index 40;*

(v) $H \cong \big(2.(\mathfrak{A}_4 \times \mathfrak{A}_4).2\big).2$ *of order is* 1152 *and index 45.*

Here we use the ATLAS [90] notations for cyclic groups: $\mathbb{Z}/n\mathbb{Z} = n$ and semi-direct products: $H \ltimes G = H : G$, 3_+^{1+2} denotes the group of order 3^3 of exponent p, $A.B$ is a group with normal subgroup isomorphic to A and quotient isomorphic to B.

We recognize a group from (i) as the stabilizer subgroup of an exceptional vector (or a line on a cubic surface). If we choose a simple root basis $(\boldsymbol{\alpha}_0, \ldots, \boldsymbol{\alpha}_5)$ such that the exceptional vector is equal to $\boldsymbol{\alpha}_5^*$, then H is generated by the reflections $s_i = s_{\boldsymbol{\alpha}_i}, i \neq 5$. It is naturally isomorphic to the Weyl group $W(D_5)$.

A group H of type (ii) is the stabilizer subgroup of a double-six. The subgroup $\mathfrak{S}_6$ permutes the lines, the subgroup 2 switches the two sixers. In the geometric root basis $\boldsymbol{\alpha}_0 = e_0 - e_1 - e_2 - e_3, \boldsymbol{\alpha}_i = e_i - e_{i+1}$, the stabilizer subgroup of the double-sixer $(e_1, \ldots, e_6; e_1', \ldots, e_6')$, where $e_i' = 2e_0 - e_1 - \ldots - e_6 + e_i$, generated by permutations of e_i's and the reflection with respect to the maximal root $2e_0 - e_1 - \ldots - e_6$.

A group of type (iv) is the stabilizer subgroup of a Steiner triad of a double-sixers.

A group of type (v) is the stabilizer subgroup of a tritangent plane (or a triple of exceptional vectors added up to 0).

Proposition 9.5.2. *$W(\mathbf{E}_6)$ contains a unique normal subgroup $W(\mathbf{E}_6)'$. It is a simple group and its index is equal to 2.*

Proof. Choose a root basis $(\alpha_0, \ldots, \alpha_5)$ in the root lattice $\mathbf{E}_6$. Let $s_0, \ldots, s_5$ be the corresponding simple reflections. Each element $w \in W(\mathbf{E}_6)$ can be written as a product of the simple reflections. Let $\ell(w)$ is the minimal length of the word needed to write w as such a product. For example, $\ell(1) = 0, \ell(s_i) = 1$. One shows that the function $\ell : W(\mathbf{E}_6) \to \mathbb{Z}/2\mathbb{Z},\ w \mapsto \ell(w) \mod 2$ is a homomorphism of groups. Its kernel $W(\mathbf{E}_6)'$ is a subgroup of index 2. The restriction of the function ℓ to the subgroup $H \cong \mathfrak{S}_6$ generated by the reflections $s_1, \ldots, s_5$ is the sign function. Suppose K is a normal subgroup of $W(\mathbf{E}_6)'$. Then $K \cap H$ is either trivial or equal to the alternating subgroup $\mathfrak{A}_6$ of index 2. It remains to use that $H \times (r)$ is a maximal subgroup of $W(\mathbf{E}_6)$ and r is a reflection which does not belong to $W(\mathbf{E}_6)'$. □

Remark 9.5.1. Recall that we have an isomorphism (9.7) of groups

$$W(\mathbf{E}_6) \cong \mathrm{O}(6, \mathbb{F}_2)^-.$$

The subgroup $W(\mathbf{E}_6)'$ is isomorphic to the commutator subgroup of $\mathrm{O}(6, \mathbb{F}_2)^-$.

Let us mention other realizations of the Weyl group $W(\mathbf{E}_6)$.

Proposition 9.5.3.

$$W(\mathbf{E}_6)' \cong \mathrm{SU}_4(2),$$

where $\mathrm{U}_4(2)$ is the group of linear transformations with determinant 1 of $\mathbb{F}_4^4$ preserving a nondegenerate Hermitian product with respect to the Frobenius automorphism of $\mathbb{F}_4$.

Proof. Let $\mathbf{F} : x \mapsto x^2$ be the Frobenius automorphism of $\mathbb{F}_4$. We view the expression

$$\sum_{i=0}^{3} x_i^3 = \sum_{i=0}^{3} x_i \mathbf{F}(x_i)$$

as a nondegenerate hermitian form in $\mathbb{F}_4^4$. Thus $\mathrm{SU}_4(2)$ is isomorphic to the subgroup of the automorphism group of the cubic surface S defined by the equation

$$t_0^3 + t_1^3 + t_2^3 + t_3^3 = 0$$

over the field $\bar{\mathbb{F}}_2$. The Weyl representation (which is defined for nonsingular cubic surfaces over fields of arbitrary characteristic) of $\mathrm{Aut}(S)$ defines a homomorphism $\mathrm{SU}_4(2) \to W(\mathbf{E}_6)$. The group $\mathrm{SU}_4(2)$ is known to be simple and of order equal to $\frac{1}{2}|W(\mathbf{E}_6)|$. This defines an isomorphism $\mathrm{SU}_4(2) \cong W(\mathbf{E}_6)'$. $\square$

Proposition 9.5.4.

$$W(\mathbf{E}_6) \cong \mathrm{SO}(5, \mathbb{F}_3), \quad W(\mathbf{E}_6)' \cong \mathrm{SO}(5, \mathbb{F}_3)^+,$$

where $\mathrm{SO}(5, \mathbb{F}_3)^+$ *is the subgroup of elements of spinor norm 1.*

Proof. Let $\bar{Q} = Q/3Q$. Since the discriminant of the lattice $\mathbf{E}_6$ is equal to 3, the symmetric bilinear form defined by

$$\langle v + 3Q, w + 3Q \rangle = -(v, w) \mod 3$$

is degenerate. It has one-dimensional radical spanned by the vector

$$v_0 = 2\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_1 + 2\boldsymbol{\alpha}_4 + \boldsymbol{\alpha}_5 \mod 3Q.$$

The quadratic form $q(v) = (v, v) \mod 3$ defines a nondegenerate quadratic form on $V = \bar{Q}/\mathbb{F}_3 v_0 \cong \mathbb{F}_3^5$. We have a natural injective homomorphism $W(\mathbf{E}_6) \to \mathrm{O}(5, \mathbb{F}_2)$. Comparing the orders, we find that the image is a subgroup of index 2. It must coincide with $\mathrm{SO}(5, \mathbb{F}_3)$. Its unique normal subgroup of index 2 is $\mathrm{SO}(5, \mathbb{F}_3)^+$. $\square$

Remark 9.5.2. Let V be a vector space of odd dimension $2k + 1$ over a finite field $\mathbb{F}_q$ equipped with a nondegenerate symmetric bilinear form. An element $v \in V$ is called a *plus vector* (resp. *minus vector*) if (v, v) is a square in $\mathbb{F}_q^*$ (resp. is not a square $\in \mathbb{F}_q^*$). The orthogonal group $\mathrm{O}(V)$ has three orbits in $|E|$: the set of isotropic lines, the set of lines spanned by a plus vector and the set of lines spanned by a minus vector. The isotropic subgroup of a non-isotropic vector v is isomorphic to the orthogonal group of the subspace $v^\perp$. The restriction of the quadratic form to $v^\perp$ is of Witt index k if v is a plus vector and of Witt index $k - 1$ if v is a minus vector. Thus the stabilizer group is isomorphic to $\mathrm{O}(2k, \mathbb{F}_q)^{\pm}$. In our case, when $k = 2$ and $q = 3$, we obtain that minus vectors correspond to cosets of roots in $\bar{Q}$, hence the stabilizer of a minus vector is isomorphic to the stabilizer of a double-six, i.e. a maximal subgroup of $W(\mathbf{E}_6)$ of index 36. The stabilizer subgroup of a plus vector is a group of index 45 and isomorphic to the stabilizer of a tritangent plane. The stabilizer of an isotropic plane is a maximal subgroup of type (iii), and the stabilizer subgroup of an isotropic line is a maximal subgroup of type (iv).

9.5.3 Automorphisms of finite order

Since any automorphism of a nonsingular cubic surface S preserves $|-K_S|$, it is induced by a projective transformation. After diagonalization we may assume that any automorphism is represented by a diagonal matrix with roots of unity as its entries.

Lemma 9.5.5. *Let $S = V(f)$ be a nonsingular cubic surface which is invariant with respect to a projective transformation σ of order $n > 1$. Then, after a linear change of variables, f is given in the following list. Also, a generator of the group $\langle\sigma\rangle$ can be defined by $[x_0, x_1, x_2, x_3] \mapsto [x_0, \zeta_n^a, \zeta_n^b x_2, \zeta_n^c x_3]$, where ζ_n is a primitive n-th root of unity:*

(i) $(n = 2)$, $(a, b, c) = (0, 0, 1)$,
$$f = t_3^2 l_1[t_0, t_1, t_2] + t_0^3 + t_1^3 + t_2^3 + at_0t_1t_2.$$

(ii) $(n = 2)$, $(a, b, c) = (0, 1, 1)$,
$$f = t_0t_2(t_2 + at_3) + t_1t_3(t_3 + bt_3) + t_0^3 + t_1^3.$$

(iii) $(n = 3)$, $(a, b, c) = (0, 0, 1)$,
$$f = t_3^3 + t_0^3 + t_1^3 + t_2^3 + at_0t_1t_2.$$

(iv) $(n = 3)$, $(a, b, c) = (0, 1, 1)$,
$$f = l_3(t_0, t_1) + m_3(t_2, t_3).$$

(v) $(n = 3)$, $(a, b, c) = (0, 1, 2)$,
$$f = l_3(t_0, t_1) + t_2t_3l_1(t_0, t_1) + t_2^3 + t_3^3.$$

(vi) $(n = 4)$, $(a, b, c) = (0, 2, 1)$,
$$f = t_3^2t_2 + t_0^3 + t_1^3 + t_2^2(t_0 + at_1).$$

(vii) $(n = 4)$, $(a, b, c) = (2, 3, 1)$,
$$f = t_0^3 + t_0t_1^2 + t_1t_3^2 + t_1t_2^2.$$

(viii) $(n = 5)$, $(a, b, c) = (4, 1, 2)$,
$$f = t_0^2t_1 + t_1^2t_2 + t_2^2t_3 + t_3^2t_0,$$

(ix) $(n = 6)$, $(a, b, c) = (0, 3, 2)$,
$$f = l_3(t_0, t_1) + t_3^3 + t_2^2(t_0 + at_1).$$

(x) $(n = 6)$, $(a, b, c) = (0, 2, 5)$,
$$f = t_0^3 + t_1^3 + t_3^2t_2 + t_2^3.$$

(xi) $(n = 6)$, $(a, b, c) = (4, 2, 1)$,
$$f = t_3^2t_1 + t_0^3 + t_1^3 + t_2^3 + \lambda t_0t_1t_2.$$

(xii) $(n = 6)$, $(a, b, c) = (4, 1, 3)$,

$$f = t_0^3 + bt_0t_3^2 + t_2^2t_1 + t_1^3.$$

(xiii) $(n = 8)$, $(a, b, c) = (4, 3, 2)$,

$$f = t_3^2t_1 + t_2^2t_3 + t_0t_1^2 + t_0^3.$$

(xiv) $(n = 9)$, $(a, b, c) = (4, 1, 7)$,

$$f = t_3^2t_1 + t_1^2t_2 + t_2^2t_3 + t_0^3.$$

(xv) $(n = 12)$, $(a, b, c) = (4, 1, 10)$,

$$f = t_3^2t_1 + t_2^2t_3 + t_0^3 + t_1^3.$$

Here the subscripts in l_i, m_i indicate the degree of the polynomial.

Proof. Choose a coordinate system where σ diagonalizes as in the statement of the lemma. Let $p_1 = [1, 0, 0, 0], \ldots, p_4 = [0, 0, 0, 1]$ be the reference points. They are fixed under the action of σ in $\mathbb{P}^3$. We will use frequently that f is of degree ≥ 2 in each variable. This follows from the assumption that the surface is nonsingular. We will also give a normal form of the equation with minimal number of parameters which is easy to get and is left to the reader.

Case 1: Two of a, b, c, say a, b, are equal to zero. Write f as a polynomial in t_3. Assume $p_3 \notin V(f)$. Then

$$f = t_3^3 + t_3^2l_1(t_0, t_1, t_2) + t_3l_2(t_0, t_1, t_2) + l_3(t_0, t_1, t_2).$$

Since f is an eigenvector with the eigenvalue equal to ζ_n^{3c} and $l_3 \neq 0$, we must have $n = 3$ and $l_1 = l_2 = 0$. This is case (iii). Assume $p_3 \in V(f)$. Then

$$f = t_3^2l_1(t_0, t_1, t_2) + t_3l_2(t_0, t_1, t_2) + l_3(t_0, t_1, t_2).$$

As above this gives $n = 2, l_2 = 0$. This is case (i).

Case 2: One of (a, b, c), say a, is equal to zero. Write f as a polynomial in the form

$$f = l_3(t_0, t_1) + t_0l_2(t_2, t_3) + t_1m_2(t_2, t_3) + m_3(t_2, t_3).$$

Assume that $l_2 = m_2 = 0$. If m_3 is of degree 3 in t_3 or t_2, say t_2, then $3b = 0 \mod n$. If l_3 is of degree 3 in t_3 too, we get $3c = 0 \mod n$, hence $n = 3$. Without loss of generality, we may assume that $(b, c) = (1, 1)$ or $(2, 1)$. In the first case m_3 is any polynomial in t_2, t_3 of degree ≥ 2 in t_2, t_3. This is case (iv). In the second case $m_3 = t_3^3 + t_2^3$. This is a special case of case (v).

If m_3 is of degree 2 in t_3, then l_3 contains $t_3^2t_2$, hence $2c + b = 0 \mod n$. This gives $n = 6, (a, b, c) = (0, 2, 5)$. This is case (x).

Assume now that l_2 or m_2 is not equal to zero. If t_2^2, t_3^2 do not enter in l_2 and m_2, then t_0t_1 must enter in one of them. This gives $b + c = 0 \mod n$. If t_2^3 or t_2^3 enters in

m_3, then $3b = 0 \mod n$ or $3c = 0 \mod n$. This gives $n = 3, (a, b, c) = (0, 1, 2)$ or $(0, 2, 1)$. This is case (v). If t_2^3, t_3^3 do not enter in m_3, then $t_2^2t_3$ and $t_2t_3^2$ both enter and we get $2b + c = b + 2c = 0 \mod n$. This again implies $n = 3$ and we are in case (v).

Now we may assume that t_2^2 enters in l_2 or m_2, then $2b = 0 \mod n$. If t_3^2 also enters in l_2 or m_2, then $2c = 0 \mod n$. This implies $n = 2$ and $m_3 = 0$. This is case (ii).

If t_3^2 does not enter in l_2 and m_2, then m_3 is of degree ≥ 2 in t_3. If t_3^3 enters in m_3, then $3c = 0 \mod n$, hence $n = 6$ and $(a, b, c) = (0, 3, 2)$. Thus

$$f = l_3(t_0, t_1) + t_2^2 m_1(t_0, t_1) + t_3^3.$$

This gives case (ix).

If t_3^2 and t_2^3 do not enter in l_2 and in m_2 but t_2t_3 enters in one of these polynomials, then we get $b + c = 0 \mod n$. If $t_3^2t_2$ enters in l_3, then $b + 2c = 0 \mod n$, hence $4c = 0$ and $n = 4, (a, b, c) = (0, 2, 1)$ or $(0, 2, 3)$. This is case (vi).

Case 3: $0, a, b, c$ are all distinct. Note that if two of (a, b, c) are equal, then, by scaling and permuting coordinates, we will be in the previous Cases. This obviously implies that $n > 3$. Also monomials $t_i^2t_j$ and $t_it_j^2$ cannot both enter in f.

Case 3a. All the reference points P_i belong to the surface.

In this case f does not contain cubes of the variables t_i and we can write

$$f = t_0^2A_1(t_1, t_2, t_3) + t_1^2B_1(t_0, t_2, t_3) + t_2^2C_1(t_0, t_1, t_3) + t_3^2D_1[t_0, t_1, t_2],$$

where A_1, B_1, C_1, D_1 are nonzero linear polynomials. Since all $0, a, b, c$ are distinct, each of these linear polynomials contains only one variable. If the coefficients at t_i and t_j contain the same variable t_k, then the plane $V(t_k)$ is tangent to the surface along a line. It is easy to see that this does not happen for a nonsingular surface. Thus, without loss of generality, we may assume that

$$f = t_0^2t_1 + t_1^2t_2 + t_2^2t_3 + t_3^2t_0.$$

Then $a+b = 2b+c-a = 2c-a = 0 \mod n$. This implies $n = 5, (a, b, c) = (4, 1, 2)$. This is case (viii).

Case 3b. Three reference points belong to the surface.

By scaling and permuting variables we may assume that p_1 does not belong to $V(f)$. The equation contains t_0^3 but does not contain the cubes of other variables. Since f is σ-invariant, t_0^2 does not enter in f. We can write

$$f = t_0^3 + t_0f_2(t_1, t_2, t_3) + f_3(t_1, t_2, t_3). \tag{9.59}$$

Each line $\ell_i = \overline{p_1, p_i}$ does not belong to the surface and contains two fixed points of σ. Suppose each line ℓ_i intersects $V(f)$ only at one point p_i. Then f_2 does not contain squares of the variables and f_3 contains squares of each variable but not cubes. Without loss of generality, we may assume that f_3 contains $t_2^2t_3$. Then $2b + c = 0 \mod n$. Since $t_2t_3^2$ does not enter in f_3, the monomial $t_3^2t_1$ must enter. This gives $2c + a = 0 \mod n$. Solving for (a, b, c) we find that $n = 9, (a, b, c) = (4, 1, 7)$. The polynomial f_2 cannot contain t_it_j and hence is equal to zero. This gives us case (xiv).

Now we are in the situation when one of the lines ℓ_i intersects $V(f)$ at a point p different from p_i. If there is no other point in the intersection, then p is a third fixed point of σ on the line. This is impossible, and therefore ℓ_i intersects the surface at three distinct points p_i, p, q. Since σ permutes P and Q, we see that the restriction of σ^2 to ℓ_i is the identity. Without loss of generality, we may assume that $i = 2$ and, hence $2a = 0 \mod n$. Obviously, $t_1^2t_2, t_1^2t_3$ do not enter in f_3 and one of t_2^2 or t_3^2 does not enter in f_2. Assume t_2^2 does not enter in f_2. Then the monomials $t_2^2t_1$ or $t_2^2t_3$ enters in g_3 with nonzero coefficient. In the first case $2b + a = 0 \mod n$. This gives $n = 4, (a, b, c) = (2, 1, 3)$ or $(2, 3, 1)$. This leads to case (vii).

In the second case, we get $2b + c = 0 \mod n$. Since $t_3^2t_2$ does not enter in f_3, $t_3^2t_1$ must enter giving $2c + a = 0 \mod n$. This easily gives $n = 8$ and $(a, b, c) = (4, 3, 2)$. This leads to case (xiii).

Case 3c. Two reference points do not belong to the surface. We may assume that p_1, p_2 are not in the surface. Thus t_0^3, t_1^3 enter in f, hence $3a = 0 \mod n$. We may assume that f is as in (9.59), where t_1 enters in f_3. Clearly, t_1^2 does not enter in g_2. If t_3^2 (or t_2^2) enters in g_2, then $2c = 0 \mod n$, and we get $n = 6, (a, b, c) = (2, b, 3)$ or $(4, b, 3)$. Since $b \neq 3$, t_2^2 does not enter in g_2. Thus $t_2^2t_3$ or $t_2^2t_1$ enter in g_3. In the first case $2b + c = 0 \mod 6$, hence $2b = 3 \mod 6$ which is impossible. Thus $t_2^2t_1$ enters giving $2b + a = 0 \mod 6$. This gives case $(a, b, c) = (4, 1, 3)$ or $(2, 5, 3)$. This is case (xii).

Now we may assume that t_3^2 and t_2^2 do not enter in f_2. If $t_2^2t_1$ enters, we are led to the previous case (xii). So we may assume that $t_2^2t_3$ enters giving $2b + c = 0$. This implies that $t_3^2t_1$ enters, hence $2c + a = 0 \mod n$. This easily gives $n = 12, (a, b, c) = (4, 1, 10)$. This is case (xv).

Case 3d. Three reference points do not belong to the surface.

We may assume that p_1, p_2, p_3 are not in the surface. Thus t_0^3, t_1^3, t_2^3 enter in f, hence $3a = 3b = 0 \mod n$. We may assume that f is as in (9.59), where t_1, t_2 enter in f_3. Clearly, t_1^2, t_2^2 do not enter in f_2. If t_3^2 enters in f_2, then $2c = 0 \mod n$, and we get $n = 6, (a, b, c) = (2, 4, 3)$ or $(4, 2, 3)$. This gives case (xi).

Assume t_3^2 does not enter in f_2. Without loss of generality, we may assume that $t_3^2t_1$ enters in f_3. This gives $2c + a = 0 \mod n$. From this follows that $n = 6$ and $(a, b, c) = (4, 2, 1)$. This case is isomorphic to case (xi).

Case 3e. No reference point belongs to the surface.

In this case each t_i^3 enters in f, hence $3a = 3b = 3c = 0 \mod n$. This is impossible for $n > 3$.

□

In the natural representation of Aut(S) in $W(\mathbf{E}_6)$ each nontrivial automorphism σ defines a conjugacy class in $W(\mathbf{E}_6)$. The following table gives the list of the conjugacy classes. This can be found in [90], [43], [277].

Here we mark with the cross the conjugacy classes realized by automorphisms of nonsingular cubic surfaces. Also $\#C(w)$ denotes the cardinality of the centralizer of an element w from the conjugacy class, Tr denotes the trace in the Picard lattice (equal to the trace in the root lattice plus 1), Char denotes the characteristic polynomial in Pic(S) and $p_e(t) = t^e + t^{e-1} + \cdots + 1$.

	Atlas	Carter	Manin	Ord	$\#C(w)$	Tr	Char
x	1A	$\emptyset$	c_{25}	6	51840	7	$(t-1)^7$
x	2A	$4A_1$	c_3	2	1152	-1	$p_1^4(t-1)^3$
x	2B	$2A_1$	c_2	2	192	3	$p_1^2(t-1)^5$
	2C	A_1	c_{16}	2	1440	5	$p_1(t-1)^6$
	2D	$3A_1$	c_{17}	2	96	1	$p_1^3(t-1)^4$
x	3A	$3A_2$	c_{11}	3	648	-2	$p_2^3(t-1)$
x	3C	A_2	c_6	3	216	4	$p_2(t-1)^5$
x	3D	$2A_2$	c_9	3	108	1	$p_2^2(t-1)^3$
x	4A	$D_4(a_1)$	c_4	4	96	3	$(t^2+1)^2(t-1)^3$
x	4B	A_1+A_3	c_5	4	16	1	$p_1p_3(t-1)^3$
	4C	$2A_1+A_3$	c_{19}	4	96	-1	$p_1^2p_3(t-1)^3$
	4D	A_3	c_{18}	4	32	3	$p_3(t-1)^4$
x	5A	A_4	c_{15}	5	10	2	$p_4(t-1)^3$
x	6A	$E_6(a_2)$	c_{12}	6	72	2	$p_2(t^2-t+1)^2(t-1)$
x	6C	D_4	c_{21}	6	36	1	$p_1^2(t^2-t+1)(t-1)^3$
x	6E	A_1+A_5	c_{10}	6	36	-1	$p_1p_5(t-1)$
x	6F	$2A_1+A_2$	c_8	6	24	0	$p_1^2p_2(t-1)^3$
	6G	A_1+A_2	c_7	6	36	2	$p_1p_2(t-1)^4$
	6H	A_1+2A_2	c_{10}	6	36	-1	$p_1p_2^2(t-1)^2$
	6I	A_5	c_{23}	6	12	1	$p_5(t-1)^2$
x	8A	D_5	c_{20}	8	8	1	$p_1(t^4+1)(t-1)^2$
x	9A	$E_6(a_1)$	c_{14}	9	9	1	$(t^6+t^3+1)(t-1)$
	10A	A_1+A_4	c_{25}	10	36	0	$p_1p_4(t-1)^2$
x	12A	E_6	c_{13}	12	12	0	$p_2(t^4-t^2+1)(t-1)$
	12C	$D_5(a_1)$	c_{24}	12	12	2	$(t^3+1)(t^2+1)(t-1)^2$

Table 9.3: Conjugacy classes in $W(\mathbf{E}_6)$

To determine to which conjugacy class our σ corresponds under the Weyl representation we use the topological Lefschetz Fixed-Point Formula.

The next Theorem rewrites the list from Lemma 9.5.5 in the same order, renaming the cases with indication to which conjugacy class they correspond. Also, we simplify the formulae for f by scaling, and reducing a cubic ternary form to the Hesse form, and a cubic binary form to sum of cubes, and a quadratic binary forms to the product of the variables. Each time we use that the forms are nondegenerate because the surface is nonsingular.

Theorem 9.5.6. *Let S be a nonsingular cubic surface admitting a non-trivial automorphism σ of order n. Then S is equivariantly isomorphic to one of the following surfaces $V(f)$ with*

$$\sigma = [x_0, \epsilon_n^a x_1, \epsilon_n^b x_2, \epsilon_n^c x_3] : \tag{9.60}$$

- $4A_1$ $(n=2)$, $(a,b,c)=(0,0,1)$,

$$f = t_3^2 f_1(t_0,t_1,t_2) + t_0^3 + t_1^3 + t_2^3 + \alpha t_0 t_1 t_2;$$

- $2A_1$ $(n=2)$, $(a,b,c)=(0,1,1)$,
$$f = t_0t_2(t_2+\alpha t_3)+t_1t_3(t_2+\beta t_3)+t_0^3+t_1^3;$$
- $3A_2$ $(n=3)$, $(a,b,c)=(0,0,1)$,
$$f = t_0^3+t_1^3+t_2^3+t_3^3+\alpha t_0t_1t_2;$$
- A_2 $(n=3)$, $(a,b,c)=(0,1,1)$,
$$f = t_0^3+t_1^3+t_2^3+t_3^3;$$
- $2A_2$ $(n=3)$, $(a,b,c)=(0,1,2)$,
$$f = t_0^3+t_1^3+t_2t_3(t_0+at_1)+t_2^3+t_3^3;$$
- $D_4(a_1)$ $(n=4)$, $(a,b,c)=(0,2,1)$,
$$f = t_3^2t_2+f_3(t_0,t_1)+t_2^2(t_0+\alpha t_1);$$
- A_3+A_1 $(n=4)$, $(a,b,c)=(2,1,3)$,
$$f = t_0^3+t_0t_1^2+t_1t_3^2+t_1t_2^2;$$
- A_4 $(n=5)$, $(a,b,c)=(4,1,2)$,
$$f = t_0^2t_1+t_1^2t_2+t_2^2t_3+t_3^2t_0;$$
- $E_6(a_2)$ $(n=6)$, $(a,b,c)=(0,3,2)$,
$$f = t_0^3+t_1^3+t_3^3+t_2^2(\alpha t_0+t_1);$$
- D_4 $(n=6)$, $(a,b,c)=(0,2,5)$,
$$f = f_3(t_0,t_1)+t_3^2t_2+t_2^3;$$
- A_5+A_1 $(n=6)$, $(a,b,c)=(4,2,1)$,
$$f = t_3^2t_1+t_0^3+t_1^3+t_2^3+\lambda t_0t_1t_2;$$
- $2A_1+A_2$ $(n=6)$, $(a,b,c)=(4,1,3)$,
$$f = t_0^3+\beta t_0t_3^2+t_2^2t_1+t_1^3;$$
- D_5 $(n=8)$, $(a,b,c)=(4,3,2)$,
$$f = t_3^2t_1+t_2^2t_3+t_0t_1^2+t_0^3;$$

- $E_6(a_1)$ $(n = 9)$, $(a,b,c) = (4,1,7)$,

$$f = t_3^2 t_1 + t_1^2 t_2 + t_2^2 t_3 + t_0^3;$$

- E_6 $(n = 12)$, $(a,b,c) = (4,1,10)$,

$$f = t_3^2 t_1 + t_2^2 t_3 + t_0^3 + t_1^3.$$

Proof. We will be computing the trace of σ^* by using the *Lefschetz fixed-point formula*

$$\sum (-)^i \mathrm{Tr}(g|H^i(X,\mathbb{Q})) = e(\mathrm{Fix}(g)).$$

We use the classification from Lemma 9.5.5.

Order 2.

In case (i), the fixed locus is the nonsingular elliptic curve given by equations $t_3 = f_3 = 0$ and isolated point $[0,0,0,1]$. The Euler-Poincaré characteristic of the fixed locus is equal to 1. Hence the trace in Pic(S) is equal to -1. This gives the conjugacy class $2A$. In case (ii), the fixed locus is the line $t_0 = t_1 = 0$ and three isolated points lying on the line $t_2 = t_3$ (not contained in the surface). The Euler-Poincaré characteristic of the fixed locus is equal to 5. Hence the trace in Pic(S) is equal to 3. This gives the conjugacy class $2A_1$.

Order 3.

In case (iii), the fixed locus is a nonsingular elliptic curve given by equations $t_3 = f_3 = 0$. The Euler-Poincaré characteristic of the fixed locus is equal to 0. Hence the trace in Pic(S) is equal to -2. This gives the conjugacy class $3A_2$.

In case (iv), the fixed locus is the set of 6 points lying on the lines $t_0 = t_1 = 0$ and $t_2 = t_3 = 0$. Here we use that the polynomials f_3, g_3 do not have multiple roots since otherwise S is singular. The Euler-Poincaré characteristic of the fixed locus is equal to 6. Hence the trace in Pic(S) is equal to 4. This gives the conjugacy class A_2.

In case $(iv)'$, the fixed locus consists of 3 points lying on the line $t_2 = t_3 = 0$. Hence the trace in Pic(S) is equal to 1. This gives the conjugacy class $2A_2$.

In case (v), the fixed locus is the set of 3 points lying on the line $t_2 = t_3 = 0$. Again we use that f_3 does not have multiple roots. The Euler-Poincaré characteristic of the fixed locus is equal to 3. Hence the trace in Pic(S) is equal to 1. This gives the conjugacy class $2A_2$.

Order 4.

In case (vi), the fixed locus is the set of 5 points lying on the lines $t_0 = t_1 = 0$ and two reference points $P_3 = [0,0,1,0]$ and $P_4 = [0,0,0,1]$. The Euler-Poincaré characteristic of the fixed locus is equal to 5. Hence the trace in Pic(S) is equal to 3. This gives the conjugacy class $D_4(a_1)$ or $4D$. To distinguish the two classes, we notice that σ^2 acts as in case (i). This implies that σ^2 belongs to the conjugacy class $4A_1$. On the other hand, the characteristic polynomial of $4D$ shows that $4D^2$ is the conjucacy class $2A_1$. Thus we have the conjugacy class $D_4(a_1)$.

In case (vii), we have three isolated fixed points $[1,0,0,0], [0,1,0,0], [0,0,1,0]$. Thus the trace is equal to 1. This gives the conjugacy class $A_1 + A_3$.

Order 5.

This is the unique conjugacy class of order 5. It is realized in case (viii). We have 4 isolated fixed points confirming that the trace is equal to 2.

Order 6.

In case (ix) we have 4 isolated fixed points so that the trace is equal to 2. This gives possible conjugacy classes $E_6(a_2), D_4, A_1 + A_2$. We know that our surface is σ^2-equivariantly isomorphic to a surface from case (iii). Thus σ^2 belongs to the conjugacy class $3A_2$. Using the characteristic polynomials we check that only the square of the conjugacy class $E_6(a_2)$ is equal to $3A_2$.

In case (x) we have 4 isolated fixed points. This gives that the trace is equal to 2. It is clear that σ^3 acts as in case (i), thus σ^3 belongs to $4A_1$. Also σ^2 acts as in case (iv). This shows that σ^2 belongs to A_2. Comparing the characteristic polynomials, this leaves only the possibility that σ belongs to D_4.

In case (xi) we have only one isolated fixed point $[0, 0, 0, 1]$. This gives that the trace is equal to -1 and hence σ belongs to $A_1 + A_5$.

In case (xii) we have 2 isolated fixed points so that the trace is equal to 2. The only conjugacy class with trace zero is $2A_1 + A_2$.

Order 8.

D_5 is the unique conjugacy class of order 8. Its trace is 1. This agrees with case (xiii), where we have 3 fixed points.

Order 9.

$E_6(a_1)$ is the unique conjugacy class of order 9. Its trace is 1. This agrees with case (xiv), where we have 3 fixed points.

Order 12.

We have 2 fixed points giving the trace of σ equal to 0. This chooses the conjugacy class E_6. □

Remark 9.5.3. Some of the conjugacy classes (maybe all ?) are realized by automorphisms of minimal resolutions of singular surfaces. Also two non-conjugate elements from Aut(S) may define the same conjugacy class in $W(\mathbf{E}_6)$. An example is an automorphism σ from the conjugacy class $3A_2$ and its square.

9.5.4 Automorphisms groups

In the following table we use the notation $\mathcal{H}_3(3)$ for the Heisenberg group of unipotent 3×3-matrices with entries in $\mathbb{F}_3$.

Theorem 9.5.7. *The following is the list of all possible groups of automorphisms of nonsingular cubic surfaces.*

Proof. Let S be a nonsingular cubic surface and G be a subgroup of Aut(S). Suppose G contains an element of order 3 from the conjugacy class A_2. Applying Theorem 9.5.6, we see that S is isomorphic to the Fermat surface $V(t_0^3 + t_1^3 + t_2^3 + t_3^3)$. It has 27 lines given by the equations $t_0 + \epsilon t_1 = 0, t_2 + \eta t_3 = 0, \epsilon^3 = \eta^3 = -1$, or their transforms under permuting the variables. It is clear that any automorphism of

Type	Order	Structure	$f(t_0,t_1,t_2,t_3)$	Parameters
I	648	$3^3:\mathfrak{S}_4$	$t_0^3+t_1^3+t_2^3+t_3^3$	
II	120	$\mathfrak{S}_5$	$t_0^2t_1+t_0t_2^2+t_2t_3^2+t_3t_1^2$	
III	108	$\mathcal{H}_3(3):4$	$t_0^3+t_1^3+t_2^3+t_3^3+6at_1t_2t_3$	$20a^3+8a^6=1$
IV	54	$\mathcal{H}_3(3):2$	$t_0^3+t_1^3+t_2^3+t_3^3+6at_1t_2t_3$	$a-a^4\neq 0$, $8a^3\neq -1$, $20a^3+8a^6\neq 1$
V	24	$\mathfrak{S}_4$	$t_0^3+t_0(t_1^2+t_2^2+t_3^2)$ $+at_1t_2t_3$	$9a^3\neq 8a$, $8a^3\neq -1$
VI	12	$\mathfrak{S}_3\times 2$	$t_2^3+t_3^3+at_2t_3(t_0+t_1)+t_0^3+t_1^3$	$a\neq 0$
VII	8	8	$t_3^2t_2+t_2^2t_1+t_0^3+t_0t_1^2$	
VIII	6	$\mathfrak{S}_3$	$t_2^3+t_3^3+at_2t_3(t_0+bt_1)+t_0^3+t_1^3$	$a^3\neq -1$
IX	4	4	$t_3^2t_2+t_2^2t_1+t_0^3+t_0t_1^2+at_1^3$	$a\neq 0$
X	4	2^2	$t_0^2(t_1+t_2+at_3)+t_1^3+t_2^3$ $+t_3^3+6bt_1t_2t_3$	$8b^3\neq -1$
XI	2	2	$t_1^3+t_2^3+t_3^3+6at_1t_2t_3$ $+t_0^2(t_1+bt_2+ct_3)$	$b^3,c^3\neq 1$, $b^3\neq c^3$, $8a^3\neq -1$

Table 9.4: Groups of automorphisms of cubic surfaces

S permutes the planes $t_i+\epsilon t_j=0$ and hence $\mathrm{Aut}(S)$ consists of permutations of the variables and multiplying the variables by cube roots of unity. This gives case I. It is easy to see that each plane $t_i+\epsilon t_j=0$ is a tritangent plane with an Eckardt point. Thus we have 18 Eckardt points, maximal possible.

Assume that G contains an element of order 5. Applying Theorem 9.5.6, we see that S is isomorphic to the *Clebsch diagonal surface*

$$t_0^2t_1+t_0t_2^2+t_2t_3^2+t_3t_1^2=0. \tag{9.61}$$

Consider the embedding of S in $\mathbb{P}^4$ given by the linear functions

$$\begin{aligned} z_0 &= t_0+t_1+t_2+t_3 \\ z_1 &= \zeta_5^3t_0+\zeta_5^4t_1+\zeta_5^2t_2+\zeta_5t_3, \\ z_2 &= \zeta_5t_0+\zeta_5^3t_1+\zeta_5^4t_2+\zeta_5^2t_3, \\ z_3 &= \zeta_5^4t_0+\zeta_5^2t_1+\zeta_5t_2+\zeta_5^3t_3, \\ z_4 &= \zeta_5^2t_0+\zeta_5t_1+\zeta_5^3t_2+\zeta_5^4t_3. \end{aligned} \tag{9.62}$$

Then one easily checks that $\sum_{i=0}^4 z_i=0$ and (9.61) implies that also $\sum_{i=0}^4 z_i^3=0$. This shows that S is isomorphic the following surface in $\mathbb{P}^4$:

$$\sum_{i=0}^4 z_i^3=\sum_{i=0}^4 z_i=0. \tag{9.63}$$

These equations exhibit an obvious symmetry which is the group $\mathfrak{S}_5$. The line

$$z_0 = z_1 + z_2 = z_3 + z_4 = 0$$

lies on S. Its $\mathfrak{S}_5$-orbit consists of 15 lines. The remaining 12 lines form a double-six. Their equations are as follows. Let ω be a primitive 5-th root of unity. Let $\sigma = (a_1, \ldots, a_5)$ be a permutation of $\{0,1,2,3,4\}$. Each line ℓ_σ spanned by the points $[\omega^{a_1}, \ldots, \omega^{\sigma_5}]$ and $[\omega^{-a_1}, \ldots, \omega^{-a_5}]$ belongs to the surface. This gives $12 = 5!/10$ different lines. One checks immdiately that two lines ℓ_σ and $\ell_{\sigma'}$ intersect if and only if $\sigma' = \sigma \circ \tau$ for some odd permutation τ. The group $\mathfrak{S}_5$ (as well its subgroup $\mathfrak{S}_4$) acts transitively on the double-six. The alternating subgroup stabilizes a sixer.

Observe that S has 10 Eckardt points $[1,-1,0,0,0]$ and other ones obtained by permutations of coordinates. Also notice that any point, say $[1,-1,0,0]$ is joined by a line in the surface to three other points $[0,0,1,-1,0], [0,0,0,1,-1], [0,0,1,0,-1]$. The graph whose vertices are Eckardt points and edges are the lines is a famous tri-valent *Petersen graph* whose group of symmetry is isomorphic to $\mathfrak{S}_5$.

Assume G is larger than $\mathfrak{S}_5$. Consider the representation of G in the symmetry group of the graph of Eckardt points. Its image is equal to $\mathfrak{S}_5$, hence its kernel is non-trivial. Let H be a maximal subgroup of $W(\mathbf{E}_6)$ which contains G. It follows from Lemma 9.5.1 that G must contain $\mathfrak{S}_6$ or an involution. The restriction of the representation to $\mathfrak{S}_6$ must be trivial, since the kernel is non-trivial and is not equal to $\mathfrak{A}_6$. This is impossible since $\mathfrak{S}_6$ contains our $\mathfrak{S}_5$. If the kernel contains an involution, then the involution fixes 10 points. Since no involutions in $W(\mathbf{E}_6)$ has trace equal to 8, we get a contradiction. Thus Aut$(S) \cong \mathfrak{S}_5$.

Assume that G contains an element $\sigma : [t_0, t_1, t_2, t_3] \mapsto [\epsilon_3 t_0, t_1, t_2, t_3]$ from the conjugacy class $3A_2$. Then we are in case (iii) of Theorem 9.5.6. The plane cubic curve $C = V(t_1^3 + t_2^3 + t_3^3 + a t_1 t_2 t_3)$ has the projective group of automorphisms isomorphic to $3^2 : 2$. Its normal subgroup 3^2 is generated by a cyclic permutation of coordinates and the transformation $[t_1, t_2, t_3] \mapsto [t_1, \epsilon_3 t_2, \epsilon_3^2 t_3]$. Together with σ this generates a group G_1 of order 54 isomorphic to $3.(3^2 : 2)$. Note that for a special value a, C may acquire an additional isomorphism of order 4 or 6). It happens when $1 - 20a^3 - 8a^6 = 0$ or $a(a^3 - 1) = 0$, respectively (see Exercises to Chapter 3). If $\zeta = \frac{-1+\sqrt{3}}{2}$ is a root of the first equation, then the extra automorphism of order 4 is given by the formula

$$[t_1, t_2, t_3] \mapsto [t_1 + t_2 + t_3, t_1 + \zeta t_2 + \zeta^2 t_3, t_1 + \zeta^2 t_2 + \zeta t_3].$$

It is easy to see that the group Aut(S) is isomorphic to $3.(3^2 : 4)$, the center is generated by the transformation which multiplies x_0 by a third root of unity.

If $a^4 = a$, the curve C is projectively isomorphic to the Fermat cubic $t_1^3 + t_2^3 + t_3^3$, hence we are in case I. According to example 9.1.2 S contains 9 tritangent planes with Eckardt points. Each plane is the preimage of a line under the projection to the plane Π containing the curve f of fixed points.

Suppose there is a symmetry τ not belonging to G_1. Since G_1 acts transitively on the set of Eckardt points, we may assume that τ fixes a tritangent plane containing an Eckardt point. Thus τ fixes the plane Π and hence is an automorphism of the plane cubic C. This proves that $G = G_1$ if C has no extra automorphisms, $G = G'$ if C has an automorphism of order 4, and S is of type I if C has an automorphism of order 6.

Assume that S contains an element σ of order 8. Then S is isomorphic to the surface from case (xiii) of Theorem 9.5.6. The only maximal subgroup of $W(\mathbf{E}_6)$ which contains an element of order 8 is a subgroup H of order 1152. As we know it stabilizes a tritangent plane. In our case the tritangent plane is $t_2 = 0$. It has the Eckardt point $x = [0,0,0,1]$. Thus $G = \mathrm{Aut}(S)$ is a subgroup of the linear tangent space T_xS. If any element of G acts identically on the set of lines in the tritangent plane, then it acts identically on the projectivized tangent space, and hence G is a cyclic group. Obviously this implies that G is of order 8. Assume that there is an element τ which permutes cyclically the lines. Let G' be the subgroup generated by σ and τ. Obviously, $\tau^3 = \sigma^k$. Since G does not contain elements of order 24, we may assume that $k = 2$ or 4. Obviously, τ normalizes $\langle\sigma\rangle$ since otherwise we have two distinct cyclic groups of order 8 acting on a line with a common fixed point. It is easy to see that this is impossible. Since $\mathrm{Aut}(\mathbb{Z}/8\mathbb{Z})) \cong (\mathbb{Z}/2\mathbb{Z})^2$ this implies that σ and τ commute. Thus $\sigma\tau$ is of order 24 which is impossible. This shows that $\mathrm{Aut}(S) \cong \mathbb{Z}/8\mathbb{Z}$.

It is easy to see that the square of the conjugacy classes $D_4, 2A_1+A_2$ is equal to A_2, the square of $E_6(a_2)$ is equal to $3A_2$, and the square of $A_1 + A_5$ is equal to $2A_2$. Also the cube of the conjugacy class $E_6(a_1)$ and the fourth power of E_6 is equal to $3A_2$. Since surfaces with automorphism of order 3 from the conjugacy classes $3A_2, A_2$, and also with an automorphisms of order 5 and 8 have been already classified we may assume that $\mathrm{Aut}(S)$ does not contain elements of order 5,8,9,12. By the previous analysis we may assume that any element of order 3 belongs to the conjugacy class $2A_2$, and elements of order 6 to the conjugacy class $E_6(a_2)$ or $A_1 + A_5$.

Assume $\mathrm{Aut}(S)$ contains an element σ from conjugacy class $2A_2$. Then the surface is σ-equivariantly isomorphic to the surface from case (v) of Theorem 9.5.6.

$$t_2^3 + t_3^3 + t_2t_3t_0 + f_3(t_0,t_1) = 0.$$

We can reduce this equation to the form

$$t_2^3 + t_3^3 + t_2t_3(t_0 + at_1) + t_0^3 + t_1^3 = 0. \tag{9.64}$$

The fixed points of σ are the points $q_i = [a_i, b_i, 0, 0]$, where $f_3(a_i, b_i) = 0$. Observe that we have 3 involutions $\sigma_i, i = 0, 1, 2$, defined by

$$[t_0, t_1, t_2, t_3] \to [t_0, t_1, \zeta_3^i t_3, \zeta_3^{2i} t_2].$$

The set of fixed points of σ_i is the nonsingular plane section $t_3 = \zeta_i^i t_2$ and an isolated fixed point $p_i = [0,0,1,-\zeta_3^i]$. Thus each σ_i belongs to the conjugacy class $4A_1$. The point p_i is an Eckardt point in the tritangent plane $t_0 - \zeta_3^i t_2 - \zeta_3^{2i} t_3 = 0$. Notice that they lie on the line $t_0 = t_1 = 0$. This line is uniquely determined by σ, it is spanned by isolated fixed point of σ in $\mathbb{P}^3$. It is immediately checked that $\sigma_i \circ \sigma_j = (\sigma_j \circ \sigma_i)^{-1} = \sigma^{i+2j}$ for $i \ne j$. This implies that the group G_1 generated by $\sigma, \sigma_0, \sigma_1, \sigma_2$ is isomorphic to $\mathfrak{S}_3$. The element σ belongs to a unique such group determined by the line ℓ. The elements of order 2 in G_1 correspond to the Eckardt points on l.

Suppose one of the fixed point of σ, say q_1, is an Eckardt point. A straightforward computation shows that this happens only if in equation (9.64) $b^3 = -1$. Also it shows that only one of the fixed points could be an Eckardt point.

Let P_3 be the 3-Sylow subgroup of $G = \mathrm{Aut}(S)$. It is a cyclic group of order 3. Since $\#\mathrm{Aut}(S) = 2^a \cdot 3^b$, the Sylow Theorems gives that the number of 3-Sylow subgroups divides $2^a 3^b$ and $\equiv 1 \mod 3$. This shows that this number is equal to 2^{2k}. If $k > 1$, then G is contained in a maximal subgroups of $W(\mathbf{E}_6)$ of order divisible by 2^4. This is a group isomorphic to $2^4 : \mathfrak{S}_5$ or $\mathfrak{S}_6 \times 2$. In the first case G stabilizes a line in S, and then any element of order 3 has 2 fixed points on this line. But, as we saw in above the fixed points of an element of order 3 do not lie on a line contained in the surface. In the second case, we use that any subgroup of $\mathfrak{S}_6$ containing 16 subgroups of order 3 must coincide with $\mathfrak{S}_6$. Certainly, it is impossible. Thus $k = 0$ or 1.

If $k = 0$, then G has a unique cyclic subgroup $\langle\sigma\rangle$ of order 3. So, either $\mathrm{Aut}(S) = G_1 \cong \mathfrak{S}_3$, or G contains an involution $\tau \notin G_1$. If $G \neq \mathfrak{S}_3$, the natural homomorphism $G/\langle\sigma\rangle \to \mathrm{Aut}(\langle\sigma\rangle) \cong \mathbb{Z}/2\mathbb{Z}$ has a non-trivial kernel of order 2^{a-1}. Let τ be an element of order 2 from the kernel. Since it commutes with σ, it leaves invariant the set of 3 collinear fixed points of σ. Thus it fixes $a = 1$ or 3 fixed points of σ. If τ is of type $2A_1$, then it has 5 isolated fixed points. The group $\langle\sigma\rangle$ leaves this set invariant and has 2 or 5 fixed points in this set. This shows that τ must be of type $4A_1$ and hence its isolated fixed point is one of the fixed points of σ which is an Eckardt point. As we have observed earlier, there could be only one such point. Hence there is only one additional element of order 2. The line joining this point with an Eckardt point p_i must be contained in S, since otherwise, by Proposition 9.1.14 we have a third Eckardt point on this line. Thus τ commutes with any involution σ_i in G_1. Hence $\mathrm{Aut}(S) \cong \mathfrak{S}_3 \times 2$. The involution τ fixes the isolated fixed points of each σ_i. This shows that in the one-dimensional subspace $x_0 = x_1 = 0$, it has 3 fixed points. This implies that it is the identity in this subspace. Thus τ acts nontrivially only on the variables t_0, t_1. This implies that $b = 1$ and the equation of S is of type VI. It is easy to see that automorphism of order 6 in G belongs to the conjugacy class $A_1 + A_5$.

If $k = 1$, G is isomorphic to a transitive subgroup of $\mathfrak{S}_4$ which contains an $\mathfrak{S}_3$. It must be isomorphic to $\mathfrak{S}_4$. Each subgroup of $\mathrm{Aut}(S)$ isomorphic to $\mathfrak{S}_3$ defines a line with 3 Eckardt points. Since any two such subgroups have a common element of order 2, each line intersects other 3 lines at one point. This shows that the four lines are coplanar and form a complete quadrangle in this plane. Also, since each of the three diagonals d_i has only two Eckardt points on it, we see that each diagonal is contained in the surface. Now choose coordinates such that the plane of the quadrangle has equation $t_0 = 0$ and the diagonals have the equations $t_0 = t_i = 0$. The equation of the surface must now look as follows.

$$at_0^3 + t_0^2 f_1(t_1, t_2, t_3) + t_0 f_2(t_1, t_2, t_3) + ct_1t_2t_3 = 0.$$

The group $\mathrm{Aut}(S)$ leaves the quadrangle invariant and hence acts by permuting the coordinates t_1, t_2, t_3 and multiplying them by ± 1. This easily implies that the equation can be reduced to the form of type V.

The surface with automorphism group isomorphic to $\mathfrak{S}_3$ has equation of type VIII.

Assume that $\mathrm{Aut}(S)$ contains an element σ from conjugacy class $2A_1$. Then the equation of the surface looks like

$$at_0t_2t_3 + t_1(t_2^2 + t_3^2 + bt_2t_3) + t_0^3 + t_1^3 = 0.$$

It exhibits an obvious symmetry of order 3 defined by

$$[t_0, t_1, t_2, t_3] \mapsto [t_0, t_1, \zeta_2 t_2, \zeta_2^2 t_3].$$

Thus we are in one of the above cases.

Suppose $\mathrm{Aut}(S)$ contains an element σ of order 4. If σ belongs to the conjugacy class 4B, then σ^2 belongs to $2A_1$ and hence this case has been already considered. If σ belongs to $4A_1$ then the equation of the surface looks like

$$t_2 t_3^2 + t_0^3 + a t_1^3 + t_2^2(t_0 + t_1) = 0.$$

Here we have to assume that the surface is not isomorphic to the surface of type VII. It follows from the proof of the next Corollary that in all previous cases, except type VII, the automorphism group is generated by involutions of type $4A_1$. Thus our surface cannot be reduced to one of the previous cases.

Finally it remains to consider the case when only involutions of type $4A_1$ are present. Suppose we have 2 such involution. They define two Eckardt points p_1 and p_2. In order the involution commute the line joining the two points must be contained in S. Suppose we have a third involution defining a third Eckardt point p_3. Then we have a tritangent plane formed by the lines $\overline{p_i, p_j}$. Obviously, it must coincide with each tritangent plane corresponding to the Eckardt points p_i. This contradiction shows that we can have at most 2 commuting involutions. This gives the last two cases of our theorem. The condition that there is only one involution of type $4A_1$ is that the line $l_1(t_0, t_1, t_2) = 0$ does not pass through a flex point of $f_3(t_0, t_1, t_2) = 0$. $\square$

Corollary 9.5.8. *Let* $\mathrm{Aut}(S)^o$ *be the subgroup of* $\mathrm{Aut}(S)$ *generated by involutions of type* $4A_1$*. Then* $\mathrm{Aut}(S)^o$ *is a normal subgroup of* $\mathrm{Aut}(S)$ *such that the quotient group is either trivial or a cyclic group of order 2 or 4. The order is 4 could occur only for the surface of type VII. The order 2 occurs only for surfaces of type X.*

Proof. We do it case by case. For surfaces of type I, the group $\mathrm{Aut}(S)$ is generated by transformations of type

$$[t_0, t_1, t_2, t_3] \mapsto [t_0, t_1, \epsilon t_3, t_2],$$

where $\epsilon^3 = 1$. It is easy to see that it is an involution of type $4A_1$ corresponding to the Eckardt point $[0, 0, 1, -\epsilon]$.

For surfaces of type II given by equation (9.63), the group $\mathrm{Aut}(S)$ is generated by transpositions of coordinates. They correspond to involutions of type $4A_1$ associated with Eckardt points of type $[1, -1, 0, 0, 0]$.

In the case of surfaces of type III, we use that a line in $\mathbb{P}^3$ joining 2 Eckardt points contains the third Eckardt point. Thus any such line generate a subgroup isomorphic to $\mathfrak{S}_3$. We have 12 lines which contain 9 flex points. They are the projections of these lines in $\mathbb{P}^2$ from the center of projection $[1, 0, 0, 0]$. One can show that the group generated by these 12 subgroups must coincide with the whole group.

The remaining cases follow from the proof of the theorem.

$\square$

Exercises

9.1 Let $Y \subset \mathbb{P}^4$ be the image of $\mathbb{P}^2$ under a rational map given by the linear system of conics through a fixed point p.

(i) Show that Y is a surface of degree 3 and its projection from a general point O in $\mathbb{P}^4$ is a non-normal cubic surface in $\mathbb{P}^3$ of type (i) from Theorem 9.2.1.

(ii) Show that the projection of Y from a point O lying in the plane spanned by the image of the exceptional curve of the blow-up of $\mathbb{P}^2$ at p and the image of a line through p is a non-normal cubic surface of type (ii) from Theorem 9.2.1.

(iii) Show that any non-normal cubic surface in $\mathbb{P}^3$ which is not a cone can be obtained in this way.

9.2 Show that the dual of the 4-nodal cubic surfrace is isomorphic to the quartic surface given by the equation

$$\sqrt{t_0} + \sqrt{t_1} + \sqrt{t_2} + \sqrt{t_3} = 0.$$

9.3 Let $\tau : (x, y) \mapsto (x^{-1}, y^{-1})$ be the standard Cremona transformation. Show that τ extends to a biregular automorphism σ of a weak Del Pezzo surface S of degree 6 and the orbit space $S/\langle\sigma\rangle$ is isomorphic to a 4-nodal cubic surface.

9.4 Show that a cubic surface can be obtained as the blow-up of 5 points on $\mathbb{P}^1 \times \mathbb{P}^1$. Find the conditions on the 5 points such that the blow-up is isomorphic to a nonsingular cubic surface.

9.5 Compute the number of m-tuples of skew lines on a nonsingular surface for $m = 2, 3, 4, 5$.

9.6 Suppose a quadric intersects a cubic along the union of three conics. Show that the three planes defined by the conics pass through three lines in a tritangent plane.

9.7 Let Γ and Γ' be two rational normal cubics in $\mathbb{P}^3$ containing a common point p. For a general plane Π through p let $\Pi \cap \Gamma = \{p, p_1, p_2\}, \Pi \cap \Gamma' = \{p, p_1', p_2'\}$ and $f(p) = \overline{p_1, p_1} \cap \overline{p_1', p_2'}$. Consider the set of planes through p as a hyperplane H in the dual space $\check{\mathbb{P}}^3$. Show that the image of the rational map $H - \to \mathbb{P}^3, \Pi \mapsto f(\Pi)$ is a nonsingular cubic surface and every such cubic surface can be obtained in this way.

9.8 Show that the linear system of quadrics in $\mathbb{P}^3$ spanned by quadrics which contain a degree 3 rational curve on a nonsingular cubic surface S can be spanned by the quadrics defined by the minors of a matrix defining a determinantal reprfesentation of S.

9.9 Show that the linear system of cubic surfaces in $\mathbb{P}^3$ containing 3 skew lines defines a birational map from $\mathbb{P}^3$ to $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

9.10 Show that non-normal cubic surfaces are scrolls, i.e. contain a one-dimensional family of lines.

9.11 Show that all singular surfaces of type $VII, X, XI, XIII - XXI$ are isomorphic and there are two non-isomorphic surfaces of type XII.

9.12 Prove that the linear system of cubic surfaces in $\mathbb{P}^3$ containing three skew lines defines a birational map from $\mathbb{P}^3$ to $\mathbb{P}^7$ whose image is equal to the Segre variety $(\mathbb{P}^1)^3$.

9.13 Compute the number of determinantal representations of a singular cubic surface.

9.14 Find determinantal representations of an irreducible non-normal cubic surface.

9.15 Let ℓ be a line on a cubic surface with canonical singularities and E be its proper inverse transform on the corresponding weak Del Pezzo surface X. Let $\mathcal{N}$ be the sublattice of Pic(X)

spanned by irreducible components of exceptional divisors of $\pi : X \to S$. Define the multiplicity of ℓ by

$$m(l) = \frac{\#\{\sigma \in \mathrm{O}(\mathrm{Pic}(X)) : \sigma(E) - E \in \mathcal{N}\}}{\#\{\sigma \in \mathrm{O}(\mathrm{Pic}(X)) : \sigma(E) = E\}}.$$

Show that the sum of the multiplicities is always equal to 27.

9.16 Show that the 24 points of intersection of a Schur quadric with the corresponding double-six lie on the Hessian of the surface ([15], vol. 3, p. 211).

9.17 Consider a Cayley-Salmon equation $l_1l_2l_3 - l_1'l_2'l_3' = 0$ of a nonsingular cubic surface.

(i) Show that the six linear polynomials l_i, l_i' satisfy the following linear equations

$$\sum_{j=1}^{3} a_{ij}l_j = \sum_{j=1}^{3} a_{ij}'l_j' = 0,\ i = 1,2,3,$$

where

$$\sum_{i=1}^{3} a_{ij} = 0,\ j = 1,2,3, \quad a_{i1}a_{i2}a_{i3} = a_{i1}'a_{i2}'a_{i3}',\ i = 1,2,3.$$

(ii) Show that for each $i = 1,2,3$ the nine planes

$$a_{ij}l_i - a_{ij}'l_j' = 0,\ i,j = 1,2,3$$

contain 18 lines common to three planes. The 18 lines obtained in this way form three double-sixers associated to the pair of conjugate triads defined by the Cayley-Salmon equation.

(iii) Show that the Schur quadrics defined by the three double-sixers can be defined by the equations

$$\sum_{j=1}^{3} a_{2j}a_{3j}l_j^2 - \sum_{j=1}^{3} a_{2j}a_{3j}{l_j'}^2 = 0,$$

$$\sum_{j=1}^{3} a_{1j}a_{3j}l_j^2 - \sum_{j=1}^{3} a_{1j}a_{3j}{l_j'}^2 = 0,$$

$$\sum_{j=1}^{3} a_{1j}a_{2j}l_j^2 - \sum_{j=1}^{3} a_{1j}a_{2j}{l_j'}^2 = 0$$

([127]).

9.18 ([128]) Prove the following theorem of Schläfli: Given five skew lines in $\mathbb{P}^3$ and a line intersecting them all, there exists a unique cubic surface that contains a double-sixer including the seven lines.

9.19 Consider the Cremona hexahedral equations $\sum x_i^3 = \sum x_i = 0$ and $x_i - x_j = 0$. Show that these equations define a 4-nodal cubic surface.

9.20 Show that the pull-back of a bracket-function (ijk) under the Veronese map is equal to $(ij)(jk)(ik)$.

9.21 Show that the condition that 6 points in $\mathbb{P}^2$ lie on a conic is $(134)(156)(235)(246) - (135)(146)(234)(256 = 0$.

9.22 Let S be the cubic surface obtained by blowing up a semi-stable point set $(p_1, \ldots, p_6)$ in $\mathbb{P}^2$. Use the Clebsch transference principle to give the following interpretation of the morphism $\Psi : P_2^6 \to P_1^6 \cong \mathcal{S}_3 \subset \mathbb{P}^5$.

(i) Show that the projection of the set $(p_1,\ldots,p_6)$ to $\mathbb{P}^1$ with center at a general point $x \in \mathbb{P}^2$ is a set of distinct points in $\mathbb{P}^1$ whose orbit in P_1^6 depends only on the image of $(p_1,\ldots,p_6)$ in P_2^6.

(ii) Show that the projection map extends to a morphism from S to P_a^6 that coincides with the map Ψ.

(iii) Show that the image $(q_1,\ldots,q_6)$ of a points set $(p_1,\ldots,p_6)$ is a set such that the pairs $(q_1,q_2),(q_3,q_4),(q_5,q_6)$ are orbits of an involution of $\mathbb{P}^1$ if and only if the triples $(p_1,p_3,p_5),(p_1,p_4,p_6),(p_2,p_3,p_6)$ and (p_2,p_4,p_5) are on lines which form a quadrilateral.

9.23 Show that the Segre cubic primal is isomorphic to a tangent hyperplane section of the cubic fourfold with 9 lines given by the equation $xyz - uvw = 0$ (*Perazzo primal* [310], [16]).

9.24 Show that the dual surface of the Segre 4-nodal cubic is a Steiner quartic surface surface with equation $\sqrt{x}+\sqrt{y}+\sqrt{z}+\sqrt{z}=0$, cleared of radicals.

9.24 Consider the following *Cayley's family of cubic surfaces* in $\mathbb{P}^3$ with parameters l,m,n,k

$$w[x^2+y^2+z^2+w^2+(mn+\frac{1}{mn})yz+(ln+\frac{1}{ln})xz+(lm+\frac{1}{lm})xy$$

$$+(l+\frac{1}{l})xw+(m+\frac{1}{m})yw+(n+\frac{1}{n})zw]+kxyz=0.$$

Find the equations of 45 tritangent planes whose equations depend rationally on $l.m.n,k$.

9.25 Show that the polar quadric of a nonsingular cubic surface with respect to an Eckardt point is equal to the union of two planes.

9.26 Show that the equation of the dual of a nonsingular cubic surface can be written in the form $A^3+B^2=0$, where A and B are homogeneous forms of degree 4 and 6, respectively. Show that the dual surface has 27 double lines and a curve of degree 24 of singularities of type A_2.

9.27 Show that a plane section of the Hessian of a general cubic surface is a smooth Lüroth quartic.

9.28 Show that the degree of the dual of the Hessian of a general cubic surface is equal to 16.

Historical Notes

Good sources for the references here are [208], [280], and [308]. According to [280], the study of cubic surfaces originates from the work of J. Plücker [315] on intersection of quadrics and cubics and L. Magnus [276] on maps of a plane by a linear system of cubics. However, it is customary to think that the theory of cubic surfaces starts from Cayley's and Salmon's discovery of 27 lines on a nonsingular cubic surface [50], [349] (see the history of discovery in [357], n. 529a, p. 183). Salmon's proof was based on his computation of the degree of the dual surface [348] and Cayley's proof uses the count of tritangent planes through a line which we gave in the text. It is reproduced in many modern discusssions of cubic surfaces (e.g. [328]). The number of tritangent planes was computed by [349] and Cayley [50]. Cayley gives an explicit four-dimensional family of cubic surfaces with a fixed tritangent plane. In 1851 J. Sylvester claims without proof that a general cubic surface can be written uniquely as a sum of 5 cubes of linear forms [403]. This fact was proven 10 years later by A. Clebsch [76]. In 1854 L. Schläfli discovers 36 double-sixes on a nonsingular cubic surface. This and other results about cubic surfaces were published later in [361]. In 1855 H. Grassmann proves that three collinear nets of planes generate a cubic surface [194]. The fact that a general cubic surface can be obtained in this way (this implies that it can be obtained as the image of the projeective plane under a rational map given by cubic surfaces) has a long history. In 1862 F. August proves that a general cubic surface can be generated by three pencils of planes [13]. L. Cremona shows in [106] that this implies that a general cubic surfaces admits Grassmann's generation. In 1904 R. Sturm in 1904 pointed out that Cremona's proof had a gap. The gap was fixed by C. Segre in [381]. In the same paper Segre proves that any normal cubic surface which does no contain a singularity of type E_8 has a linear determinantal representation. In 1956 J. Steiner introduces what we called Steiner systems of lines [397]. This gives 120 essentially different Cayley-Salmon equations of a nonsingular cubic surface. The existence of which was first shown by Cayley [50] and Salmon [349].

Cubic surfaces with a double line were classified in 1862 by A. Cayley [57] and, via a geometric approach, by L. Cremona [102]. In 1863 L. Schläfli [360] classified singular cubic surfaces with isolated singularities, although most of these surfaces were already known to G. Salmon [349]. The old notations for A_k-singularities are C_2 for A_1 (conic-node), B_{k+1} (biplanar nodes) for $A_k, k, > 1$ and U_{k+1} (uniplanar node) for D_k. The subscript indicates the decrease of the class of the surface. In [65] Cayley gives a combinatorial description of the sets of lines and tritangent planes on singular surfaces. He also gives the equations of the dual surfaces. Even before the discovery of 27 lines, in a paper of 1844 [48], Cayley studied what we now call the Cayley 4-nodal cubic surface. He finds its equation and describes its plane sections which amounts to describe its realization as the image of the plane under the map given by the linear system of cubic curves passing through the vertices of a complete quadrilateral. Schläfli and later F. Klein [248] and L. Cremona [106] also studied the reality of singular points and lines.

In 1866 A. Clebsch proves that a general cubic surface can be obtained as the image of a birational map from the projective plane given by cubics through 6 points [78]. Using this he shows that the Schläfli notation a_i, b_i, c_{ij} for 27 lines correspond to the images of the exceptional curves, conics through 5 points and lines through two points. This important result was independently proven by L. Cremona in his memoir [106] of 1868 that got him the prize (shared with R. Sturm) offered by R. Steiner through the Royal Academy of Sciences of Berlin in 1864 and awarded in 1866. Some of the results from this memoir are discussed from a modern point of view in [138]. Many results from Cremona's memoir are independently proved by R. Sturm [398], and many of them were announced by J. Steiner (who did not provide proofs). In particular, Cremona proves the result, anticipated in the work of Magnus, that any cubic surface can be obtained as the image of a plane under the cubo-cubic birational transformation of $\mathbb{P}^3$. Both of the memoirs had a lengthy discussion of Steiner systems of tritangent planes. We refer to [138] for a historical discussion of Cremona's work on cubic surfaces.

The Cremona hexahedral equations were introduced by L. Cremona in [110] in 1878. Although known to T. Reye [332] (in geometric form, no equations can be found in his paper), Cremona was the first who proved that the equations are determined by a choice of a double-six. The invariant theory of Cremona hexaedral equations was studied by A. Coble in [85]. The Segre cubic arised in the work of C. Segre on cubic threefolds with singular points. Its realization as the GIT-moduli space of ordered sets of six points in $\mathbb{P}^1$ is due to Coble.

F. Eckardt gives a complete classification of cubic surfaces with Eckardt points (called Ovalpoints in [358]) in terms of their Hessian surface [149]. He also considers singular surfaces. A modern account of this work can be found [112]. The Clebsch Diagonalfläche with 10 Eckardt points was first studied by A. Clebsch in [81]. It has an important role in Klein's investigation of the Galois group of a quintic equations [251]. The group of automorphisms of a nonsingular cubic surfaces was computed by S. Kantor [241] and A. Wiman [426].

In 1897 J. Hutchinson showed in [229] that the Hessian surface of a nonsingular cubic surface could be isomorphic to the Kummer surface of the Jacobian of a genus 2 curve. This happens if the invariant $I_8I_{24} + 8I_{32}$ vanishes [343]. The group of automorphisms of the Hessian of a cubic surface was described only recently [135].

The relationship of the Gosset polynomial 2_{21} to 27 lines on a cubic surface was first discovered in 1910 by P. Schoute [363] (see [412]). The Weyl group $W(\mathbf{E}_6)$ as the Galois group of 27 lines was first studied by C. Jordan. Together with the group of 28 bitangents of a plane quartic isomorphic to $W(\mathbf{E}_7)$, it is discussed in many classical text-books in algebra (e.g. [423], B. II, [124]). S. Kantor [241] realized the Weyl group $W(\mathbf{E}_n)$ as groups of linear transformations preserving a quadratic form of signature $(1, n)$ and a linear form. A Coble [85], Part II, was the first who showed that the group is generated by the permutations group and one additional involution. So we should credit him the discovery of the Weyl groups as reflection groups. Apparently independently of Coble, this fact was rediscovered by P. Du Val [144]. We refer to [37] for the history of Weyl groups, reflection groups and root systems. Note that the realization of the Weyl group as a reflection group in the theory of Lie algebras was obtained by H. Weyl in 1928, ten years later after Coble's work.

The Gosset polytopes were discovered in 1900 by T. Gosset [190]. The notation n_{21} belongs to him. They had been later rediscovered by E. Elte and H.S.M. Coxeter (see [101]) but only Coxeter realized that their groups of symmetries are reflection groups. The relationship between the Gosset polytopes n_{21} and curves on Del Pezzo surfaces of degree $5 - n$ was found by Du Val [144]. This fundamental paper is the origin of a modern approach to the study of Del Pezzo surfaces by means of root systems of finite-dimensional Lie algebras [121], [277].

Yu. Manin's book [277] is a good source on cubic surfaces over non-algebraically closed field and B. Segre's book [383] has a lot of information about real cubic surfaces.

Chapter 10

Geometry of Lines

10.1 Grassmannians of lines

10.1.1 Generalities about Grassmannians

Let V be a vector space of dimension n. Let us recall some basic facts about Grassmann varieties of m-dimensional subspaces of V and fix the notations. We will denote by $G(m, V)$ the variety of m-dimensional subspaces of V and by $G(V, m)$ the variety of equivalence classes of m-dimensional quotients of V. By taking the dual of the quotient map we will often identify $G(m, V)$ with $G(V^\vee, m)$. In this notation, $G_1(V) = |V|$ and $G(V, 1) = \mathbb{P}(V)$. Also, by taking the kernel of the quotient or taking the dual of a subspace we can identify $G(m, V)$ with $G(n - m, V^\vee)$. Thus there are natural isomorphisms of varieties

$$G(m, V) \cong G(V^\vee, m) \cong G(n - m, V^\vee) \cong G(V, n - m).$$

We will also use the notation $G_{m-1}(|V|)$ for $G(m, V)$ and $G_{m-1}(\mathbb{P}(V))$ for $G(V, m)$. If we fix coordinates in V to identify V with $\mathbb{C}^n$ we write $G(m, n)$ instead of $G(m, \mathbb{C}^n)$ and, similarly, $G(n, m)$ for $G(\mathbb{C}^n, m)$.

Passing to the exterior powers, an inclusion of subspaces $L \hookrightarrow V$ defines a line $\bigwedge^m L \hookrightarrow \bigwedge^m V$ in $\mathbb{P}(\bigwedge^m V^\vee) = |\bigwedge^m V|$. This defines the *Plücker embedding* $G(m, V) \hookrightarrow \mathbb{P}(\bigwedge^m V^\vee)$. In coordinates, a point in $G(m, n)$ is represented by a matrix A of size $m \times n$ and rank m. Its rows are formed by a basis of the subspace. The corresponding point $[A]$ in $G(m, n)$ is the orbit of A with respect of the action of the group $\mathrm{GL}(m)$ by left multiplication. The maximal minors $p_I = |A_I|, I = (i_1 < \ldots < i_m)$ of the matrix A formed by the columns with indices from the subset I are the coordinates of $[A]$ in the Plücker embedding of $G(m, n)$. They are called the *Grassmannian* (or *Plücker*) *coordinates* of $[A]$. Adding one additional row to A formed by an element of the basis, we obtain by equating to zero the maximal minors the *Plücker relations*

$$\sum_{k=1}^{n} (-1)^k p_{i_1,\ldots,i_{m-1},j_k} p_{j_1,\ldots,j_{k-1},j_{k+1},\ldots,j_{m+1}} = 0, \tag{10.1}$$

where $(i_1,\ldots,i_{m-1})$ and $(j_1,\ldots,j_{m+1})$ are two strictly increasing subsets of $[1,n]$. These relations are easily obtained by considering the left-hand-side expression as an alternating $(m+1)$-multilinear function on $(\mathbb{C}^m)^{m+1}$ whose arguments are columns of a general $m\times n$-matrix with indices in $j_1,\ldots,j_{m+1}$. It is known that these equations define $G(m,n)$ scheme-theoretically in $\mathbb{P}^{\binom{n}{m}-1}$.

All matrices with $|A_{ij}|\neq 0$ for a fixed pair (i,j) form a Zariski open subset of $G(m,n)$ isomorphic to the affine space $\mathbb{A}^{m(n-m)}$. The coordinates in this space are the entries of $A_{ij}^{-1}A$ taken from the columns $A_k, k\neq i,j$. This shows that $G(m,n)$ is a smooth rational variety of dimension $m(n-m)$.

The Grassmann variety $G=G(V,m)$ represents a functor $\mathbf{G}(V,m)$ which assigns to a scheme S the set of pairs $(\mathcal{E},\sigma)$ consisting of a rank m vector bundles $\mathcal{E}$ together with a surjective map of sheaves $\sigma: V\otimes\mathcal{O}_S\to\mathcal{E}$ up to a natural equivalence of surjections. Given such a pair $(\mathcal{E},\sigma)$, one defines a morphism $S\to G(V,m)$ by assigning to a point $s\in S$, the surjection $V\to\mathcal{E}(x)$ or a point of $G(n-m,V)$ by assigning the kernel of this surjection. The universal surjection in the sense of representable functors is defined by a vector bundle $V\otimes\mathcal{O}_G\to\mathcal{Q}_G$, where $\mathcal{Q}_G$ is a vector bundle of rank m over G, called the *universal quotient bundle* over G. Its kernel is denoted by $\mathcal{S}_G$ and is called the *universal subbundle* over G. By definition we have an exact sequence of locally free sheaves (the *tautological exact sequence* on G)

$$0\to\mathcal{S}_G\to V\otimes\mathcal{O}_G\to\mathcal{Q}_G\to 0. \tag{10.2}$$

and its dual exact sequence

$$0\to\mathcal{Q}_G^\vee\to V^\vee\otimes\mathcal{O}_G\to\mathcal{S}_G^\vee\to 0. \tag{10.3}$$

The surjection $V^\vee\otimes\mathcal{O}_G\to\mathcal{S}_G^\vee$ is the universal surjection for the functor $\mathbf{G}(V^\vee,n-m)$.

Passing to the exterior powers we obtain the surjections $\bigwedge^m V\otimes\mathcal{O}_G\to\bigwedge^m\mathcal{Q}_G$ and $\bigwedge^m V^\vee\otimes\mathcal{O}_G\to\bigwedge^m\mathcal{S}_G^\vee$. The first one defines a closed embedding

$$G=\mathbb{P}(\bigwedge^n\mathcal{Q}_G)\to\mathbb{P}(\bigwedge^m V)\times G\to\mathbb{P}(\bigwedge^m V),$$

the second one defines a closed embedding

$$G=\mathbb{P}(\bigwedge^m\mathcal{S}_G^\vee)\to\mathbb{P}(\bigwedge^{n-m}V^\vee)\times G\to\mathbb{P}(\bigwedge^{n-m}V^\vee).$$

The second one coincides with the Plücker embedding of $G(n-m,V)$ from above, the first one is the Plücker embedding of $G(m,V^\vee)$.

Here we use the notation $\mathbb{P}(\mathcal{E})$ for the projectivization of the vector bundle $\mathcal{E}$ over a scheme Y. As usual we identify vector bundles $\mathbb{V}(\mathcal{E})$ with their locally free sheaves of sections $\mathcal{E}$ so that $\mathbb{V}(\mathcal{E})=\mathrm{Spec}(S^\bullet\mathcal{E}^\vee)$, where $S^\bullet$ denotes the graded symmetric algebra functor (see [206]). So

$$\mathbb{P}(\mathcal{E}):=\mathrm{Proj}(S^\bullet\mathcal{E}^\vee).$$

We will use the notation

$$|\mathcal{E}| := \mathbb{P}(\mathcal{E}^\vee).$$

A map of locally sheaves $\alpha : \mathcal{E} \to \mathcal{E}'$ defines a canonical homomorphism of symmetric algebras $S^\bullet\mathcal{E}'^\vee \to S^\bullet\mathcal{E}'^\vee$ and the corresponding morphism of vector bundles $\mathbb{V}(\mathcal{E}') \to \mathbb{V}(\mathcal{E})$. Passing to the projective spectra, we get only a rational map $\mathbb{P}(\mathcal{E}') \to \mathbb{P}(\mathcal{E})$. However, if α is surjective, the rational map is a closed embedding morphism. This what we used in the above descriptions of the Plücker embeddings.

Let $\pi : \mathbb{P}(\mathcal{E}) \to Y$ be the canonical structure morphism of a Y-scheme. There exists a unique invertible sheaf (line bundle) $\mathcal{L}$ on $\mathbb{P}(\mathcal{E})$ together with a surjective morphism of sheaves $\pi^*\mathcal{E} \to \mathcal{L}$ such that the corresponding morphism

$$\mathcal{E} \cong \pi_*\pi^*\mathcal{E} \to \pi_*(\mathcal{L})$$

is an isomorphism (see [206]). This sheaf is denoted by $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. Note that, for any invertible sheaf $\mathcal{M}$, the projective bundles $\mathbb{P}(\mathcal{E})$ and $\mathbb{P}(\mathcal{E} \otimes \mathcal{M})$ are isomorphic and

$$\mathcal{O}_{\mathbb{P}(\mathcal{E}\otimes\mathcal{M})} \cong \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes \pi^*\mathcal{M}.$$

When Y is a point and $\mathcal{E}$ is a vector space E, then we obtain the usual identification of the space of global sections of the sheaf $\mathcal{O}_{\mathbb{P}(E^\vee)}(1)$ with the vector space E.

The construction of the projective bundle $\mathbb{P}(\mathcal{E})$ considered as a relative notion of a projective space is generalized to the notion of a relative Grassmannian $G(\mathcal{E}, m)$. If $f : S \to Y$ is a Y-scheme, then S-points of $G(\mathcal{E}, m)$ are surjective maps of locally free sheaves $f^*\mathcal{E} \to \mathcal{F}$, where $\mathcal{F}$ is a locally free sheaf of rank m on S. We have $G(\mathcal{E}, 1) = \mathbb{P}(\mathcal{E})$. By definition, $G(m, \mathcal{E}) = G(\mathcal{E}^\vee, m)$. There is also a universal surjective map $\pi^*(\mathcal{E}) \to \mathcal{F}$, where $\pi : G(\mathcal{E}, m) \to Y$ is the structure morphism and $\mathcal{F}$ is a locally free sheaf of rank m over $G(\mathcal{E}, m)$. The surjection $\bigwedge^m \mathcal{E} \to \bigwedge^m \mathcal{F}$ defines a closed embedding $G(\mathcal{E}, m) \hookrightarrow \mathbb{P}(\pi_* \bigwedge^m \mathcal{F}) \subset \mathbb{P}(\mathcal{E})$. This is the relative Plücker embedding. We denote the sheaf $\bigwedge^m \mathcal{F}$ by $\mathcal{O}_{G(\mathcal{E},m)}(1)$. In the case when $m = 1$, $G(\mathcal{E}, 1) = \mathbb{P}(\mathcal{E})$ and the sheaf $\mathcal{O}_{G(\mathcal{E},m)}(1)$ coincides with the sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. P

Let $G = G(V, m) = G(n-m, V)$. The surjection $V^\vee \otimes \mathcal{O}_G \to \mathcal{S}_G^\vee$ defines the closed embedding $\mathbb{P}(\mathcal{S}_G^\vee) \hookrightarrow \mathbb{P}(V^\vee \otimes \mathcal{O}_G) = |V| \times G$. Its image is the *incidence variety*

$$Z_G = \{(x, \Pi) \in |V| \times G(n-m, V) : x \in \Pi\}.$$

Let

$$p : Z_G \to |V|, \quad q : Z_G \to G(n-m, V)$$

be the corresponding projections. By definition the projection q is the projective bundle $\mathbb{P}(\mathcal{S}_G^\vee) = |\mathcal{S}_G|$.

The fibre of the projection p over a point $x = [v] \in |V|$ is a surjection of V with kernel containing v. It can be identified with a point in the Grassmannian $G(V/\mathbb{C}v, m-1) = G(n-m-1, V/\mathbb{C}v)$. Recall that the quotient space $V/\mathbb{C}v, v \in V$ are the fibres of the quotient sheaf $V \otimes \mathcal{O}_{|V|}/\mathcal{O}_{|V|}(-1)$ which is isomorphic to the twisted tangent sheaf $\mathcal{T}_{|V|}(-1)$ via the Euler exact sequence

$$0 \to \mathcal{O}_{|V|} \to \mathcal{O}_{|V|}(1) \otimes V \to \mathcal{T}_{|V|} \to 0.$$

Assume $m = n-2$, then $G(V/\mathbb{C}v, m) \cong \mathbb{P}(V/\mathbb{C}v)$. This gives

$$Z_G \cong \mathbb{P}(\mathcal{T}_{|V|}(-1)^\vee) = \mathbb{P}(\Omega^1_{|V|}(1)). \tag{10.4}$$

We omit a rather technical proof of the following.

Proposition 10.1.1. *The projection $p : Z_G \to |V|$ is isomorphic to the relative Grassmannian $G(\Omega^1_{|V|}, m)$.*

The universal surjection for the relative Grassmannian $Z_G \to |V|$ is equal to $p^*\Omega^1_{|V|}(1) \to q^*\mathcal{Q}_G$. Thus

$$\mathcal{O}_{G(\Omega^1_{|V|},m)}(1) \cong q^*(\bigwedge^m \mathcal{Q}_G) \tag{10.5}$$

$$\mathcal{O}_{\mathbb{P}(\mathcal{S}_G^\vee)}(1) \cong p^*\mathcal{O}_{|V|}(1). \tag{10.6}$$

Let us compute the canonical sheaf ω_G of G.

Lemma 10.1.2. *Let $\mathcal{T}_G$ be the tangent bundle of $G = G_r(|V|)$. There is a natural isomorphism of sheaves*

$$\begin{aligned} \mathcal{T}_G &\cong \mathcal{S}_G^\vee \otimes \mathcal{Q}_G, \\ \omega_G &\cong \mathcal{O}_G(-n), \end{aligned}$$

where $\mathcal{O}_G(1)$ is taken with respect to the Plïcker embedding.

Proof. The first formula was an Exercise from Chapter 2, so we give its solution. It is easy to see (same as for the projective space) that the tangent space $\mathrm{T}_\ell G$ is canonically isomorphic to $\mathrm{Hom}(L, V/L) \cong L^\vee \otimes V/L = (\mathcal{S}_G)(\ell)^* \otimes (\mathcal{Q}_G)(\ell)$, where $\ell = |L|$. One can show that this isomorphism can be extended to the isomorphism of sheaves (see the details in [7]). Globalizing we easily get the first isomorphism.

Since $\bigwedge^m V \to \bigwedge^m S_G^\vee$ defines the Plücker embedding, we have

$$c_1(\mathcal{S}_G^\vee) = c_1(\mathcal{O}_G(1)).$$

Now the second isomorphism follows from a well-known formula for the first Chern class of tensor product of vector bundles (see [206], Appendix A), where we use that $\bigwedge^m S_G \vee \cong \mathcal{O}_G(1)$. □

Recall the relative Euler sequence for the projective bundle $\mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is a locally free sheaf of rank r over a variety Y.

$$0 \to \mathcal{O}_{\mathbb{P}(\mathcal{E})} \to \pi^*(\mathcal{E})(1) \to \mathcal{T}_{\mathbb{P}(\mathcal{E})/Y} \to 0 \tag{10.7}$$

where as usual $\mathcal{F}(m)$ means $\mathcal{F} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)^{\otimes m}$.

This gives the following formula for the relative canonical sheaf of $\mathbb{P}(\mathcal{E})$

$$\omega_{\mathbb{P}(\mathcal{E})/Y} \cong \pi^*(\bigwedge^r \mathcal{E})(-r), \tag{10.8}$$

Applying this formula to the projective bundle $Z_G = \mathbb{P}(\mathcal{S}_G^\vee)$ we obtain

$$\begin{aligned} \omega_{Z_G/G} &\cong q^*(\bigwedge^m \mathcal{S}_G^\vee) \otimes p^*\mathcal{O}_{|V|}(-m) \cong q^*\mathcal{O}_G(1) \otimes p^*\mathcal{O}_{|V|}(-m) \quad (10.9)\\ \omega_{Z_G} &\cong \omega_{Z_G/G} \otimes q^*(\omega_G) \cong q^*\mathcal{O}_G(-n) \otimes p^*\mathcal{O}_{|V|}(-m)\\ \omega_{Z_G/|V|} &\cong \omega_{Z_G} \otimes p^*\omega_{|V|}^{-1} \cong q^*\mathcal{O}_G(-n) \otimes p^*\mathcal{O}_{|V|}(n-m) \end{aligned}$$

10.1.2 Schubert varieties

Let us recall some fact about the cohomology ring $H^*(G,\mathbb{Z})$, where $G = G_r(\mathbb{P}^N)$ (see [173], Chapter 14).

Fix a flag

$$A_0 \subset A_1 \subset \ldots \subset A_r \subset \mathbb{P}^N$$

of subspaces of dimension $a_0 < a_1 < \ldots < a_r$, and define the *Schuber variety*

$$\Omega(A_0, A_1, \ldots, A_r) = \{\Pi \in G : \dim \Pi \cap A_i \geq i, i = 0, \ldots, r\}.$$

This is a closed subvariety of G of dimension $\sum_{i=0}^r (a_i - i)$. Its cohomology class $[\Omega(A_0, A_1, \ldots, A_r)]$ in $H_*(G,\mathbb{Z})$ depends only on $a_0, \ldots, a_r$. It is called a *Schubert cycle* and is denoted by $(a_0, \ldots, a_r)$. Let $a_0 = N - r - d, a_i = N - r + i, i \geq 1$. The varieties

$$\Omega(A_0) := \Omega(A_0), \ldots, A_r) = \{\Pi \in G : \Pi \cap A_0 \neq \emptyset\}$$

are called the *special Schuber varieties*. Their codimension is equal to d.

Under the Poincaré duality $H_*(G,\mathbb{Z}) \to H^*(G,\mathbb{Z})$, the cycles $(a_0, \ldots, a_r)$ are mapped to *Schubert classes* $\{\lambda_0, \ldots, \lambda_r\}$ defined in terms of the Chern classes

$$\sigma_s = c_s(\mathcal{Q}_G) \in H^{2s}(G,\mathbb{Z}),\ s = 1, \ldots, N - r$$

by the determinantal formula

$$\{\lambda_0, \ldots, \lambda_r\} = \det(\sigma_{\lambda_i + j - i})_{0 \leq i,j \leq r},$$

where $\lambda_i = N - r + i - a_i,\ i = 0, \ldots, r$. The classes σ_s are dual to the classes of special Schubert varieties $(N - r - s, N - r + 1, \ldots, N)$.

The tautological exact sequence (10.2) shows that

$$1 = (\sum c_s(\mathcal{Q}_G))(\sum c_s(\mathcal{S}_G)).$$

In particular,

$$\sigma_1 = -c_1(\mathcal{S}_G) = c_1(\mathcal{S}_G^\vee) = c_1(\mathcal{O}_G(1)).$$

A proof of the following result can be found in [173] or [221].

Proposition 10.1.3. *The cohomology ring $H^*(G.\mathbb{Z})$ is generated by the special Schubert classes $\sigma_s = \{s, 0, \ldots, 0\}$. The Schubert cycles $(a_0, \ldots, a_r)$ with $\sum_{i=0}^r (a_i - i) = d$ freely generate $H_{2d}(G,\mathbb{Z})$. The Schubert classes $\{\lambda_0, \ldots, \lambda_r\}$ with $d = \sum_{i=0}^r \lambda_i$ freely generate $H^{2d}(G,\mathbb{Z})$. In particular,*

$$\mathrm{Pic}(G) \cong H^2(G,\mathbb{Z}) = \mathbb{Z}\sigma_1.$$

It follows from the above proposition that $H^*(G,\mathbb{Z})$ is isomorphic to the Chow ring $\mathrm{CH}(G)$ of algebraic cycles on G. Under this isomorphism $H^{2m}(G,\mathbb{Z}) \cong \mathrm{CH}^m(H)$. Under the Poincaré isomorphism $\gamma \mapsto \alpha_\gamma$ the intersection form on cycles $\langle\gamma,\mu\rangle$ is defined by

$$\langle\gamma,\mu\rangle = \int_\mu \alpha_\gamma = \int_G \alpha_\gamma \wedge \alpha_\mu := \alpha_\gamma \cdot \alpha_\mu.$$

The intersection form on $\mathrm{CH}^*(G)$ is defined by the intersection form on $H^{2*}(G,\mathbb{Z})$ and is completely determined by ***Pieri's formulas***

$$\{\lambda_0,\ldots,\lambda_r\}\cdot\sigma_s = \sum\{\mu_0,\ldots,\mu_r\}, \tag{10.10}$$

where the sum is taken over all $\{\mu\}$ such that $N-r \geq \lambda_0 \geq \ldots \mu_r \geq \lambda_r$ and $\sum\lambda_i = s+\sum\mu_i$.

Here are some special cases. We set $\sigma_{s,t} = \{s,t,0,\ldots,0\}$

$$\begin{aligned}\sigma_1^2 &= \sigma_2+\sigma_{1,1},\\ \sigma_1\cdot\sigma_2 &= \sigma_3+\sigma_{2,1}\\ \sigma_1\cdot\sigma_{1,1} &= \sigma_{2,1}.\end{aligned}$$

For example, the degree of G is equal to $\sigma_1^{\dim G}$. We refer to [173], Example 14.7.11, for the following formula computing the degree of $G_r(\mathbb{P}^N)$

$$\deg G_r(\mathbb{P}^N) = \frac{1!2!\ldots\dim G!}{(N-r)!(N-r+1)!\ldots N!} \tag{10.11}$$

Example 10.1.1. Let us look at the Grassmanian $G_1(\mathbb{P}^3)$ of lines in $\mathbb{P}^3$. We know that this is a nonsingular quadric in $\mathbb{P}^5$. The Schubert class of codimension 1 is represented by the special Schubert variety $\Omega(\ell)$ of lines intersecting a given line ℓ. We have two codimension 2 Schubert cycles σ_2 and $\sigma_{1,1}$ represented by the Schubert varieties $\Omega(x)$ of lines containing a given point and Ω_Π of lines containing in a given plane Π. Each of these varieties is isomorphic to $\mathbb{P}^2$. In classical terminology $\Omega(x)$ is called an *α-plane* and Ω_Π a *β-plane*. We have one-dimensional Schubert cycle $\sigma_{2,1}$ represented by the Schubert variety $\Omega(x,\Pi)$ of lines in a plane π containing a given point $x\in\Pi$. It is isomorphic to $\mathbb{P}^1$. Finally we have a 0-dimensional Schubert variety $\{l\}$ representing lines contained in a given line ℓ. Thus

$$\mathrm{CH}(G(2,4)) = \mathbb{Z}[G]\oplus\mathbb{Z}\sigma_1\oplus(\mathbb{Z}\sigma_2+\mathbb{Z}\sigma_{1,1})\oplus\mathbb{Z}\sigma_{2,1}\oplus\mathbb{Z}[\text{point}].$$

Note that the two classes in codimension 2 represent two different rulings of the Klein quadric by planes.

We have

$$\sigma_2\cdot\sigma_{1,1} = 0,\ \sigma_2^2 = 1, \sigma_{1,1}^2 = 1. \tag{10.12}$$

Write $\sigma_1^2 = a\sigma_2+b\sigma_{1,1}$. Intersecting both sides with σ_2 and $\sigma_{1,1}$, we obtain $a = b = 1$ confirming Pieri's formula (10.10). Squaring σ_1^2, we obtain $\deg G = \sigma_1^4 = 2$, confirming the fact that G is a quadric in $\mathbb{P}^5$.

A surface S in $G_1(\mathbb{P}^3)$ is called a *congruence of lines*. Its cohomology class $[S]$ is equal to $m\sigma_2 + n\sigma_{1,1}$. The number m (resp. n) is classically known as the *order* of S (resp. *class*). It is equal to the number of lines in S passing through a general point in $\mathbb{P}^3$ (resp. contained in a general plane). The sum $m+n$ is equal to $\sigma_1 \cdot [S]$ and hence coincides with the degree of S in $\mathbb{P}^5$.

As one of many applications of Schubert calculus let us prove the following nice result which can be found in many classical text-books (first proven by L. Cremona [103]).

Theorem 10.1.4. *The number of common secants of two general Veronese cubics in $\mathbb{P}^3$ is equal to 10.*

Proof. Consider the congruence of lines formed by secants of a Veronese cubic. Through a general point in $\mathbb{P}^3$ passes one secant. In a general plane lie 3 secants. Thus the order of the congruence is equal to 1 and the class is equal to 3. Using (10.12), we see that the two congruences intersect at 10 points. □

Remark 10.1.1. Let R_1 and R_2 be two general Veronese cubic curves in $\mathbb{P}^3$ and let $\mathcal{N}_i$ be the net of quadrics through R_i. The linear system $\mathcal{W}$ of quadrics in the dual space apolar to the linear system $\mathcal{N}$ spanned by $\mathcal{N}_1$ and $\mathcal{N}_2$ is of dimension 3. The Steinerian quartic defined by this linear system contains 10 lines, the singular lines of 10 reducible quadrics from $\mathcal{W}$. The dual of these lines are the ten common secants of R_1 and R_2 (see [334], [281], [87]). Also observe that the 5-dimensional linear system $\mathcal{N}$ maps R_i to a curve C_i of degree 6 spanning the plane Π_i in $\mathcal{N}^\vee$ apolar to the plane $\mathcal{N}_j$. The ten pairs of intersection points of C_i with the ten common secants correspond to the branches of the ten singular points of C_i.

10.1.3 Secant varieties of Grassmannians of lines

From now on, we will restrict ourselves with the Grassmannian of lines.

By contraction, we can identify $\bigwedge^2 V$ with the space of linear maps $u : V^\vee \to V$ such that the transpose map ${}^t u$ is equal to $-u$. The rank of u is the rank of the map. Since ${}^t u = -u$, the rank takes even values. The image $L = u(V^\vee)$ of a map of rank 2 is a 2-dimensional subspace L of V which can be identified with a line in $|V|$. Conversely, for any 2-dimensional subspace $L \subset V$, the image of the map $\bigwedge^2 L \to \bigwedge^2 V$ is a one-dimensional subspace of $\bigwedge^2 V$ and its nonzero element u has rank equal to 2. In this way one obtains an isomorphism from the Grassmannian variety $G_1(|V|) = G(2,V)$ of lines in $|V|$ and the variety of rank 2 tensors in $\bigwedge^2 V$ up to proportionality.

After fixing a basis in V, we can identify $\bigwedge^2 V$ with the space of $n \times n$ skew-symmetric matrices $A = (p_{ij})$ and $G(2,V)$ with the locus of matrices of rank 2. The entries $p_{ij}, i < j$, define projective coordinates on $\mathbb{P}(\bigwedge^2 V)$, the *Plücker coordinates*. In particular, $G(2,V)$ is the zero set of the 4×4 pfaffians of A. In fact, a stronger assertion is true.

Proposition 10.1.5. *The homogeneous ideal of $G(2,V) \subset \mathbb{P}(\bigwedge^2 V)$ is generated by 4×4 pfaffians of a general skew-symmetric matrix of size n.*

Another way to look at $G(2,V)$ is to use the decomposition

$$V \otimes V \cong S^2V \oplus \bigwedge^2 V$$

to identify $G(2,V)$ with the projection of the Segre variety $s_2(|V|) \subset |V| \times |V|$ to $|\bigwedge^2 V|$ with center equal to the subspace $|S^2V| \subset |V \otimes V|$.

By (10.1.5) the ideal of $G(2,V)$ is generated by $\binom{n}{4}$ quadratic forms of rank 6:

$$p_{ij}p_{kl} - p_{ik}p_{jl} + p_{il}p_{jk} = \text{Pfaf}\begin{pmatrix} 0 & p_{ij} & p_{ik} & p_{il} \\ -p_{ij} & 0 & p_{jk} & p_{jl} \\ -p_{ik} & -p_{jk} & 0 & p_{kl} \\ -p_{il} & -p_{jl} & -p_{kl} & 0 \end{pmatrix}$$

with $1 \le i < j < k < l \le n$. If $n = 4$ then G is the *Klein quadric*

$$V(p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23}) \subset \mathbb{P}^5 \tag{10.13}$$

defining the Grassmannian of lines in $\mathbb{P}^3$.

The formula (10.11) for the degree of the Grassmannian gives in our special case

$$\deg G(2,n) = \frac{(2n-4)!}{(n-2)!(n-1)!} \tag{10.14}$$

One can also compute the degrees of Schubert varieties

$$\deg \Omega(a_0,a_1) = \frac{(a_0+a_1-1)!}{a_0!a_1!}(a_1-a_0). \tag{10.15}$$

Let $p \in \mathbb{P}(\bigwedge^2 V)$ and u_p be a representative of p in $\bigwedge^2 V$. We say that p is of rank r if u_p is of rank r. It is clear that u_p has rank $\le 2k$ if and only if there exist matrices $s_1,\ldots,s_k$ of rank 2 such that

$$u_p = s_1 + \cdots + s_k.$$

In other words, p has rank $\le 2k$ if and only if $p \in S$, where S is a space of dimension $k-1$ which is at least k-secant to G. This gives the following.

Proposition 10.1.6. *The variety*

$$G_k =: \{p \in \mathbb{P}(\bigwedge^2 V) : \ p \textit{ has rank} \le 2k+2\}$$

is the k-secant variety $\text{Sec}_k(G)$ *of* $G = G(2,V)$.

Let $t = [\frac{n-4}{2}]$, then t is the maximal number k such that $\text{Sec}_k G \ne \mathbb{P}(\bigwedge^2 V)$. So the Plücker space is stratified by the rank of its points and the strata are the following:

$$\mathbb{P}^{n-1} \setminus \text{Sec}_t(G),\ \text{Sec}_t(G) \setminus \text{Sec}_{t-1},\ \ldots, \text{Sec}_1 \setminus G,\ G. \tag{10.16}$$

It follows from the previous remarks that $\mathrm{Sec}_k(G) \setminus \mathrm{Sec}_{k-1}(G)$ is the orbit of a matrix of rank $2k+2$ and size n under the action of $\mathrm{GL}(n)$. Therefore,

$$\dim \mathrm{Sec}_k(G) = \dim \mathrm{GL}(n)/H_k,$$

where H_k is the stabilizer of a skew symmetric matrix of rank $2k+2$ (e.g. with the standard symplectic matrix J_{2k+2} in the left upper corner and zero elsewhere). An easy computation gives the following.

Proposition 10.1.7. *Let $0 \le k \le t$, then*

$$d_k = \dim \mathrm{Sec}_k(G) = (k+1)(2n-2k-3) - 1.$$

Let $X \subset \mathbb{P}^r$ be a reduced and nondegenerate variety: the ***k-th defect*** of X can be defined as

$$\delta_k(X) = \min\big((k+1)\dim X + k, r\big) - \dim \mathrm{Sec}_k(X),$$

which is the difference between the expected dimension of the k-secant variety of X and the effective one. We say that X is k-***defective*** if $\mathrm{Sec}_k(X)$ is a proper subvariety and $\delta_k(X) > 0$.

Example 10.1.2. Let $n = 2t+4$, then $\mathrm{Sec}_t(G) \subset \mathbb{P}(\bigwedge^2 V)$ is the pfaffian hypersurface of degree $t+2$ in $\mathbb{P}(\bigwedge^2 V)$ parameterizing singular skew-symmetric matrices (p_{ij}) of size $2t+4$. The expected dimension of $\mathrm{Sec}_t(G)$ is equal to $4t^2+8t+5$ which is larger than $\dim \mathbb{P}(\bigwedge^2 V) = \binom{2t+4}{2} - 1$. Thus $d_t(G) = \dim \mathrm{Sec}_t(G) + 1$ and $\delta_t(G) = 1$.

In the special case $n = 6$, we have $t = 2$ and $\dim G = 8$. Recall that a nondegenerate subvariety $X \subset \mathbb{P}^N$ with $\dim X = [\frac{2N}{3}] - 1$ is called a *Severi-Zak variety* if $\mathrm{Sec}_1(X) \ne \mathbb{P}^N$. There are four non-isomorphic Severi-Zak varieties and $G(2,6)$ is one of them. The other three are the Veronese surface in $\mathbb{P}^5$, the Segre variety $s_2(\mathbb{P}^2 \times \mathbb{P}^2)$ in $\mathbb{P}^8$ and the E_6-variety of dimension 16 in $\mathbb{P}^{26}$.

Using Schubert varieties one can describe the projective tangent space of $\mathrm{Sec}_k(G)$ at a given point $p \notin \mathrm{Sec}_{k-1}(G)$. Consider p as a linear map $V^\vee \to V$ and let K be its kernel. The rank of p is equal to $2k+2$. Thus the orthogonal subspace $K^\perp \subset V$ defines a linear subspace $\Lambda_p = |K^\perp|$ of $|V|$ of dimension $2k+1$. Let $\Omega(\Lambda_p)$ be the corresponding special Schubert variety and $\langle \Omega(\Lambda_p)\rangle$ be its linear span in the Plücker space.

Proposition 10.1.8.

$$\mathbb{T}_p(\mathrm{Sec}_k(G)) = \langle \Omega(\Lambda_p)\rangle.$$

Proof. Since $\mathrm{Sec}_k(G) \setminus \mathrm{Sec}_{k-1}(G)$ is a homogeneous space for $\mathrm{GL}(n)$ we may assume that the point p is represented by a bivector $\omega = \sum_{i=1}^{k+1} e_i \wedge e_{i+1}$. The corresponding subspace $K^\perp$ is spanned by $e_1, \ldots, e_{2k+2}$. A line ℓ intersects $\mathbb{P}(\Lambda_p)$ if and only if it can be represented by a bivector $v \wedge w$, where $v \in K^\perp$. Thus $W = \langle \Omega(\Lambda_p)\rangle$ is the span of bivectors $e_i \wedge e_j$, where either i or j is less or equal than $2k$. In other words, W is given by vanishing of $\binom{n-2k-2}{2}$ Plücker coordinates p_{ab}, where $a, b > 2k+2$. It is easy to see that this agrees with formula (10.1.7) for $\dim \mathrm{Sec}_k(G)$. So, it is enough to show that W is contained in the tangent space. We know that the equations of $\mathrm{Sec}_k(G)$

are given by pfaffians of size $4k+4$. Recall the formula for the pfaffians from Chapter 2, Exercise 2.1,

$$\mathrm{Pfaf}(A) = \sum_{S\in\mathcal{S}} \pm \prod_{(ij)\in S} a_{ij},$$

where S is a set of pairs $(i_1j_1),\ldots,(i_{2k+2},j_{2k+2})$ such that $1 \le i_s < j_s \le 4k+4$, $s = 1,\ldots 2k+2$, $\{i_1,\ldots,i_{2k+2},j_1,\ldots,j_{2k+2}\} = \{1,\ldots,4k+4\}$. Consider the Jacobian matrix of $\mathrm{Sec}_k(G)$ at the point p. Each equation of $\mathrm{Sec}_k(G)$ is obtained by a choice of a subset I of $\{1,\ldots,n\}$ of cardinality $4k+4$ and writing the pfaffian of the submatrix of (p_{ij}) formed by the columns and rows with indices in I. The corresponding row of the Jacobian matrix is obtained by taking the partials of this equation with respect to all p_{ij} evaluated at the point p. If $a,b \le 2k+2$, then one of the factors in the product $\prod_{(ij)\in S} p_{ij}$ corresponds to a pair (i,j), where $i,j > 2k+2$. When we differentiate with respect to p_{ab} its value at p is equal to zero. Thus the corresponding entry in the Jacobian matrix is equal to zero. So, all nonzero entries in a row of the Jacobian matrix correspond to the coordinates of vectors from W which are equal to zero. Thus W is contained in the space of solutions. $\square$

Taking $k=0$, we obtain

Corollary 10.1.9.

$$\mathbb{T}_\ell(G) = \langle\Omega(\ell)\rangle.$$

Let Λ be any subspace of $\mathbb{P}^{n-1}$ of dimension $2k+1$. Consider the set of points

$$P_\Lambda = \{p \in \mathbb{P}(\bigwedge^2 V) : \Lambda = \Lambda_p\}.$$

This is the projectivization of the linear space of skew-symmetric matrices of rank $2k+2$ with the given nullspace of dimension $2k+2$. An easy computation using the formula (10.1.7) for $d_k = \dim \mathrm{Sec}_k(G)$ shows that its dimension is equal to $(2k+1)(k+1)-1$.

Let

$$\gamma_k : \mathrm{Sec}_k(G) \setminus \mathrm{Sec}_{k-1}(G) \to G(d_k+1, \bigwedge^2 V), \quad d_k = \dim \mathrm{Sec}_k(G),$$

be the *Gauss map* which assigns to a point its embedded tangent space. Applying Proposition 10.1.8, we obtain

Corollary 10.1.10.

$$\gamma_k^{-1}(\langle\Omega(\Lambda)\rangle) = P_\Lambda.$$

In particular, any hyperplane in the Plücker space containing $\Omega(\Lambda)$ *is tangent to* $\mathrm{Sec}_k(G)$ *along the subvariety* P_Λ *of dimension* $(2k+1)(k+1)-1$.

Example 10.1.3. Let $n = 6$. The secant variety $\mathrm{Sec}_1(G)$ is a cubic hypersurface in $\mathbb{P}^{14}$ defined by the pfaffian of 6×6 skew-symmetric matrix whose entries are Plücker coordinates p_{ij} . The Gauss map is the restriction to $\mathrm{Sec}_1(G)$ of the polar map $\mathcal{P} : \mathbb{P}^{14} \dashrightarrow \check{\mathbb{P}}^{14}$ given by the partials of the cubic. The singular locus of $\mathrm{Sec}_1(G)$ is G. The Plücker equations of G are the partials of the pfaffian cubic hypersurface. The map $\mathcal{P}$ is a Cremona transformation in $\mathbb{P}^{14}$ defined by the linear system of quadrics defining the Plücker equations of G. It can be resolved by blowing up G and then blowing down the proper inverse transform of $\mathrm{Sec}_1(G)$ to a subvariety isomorphic to G^*, where $G^* = G(2, V^\vee)$. The image of the exceptional locus of the blow-up is equal to $\mathrm{Sec}_1(G^*)$. Three other Severi-Zak varieties define a similar Cremona transformation (of $\mathbb{P}^5$, $\mathbb{P}^8$ and $\mathbb{P}^{26}$). It is given by the partials of the cubic form defining the first secant variety.

Let X be a subvariety of G, and Z_X be the preimage of X under the projection $q : Z_G \to G$. The image of Z_X in $\mathbb{P}^{n-1}$ is the union of lines $\ell \in X$. We will need the description of its set of nonsingular points.

Proposition 10.1.11. *The projection $p_X : Z_X \to \mathbb{P}^n$ is smooth at (x, l) if and only if*

$$\dim_l \Omega(x) \cap \mathbb{T}_l(X) = \dim_{(x,\ell)} p_X^{-1}(x)$$

Proof. Let $(x, l) \in Z_X$ and let F be the fibre of $p_X : Z_X \to \mathbb{P}^n$ passing through the point (x, l) identified with the subset $\Omega(x) \cap X$ under the projection $q : Z_X \to G$. Then

$$\mathbb{T}_{x,l}(F) = \mathbb{T}_l(\Omega(x)) \cap \mathbb{T}_l(X) = \Omega(x) \cap \mathbb{T}_l(X). \tag{10.17}$$

This proves the assertion. □

Corollary 10.1.12. *Let $Y = p_X(Z_X) \subset \mathbb{P}^{n-1}$ be the union of lines $l \in X$. Assume X is nonsingular and $p_X^{-1}(x)$ is a finite set. Suppose $\dim_l \Omega(x) \cap \mathbb{T}_l(X) = 0$ for some $l \in X$ containing x. Then x is nonsingular as a point of Y.*

10.2 Linear complexes of lines

10.2.1 Linear complexes and apolarity

An effective divisor $D \subset G = G(2, n)$ is called a *complex of lines*. Since we know that $\mathrm{Pic}(G)$ is generated by $\mathcal{O}_G(1)$ we see that

$$D \in |\ \mathcal{O}_G(d)\ |$$

for some $d \geq 1$.The *degree* of D is d.

An example of a complex of degree d in $G(2, n)$ is the *Chow form* of a subvariety $X \subset \mathbb{P}^{n-1}$ of codimension 2 (see [183]). It parametrizes lines which have non-empty intersection with X. Its degree is equal to the degree of X. When X is linear, this is of course the special Schubert variety $\Omega(X)$.

A *linear complex* is a complex of degree one, that is a hyperplane section $X = H \cap G$ of G. If no confusion arises we will sometimes identify X with the corresponding hyperplane $\langle H \rangle$. A linear complex is called *special* if it is tangent to G at some point $\ell \in G$, i.e. $X = \Omega(\ell)$.

If we write $\mathbb{P}^{n-1} = |V|$ for some vector space V of dimension n, then a linear complex X is defined by a linear form on $\bigwedge^2 V$, i.e. an element $\omega \in \bigwedge^2 V^\vee$. We write $X = X_\omega$. In coordinates, a linear complex (or, more precisely, the corresponding hyperplane) can be written as

$$\sum_{1\le i<j\le n} a_{ij}p_{ij} = 0.$$

For example, the complex $p_{ij} = 0$ parametrizes the lines intersecting the coordinate $(n-3)$-plane $x_k = 0, k \ne i,j$, in $\mathbb{P}^{n-1}$.

Remark 10.2.1. Recall from Remark 10.2.4 that we have a natural isomorphism

$$\bigwedge^2 V^\vee \cong H^0(|V|, \Omega^1_{|V|}(1)).$$

Also we know that the projection $q: Z \to |V| \cong \mathbb{P}^{n-1}$ is isomorphic to $\mathbb{P}(\Omega^1_{|V|}(1))$. Thus a linear line complex can be viewed as a divisor in the linear system $|\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|$, where $\mathcal{E} = \Omega^1_{|V|}(1))$. The projective bundle $\mathbb{P}(\mathcal{E})$ is isomorphic to the projectivization of the tangent bundle $\mathbb{V}(\Omega^1_{|V|})$. As usual, we can choose local coordinates $z_1,\ldots,z_{n-1}$ in $|V|$ defining the basis $(\frac{\partial}{\partial z_1},\ldots,\frac{\partial}{\partial z_{n-1}})$ in tangent spaces, then the a linear line complex $\omega \in \bigwedge^2 V^\vee$ is locally given by an expression

$$\sum_{i=1}^{n-1} A_i(z_1,\ldots,z_{n-1})dz_i = 0,$$

where dz_i are coordinates in the tangent space. This equation is called the *Pfaff partial differential equation*. More generally, any line complex of degree d can be considered as the zero set of a section of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(d)$ and can be locally defined by the *Monge's partial differential equation*

$$\sum_{i_1+\ldots+i_{n-1}=d}^{n} A_{i_1,\ldots,i_n} dz_1^{i_1}\ldots dz_{n-1}^{i_{n-1}} = 0.$$

If $F(\ldots,p_{ij},\ldots) = 0$ is the equation of the complex in Plücker coordinates, we dehomogenize it, by taking homogeneous coordinates $[x_1,\ldots,x_n] = (1, z_1,\ldots,z_{n-1})$ in $\mathbb{P}^{n-1}$ and replace the Plúcker coordinates with $p_{1,j} = dz_j, p_{ij} = z_i dz_j - z_j dz_i, i > 1$. The resulting equation become a Monge's partial differential equations. Recall that a solution of a Monge equation is a curve $z_i = \phi_i(t)$ such that replacing z_i with $\phi_i(t)$ and dz_i with $\frac{d\phi_i}{dt}$ we get the identity (in some open set). The tangent line to the curve at a point z defines a point in $q^{-1}(z)$ belonging to the divisor $p^{-1}(V(F))$, where p" $Z \to G_1(\mathbb{P}^{n-1})$ is the natural projection.

The projective equivalence classes of linear complexes coincide with the orbits of $\mathrm{GL}(V)$ acting naturally on $|\bigwedge^2 V^\vee|$. The $\mathrm{GL}(V)$-orbit of a linear complex X_ω is uniquely determined by the *rank* $2k$ of ω. We will identify ω with the associated linear map $V \to V^\vee$. Let Ker(ω) be the radical of the bilinear form ω (or the kernel of the corresponding linear map $V \to V^\vee$) and

$$C_\omega = |\mathrm{Ker}(\alpha_\omega)|. \tag{10.18}$$

It is called the *center* of a linear complex H. We have encountered with this in Chapter 2. This is a linear subspace of $\mathbb{P}(V)$ of dimension $n-1-2k$, where $2k$ is the rank of X_ω.

Proposition 10.2.1. *Let X_ω be a linear complex and C_ω be its center. Then*

$$\Omega(C_\omega) \subset X_\omega,$$

$$G_1(C_\omega) = \mathrm{Sing}(X_\omega).$$

Proof. Since $\mathrm{GL}(V)$ acts transitively on the set of linear complexes of equal rank, we may assume that $\omega = \sum_{i=1}^k e_i^* \wedge e_{k+i}^*$, where $e_1^*, \ldots, e_n^*$ is a basis of $V^\vee$ dual to a basis $e_1, \ldots, e_n$ of V. The linear space $\mathrm{Ker}(\omega)$ is spanned by $e_i, i > 2k$. A line l intersects C_ω if and only if it can be represented by a bivector $v \wedge w \in \bigwedge^2 V$, where $v \in C_\omega$. The linear span of $\Omega(C_\omega)$ is spanned by bivectors $e_i \wedge e_j$, where $i < 2k$. It is obvious that it is contained in the hyperplane $\langle X_\omega \rangle \subset \bigwedge^2 V$ defined by $\omega = 0$. This checks the first assertion.

It follows from Corollary 10.1.9 that

$$\ell \in \mathrm{Sing}(X_\omega) \Longleftrightarrow \mathbb{T}_\ell(G) \subset X_\omega \Longleftrightarrow \Omega(\ell) \subset X_\omega.$$

Suppose $\Omega(\ell) \subset X_\omega$ but ℓ does not belong to C_ω. We can find a point in ℓ represented by a vector $v = \sum a_i e_i$, where $a_i \neq 0$ for some $i \leq 2k$. Then the line represented by a 2-vector $v \wedge e_{k+i}$ intersects ℓ but does not belong to X_ω (since $\omega(v \wedge e_{k+i}) = a_i \neq 0$). Thus $\Omega(\ell) \subset X_\omega$ implies $\ell \subset C_\omega$. Conversely, this inclusion implies $\Omega(\ell) \subset \Omega(C_\omega) \subset X_\omega$. This proves the second assertion. $\square$

It follows from the proposition that any linear complex is singular unless its rank is equal to $2[\frac{n}{2}]$, maximal possible. Thus the set of hyperplanes in the Plücker space which are tangent to G can be identified with the set of linear complexes of rank $\leq 2[\frac{n-2}{2}]$. Consider $G(2, V^\vee)$ in its Plücker embedding in $\mathbb{P}(\bigwedge^2 V)$. Exchanging the roles of V and $V^\vee$, we obtain the following beautiful result.

Corollary 10.2.2. *Let $t = [\frac{n-4}{2}]$, then $Sec_t(G)$ is equal to the dual variety of the Grassmannian $G(2, V^\vee)$ in $\mathbb{P}(\bigwedge^2 V)$.*

When $n = 4, 5$ we see that $G(2, V)$ is dual to $G(2, V^\vee)$. When $n = 6$ we obtain that the dual of $G(2, V^\vee)$ is equal to $\mathrm{Sec}_1(G(2, V))$. This agrees with Example 10.1.3.

Let $\alpha_\omega : V \to V^\vee$ be defined by a skew-symmetric bilinear form ω on V. For any linear subspace E of V let

$$E_\omega^\perp = \alpha_\omega(\Lambda)^\perp = \{w \in V : \omega(v, w) = 0, \forall v \in E\}.$$

For any subspace $\Lambda = \mathbb{P}(E) \subset \mathbb{P}^n$ let

$$i_\omega(\Lambda) = \mathbb{P}(E_\omega^\perp).$$

It is clear that $[v \wedge w] \in G$ belongs to X_ω if and only if $\omega(v, w) = 0$, and hence if and only if $v, w \in E^\perp$, where E is the span of v and w. Thus

$$X_\omega = \{\ell \in G : \ell \subset i_\omega(\ell)\}. \tag{10.19}$$

Clearly $i_\omega(\Lambda)$ contains the center $C_\omega = \mathbb{P}(\mathrm{Ker}(\alpha_\omega))$ of H. Its dimension is equal to $n+1-\dim\Lambda+\dim\Lambda\cap C_\omega$.

Since ω is skew-symmetric, for any point $x \in \mathbb{P}(E)$,

$$x \in i_\omega(x).$$

When ω is nonsingular, we obtain a bijective correspondence between points and hyperplanes classically known as a *null-system*.

In the special case when $n = 3$ and $C_\omega = \emptyset$ this gives the *polar duality* between points and planes. The plane $\Pi(x)$ corresponding to a point x is called the *null-plane* of x. The point x_Π corresponding to a plane Π is called the *null-point* of Π. Note that $x \in \Pi(x)$ and $x_\Pi \in \Pi$. Also in this case the lines ℓ and $i_\omega(\ell)$ are called *polar lines*. They never intersect unless they coincide.

We also have a correspondence between lines in $\mathbb{P}^3$

$$i_\omega : G_1(\mathbb{P}^3) \to G_1(\mathbb{P}^3), \quad \ell \mapsto i_\omega(\ell).$$

Note that the lines ℓ and $i_\omega(\ell)$ are always skew or coincide. The set of fixed points of i_ω on $G_1(\mathbb{P}^3)$ is equal to X_ω. It is easy to see that i_ω corresponds to the projection of the Klein quadric $G = G_1(\mathbb{P}^3)$ in $\mathbb{P}^5$ from the point c dual to the hyperplane $\langle X_\omega\rangle$ with respect to G. Thus

$$G/(i_\omega) \cong \mathbb{P}^4.$$

The hyperplane $\langle X_\omega\rangle$ is the polar hyperplane $P_c(G)$. The ramification divisor of the projection $G \to \mathbb{P}^4$ is the linear compex $X_\omega = P_c(G) \cap G$. The branch divisor is a quadric in $\mathbb{P}^4$.

Proposition 10.2.3. *Let X_ω be a nonsingular linear complex in $G(2,n)$. Let ℓ be a line in $\mathbb{P}^{n-1}$. Then any line $\ell' \in X_\omega$ intersecting ℓ also intersects $i_\omega(\ell)$. The linear complex X_ω consists of lines intersecting the line ℓ and the codimension 2 subspace $i_\omega(\ell)$.*

Proof. Let $x = \ell \cap \ell'$. Since $x \in \ell'$, we have $\ell' \subset i_\omega(\ell') \subset i_\omega(x)$. Since $x \in \ell$, we have $i_\omega(\ell) \subset i_\omega(x)$. Thus $i_\omega(x)$ contains ℓ' and $i_\omega(\ell)$. Since X_ω is nonsingular, $\dim i_\omega(x) = n-2$, hence the line ℓ' intersects the $(n-3)$-plane $i_\omega(\ell)$.

Conversely, suppose ℓ' intersects ℓ at a point x and intersects $i_\omega(\ell)$ at a point x'. Then $x, x' \in i_\omega(\ell')$ and hence $\ell' = \overline{x, x'} \subset i_\omega(\ell')$. Thus ℓ' belongs to X_ω. □

Definition 10.1. *A linear complex X_ω in $|\bigwedge^2 V|$ is called* apolar *to a linear complex X_{ω^*} in $|\bigwedge^2 V^\vee|$ if $\omega^*(\omega) = 0$.*

In the case $n = 3$, we can identify $|\bigwedge^2 V|$ with $|\bigwedge^2 V^\vee|$ by using the polarity defined by the Klein quadric. Thus we can speak about apolar linear complexes in $\mathbb{P}^3$. In Plücker coordinates, this gives the relation

$$a_{12}b_{34} + a_{13}b_{24} - a_{14}b_{23} + a_{23}b_{14} - a_{24}b_{13} + a_{34}b_{12} = 0. \tag{10.20}$$

Lemma 10.2.4. *Let X_ω and $X_{\omega'}$ be two nonsingular linear compexes in $\mathbb{P}^3$ and $A = \alpha_\omega$ and $B = \alpha_{\omega'} : V \to V^\vee, B : V \to V^\vee$ be the corresponding linear maps. Then X_ω and $X_{\omega'}$ are apolar to each other if and only if $g = B^{-1} \circ A$, considered as a transformation of $|V|$, satisfies $g^2 = 1$.*

Proof. Take two skew lines ℓ, ℓ' in the intersection $X_\omega \cap X_{\omega'}$. Choose coordinates in V such that ℓ and ℓ' are two opposite edges of the coordinate tetrahedron $V(t_0t_1t_2t_3)$, say $\ell : t_0 = t_2 = 0$, and $\ell' : t_1 = t_3 = 0$. Then the linear complexes have the following equations in Plücker coordinates

$$H : ap_{12} + bp_{34} = 0; \quad H' : cp_{12} + dp_{34} = 0.$$

The condition that X_ω and $X_{\omega'}$ are apolar is $ad + bc = 0$. Now we have

$$A = \begin{pmatrix} 0 & a & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & -b & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & c & 0 & 0 \\ -c & 0 & 0 & 0 \\ 0 & 0 & 0 & d \\ 0 & 0 & -d & 0 \end{pmatrix},$$

$$B^{-1} = \begin{pmatrix} 0 & -c^{-1} & 0 & 0 \\ c^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -d^{-1} \\ 0 & 0 & d^{-1} & 0 \end{pmatrix}.$$

This gives

$$AB^{-1} = \begin{pmatrix} a/c & 0 & 0 & 0 \\ 0 & a/c & 0 & 0 \\ 0 & 0 & b/d & - \\ 0 & 0 & 0 & b/d \end{pmatrix} = \frac{a}{c} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

This shows that $(AB^{-1})^2$ defines the identical transformation of $|V|$. It is easy to see that conversely, this implies that $ad + bc = 0$. $\square$

In particular, a pair of apolar linear complexes defines an involution of $|V|$. Any pair of linear complexes defines a projective transformation of $|V|$ as follows. Take a point x, define its null-plane $\Pi(x)$ with respect to ω and then take its null-point y with respect to ω'. For apolar complexes we must get an involution. That is, the null-plane of y with respect to ω must coincide with the null-plane of x with respect to ω'.

Since any set of mutually apolar linear complexes is linearly independent, we see that the maximal number of mutually apolar linear complexes is equal to 6. If we choose these complexes as coordinates z_i in $\bigwedge^2 V$ we can write the equation of the Klein quadric as the sum

$$Q = \sum_{i=0}^{5} z_i^2.$$

Since each pair of apolar linear complexes defines an involution in $\mathbb{P}(\bigwedge^2 V)$ we obtain 15 involutions. They form an elementary abelian group of order 2^4 of projective transformations in $\mathbb{P}^3$. This group is called the *Heisenberg group*. The group originates from a linear non-abelian group $\mathcal{H}_2'$ of order 32, a central extension of $\mathcal{H}_2$

$$1 \to \mu_2 \to \mathcal{H}_2' \to \mathcal{H}_2 \to 1.$$

An example of six mutually apolar linear complexes is the set

$$(z_0,\ldots,z_5) = (p_{12}+p_{34}, i(p_{34}-p_{12}), p_{13}-p_{24}, -i(p_{24}+p_{13}), p_{14}+p_{23}, i(p_{23}-p_{14})),$$

where $i=\sqrt{-1}$. These coordinates in the Plücker space are called the *Klein coordinates*.

A set of six mutually apolar linear complexes define a (16_6)-configurations of points and planes. It is formed by 16 points and 16 planes in $\mathbb{P}^3$ such that each point is a null-point of 6 planes, each with respect to one of the six complexes. Also each plane is a null-plane of 6 points with respect to one of the six complexes. To construct such a configuration one can start from any point $p_1=[a_0,a_1,a_2,a_3]\in\mathbb{P}^3$ such that no coordinate is equal to zero. Assume that our six apolar complexes correspond to Klein coordinates. The first complex is $p_{12}+p_{34}=e_1^*\wedge e_2^*+e_3^*\wedge e_4^*$. It transforms the point p_1 to the plane $-a_1t_0+a_0t_1+a_3t_2-a_2t_3=0$. Taking other coordinates we get 5 more null-planes

$$\begin{aligned}
a_1t_0-a_0t_1+a_3t_2-a_2t_3 &= 0,\\
a_2t_0-a_3t_1-a_0t_2+a_1t_3 &= 0,\\
a_2t_0+a_3t_1-a_0t_2-a_1t_3 &= 0,\\
a_3t_0+a_2t_1-a_1t_2-a_0t_3 &= 0,\\
-a_3t_0+a_2t_1-a_1t_2+a_0t_3 &= 0.
\end{aligned}$$

Next we take the orbit of p_1 with respect to the Heisenberg group. It consists of 16 points. Computing the null-planes of each point we find altogether 16 planes forming with the 16 points a (16_6)-configurations. The following table gives the coordinates of the 16 points.

a_0,a_1,a_2,a_3	a_1,a_0,a_3,a_2	$a_0,-a_1,a_2,-a_3$	$a_1,-a_0,a_3,-a_2$
a_2,a_3,a_0,a_1	a_3,a_2,a_1,a_0	$a_2,-a_3,a_0,-a_1$	$a_3,-a_2,a_1,-a_0$
$a_0,a_1,-a_2,-a_3$	$a_1,a_0,-a_3,-a_2$	$a_0,-a_1,-a_2,a_3$	$a_1,-a_0,-a_3,a_2$
$a_2,a_3,-a_0,-a_1$	$a_3,a_2,-a_1,-a_0$	$a_2,-a_3,-a_0,a_1$	$a_3,-a_2,-a_1,a_0$

A point $(\alpha,\beta,\gamma,\delta)$ in this table is contained in 6 planes $at_0+bt_1+ct_2+dt_3=0$, where (a,b,c,d) is one of the following

$$(\delta,-\gamma,\beta,-\alpha),\ (\delta,\gamma,-\beta,-\alpha),\ (\gamma,\delta,-\alpha,-\beta),$$

$$(-\gamma,\delta,\alpha,-\beta),\ (-\beta,\alpha,\delta,-\gamma),\ (\beta,-\alpha,\delta,-\gamma).$$

Dually, a plane $\alpha t_0+\beta t_1+\gamma t_2+\delta t_3=0$ contains 6 points $[a,b,c,d]$, where (a,b,c,d) is as above.

Note that one checks directly that the six null-points of each of the 16 planes of the configuration lie on a conic. So we have a configuration of 16 conics in $\mathbb{P}^3$ each contains 6 points of the configuration. Also observe that any two conics intersect at 2 points.

There is a nice symbolic way to exhibit the (16_6)-configuration. After we fix an order on a set of 6 mutually apolar linear complexes we can identify nonzero elements of the Heisenberg group $\mathcal{H}_2$ with 2-element subsets of the set $\{1,2,3,4,5,6\}$ with addition defined by the symmetric sum where we replace a subset of cardinality 4 with its complementary subset. The empty set corresponds to the zero. A subset of two elements $\{i, j\}$ corresponds to the involution defined by a pair of apolar complexes. We take the ordered set of apolar linear complexes defined by the Klein coordinates. First we match the orbit of the point $[a_0, a_1, a_2, a_3]$ from the table from above with the first of the following tables. To find the 6 planes which contain a point from the (ij)-th spot we look at the same spot in the second of the following tables. Take the involutions in the i-th row and j-th column but not at the (ij)-spot. These involutions are matched with the planes containing the point. As always we identify a plane $a_0t_0 + a_1t_1 + a_2t_2 + a_3t_3$ with the point $[a_0, a_1, a_2, a_3]$. For example, the point $\emptyset$ is contained in 6 planes $(15), (13), (26), (46), (24), (35)$. Conversely, take a plane corresponding to the (ij)-th spot in the second table. The point contained in this plane can be found in the same row and the same column in the first table excluding the (ij)-th spot. For example, the plane $\emptyset$ contains the points $(45), (34), (35), (16), (12), (26)$.

$$
\begin{matrix} \emptyset & (45) & (34) & (35) \\ (16) & (23) & (25) & (24) \\ (12) & (36) & (56) & (46) \\ (26) & (13) & (15) & (14) \end{matrix}
\qquad
\begin{matrix} (14) & (15) & (13) & (26) \\ (46) & (56) & (36) & (12) \\ (24) & (25) & (23) & (16) \\ (35) & (34) & (45) & \emptyset \end{matrix}.
$$

Another way to remember the rule of the incidence is a follows. A point corresponding to an involution (ab) is contained in a plane corresponding to an involutiuon (cd) if and only if

$$(ab) + (cd) + (24) \in \{\emptyset, (16), (26), (36), (46), (56)\}.$$

Consider a regular map $\mathbb{P}^3 \to \mathbb{P}^4$ defined by the polynomials

$$x^4 + y^4 + z^4 + w^4,\ x^2w^2 + y^2z^2,\ y^2w^2 + x^2z^2,\ z^2w^2 + x^2y^2,\ xyzw.$$

Observe that this map is invariant with respect to the action of the Heisenberg group $\mathcal{H}_2$. So, it defines a regular map

$$\Phi : \mathbb{P}^3/\mathcal{H}_2 \to \mathbb{P}^4.$$

Proposition 10.2.5. *The map Φ defines an isomorphism*

$$\mathbb{P}^3/\mathcal{H}_2 \cong \mathcal{I}_4,$$

where $\mathcal{I}_4$ is a quartic hypersurface given by the equation

$$X^2V^2 - 2XYZW + Y^2Z^2 + Y^2W^2 - Y^2V^2 + Z^2W^2 - Z^2V^2 - W^2V^2 + V^4 = 0. \quad (10.21)$$

Proof. Since the map is given by 5 polynomials of degree 4, the degree of the map times the degree of the image must be equal to 4^3. We know that its degree must be multiple of 16, this implies that either the image is $\mathbb{P}^3$ or a quartic hypersurface. Since the polynomials are linearly independent the first case is impossible. A direct computation gives the equation of the image. $\square$

A quartic hypersurface projectively isomorphic to the hypersurface $\mathcal{I}_4$ will be called an *Igusa-Richmond quartic primal*. We will see a little later that it is the dual hypersurface of the Segre cubic primal.

Note that the fixed-point set of each non-trivial element of the Heisenberg group $\mathcal{H}_2$ consists of two skew lines given by equations

$$t_0 \pm t_1 = t_2 \pm t_3 = 0,\ t_0 \pm it_1 = t_2 \pm it_3 = 0,\ t_0 = t_1 = 0,$$

and equations obtained from those by permuting coordinates. Each line has a stabilizer subgroup of index 2. Thus the images of the 30 lines is the set of 15 double lines on $\mathcal{I}_4$. The stabilizer subgroup acts on the line as a Klein group 2^2. It has 6 points with non-trivial stabilizer of order 2. Altogether we have $30 \times 6 = 180$ such points which form 15 orbits. These orbits and the double lines from a (15_3)-configuration. The local equation of $\mathcal{I}_4$ at one of these orbits is $v^2 + xyz = 0$.

It was shown by J. Igusa that the quartic hypersurface defined by equation (10.21) is isomorphic to a compactification $\overline{\mathcal{A}}_2(2)$ of the moduli space of principally polarized abelian surfaces with level 2 structure. We refer to [130], [180], [177] for a moduli-theoretical interpretation of the $\mathcal{H}_2$-equivariant map $\Phi : \mathbb{P}^3 \to \mathcal{I}_4$.

10.2.2 6 lines

We know that any 5 lines in $\mathbb{P}^3$ are contained in a linear complex. In fact, in a unique linear complex when the lines are linearly independent as vectors in $\bigwedge^2 V$. A set of 6 lines is contained in a linear complex only if they are linearly dependent. The 6×6 matrix of its polar coordinates must have a nonzero determinant. An example of 6 dependent lines is the set of lines intersecting a given line ℓ. They are contained in the linear span of the Schubert linear complex $\Omega(\ell)$. We will give a geometric characterization of a set of 6 linearly dependent lines which contains a subset of 5 linearly independent lines.

Lemma 10.2.6. *Let $\sigma : \mathbb{P}^1 \to \mathbb{P}^1$ be an involution. Then its graph is an irreducible curve $\Gamma_g \subset \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(1,1)$ such that $\iota(\Gamma_g) = \Gamma_g$, where ι is the automorphism $(x,y) \mapsto (y,x)$. Conversely, any curve on $\mathbb{P}^1 \times \mathbb{P}^1$ with these properties is equal to the graph of some involution.*

Proof. This is easy and left to the reader. □

Corollary 10.2.7. *Let σ, τ be two different involutions of $\mathbb{P}^1$. Then there exists a unique common orbit $\{x,y\}$ with respect to σ and τ.*

We will need the following result of M. Chasles.

Theorem 10.2.8. *Let Q be a nondegenerate quadric in $\mathbb{P}^3$ and σ be an automorphism of order 2 of Q which is the identity on one of the rulings. Then the set of lines in $\mathbb{P}^3$ which are either contained in this ruling or intersect an orbit of lines in the second ruling form a linear complex. Conversely, any linear complex is obtained in this way from some pair (Q, σ).*

Proof. Consider the set X of lines defined as in the first assertion of the Theorem. Take a general plane Π and a point $x \in \Pi$. Consider the Schubert variety $\Omega(x, \Pi)$. It is a line in the Plücker space. The plane interesects Q along a conic C. Each line from $\Omega(x, \Pi)$ intersects C at two points. This defines an involution on C. Each line from the second ruling intersects C at one point. Hence σ defines another involution on C. By Corollary 8.21 there is a unique common orbit. Thus there is a unique line from $\Omega(x, \Pi)$ which belongs to X. Thus X is a linear complex.

Let ℓ_1, ℓ_2, ℓ_3 be any three skew lines in X. Let Q be a quadric containing these lines. It is obviously nonsingular. The lines belong to some ruling of Q. Take any line ℓ from the other ruling. Its polar line $\ell' = i_H(\ell)$ intersects ℓ_1, ℓ_2, ℓ_3 (because it is skew to ℓ or coincides with it). Hence ℓ' lies on Q. Now we have an involution on the second ruling defined by the polarity with respect to X. If $m \in X$ and is not contained in the first ruling, then m intersect a line ℓ from the second ruling, by Proposition 10.2.3, it also intersects ℓ'. This is the description of X from the assertion of the Theorem. □

Remark 10.2.2. Let C be the curve in $G(2, 4)$ parameterizing lines in a ruling of the quadric Q. Take a general line ℓ in $\mathbb{P}^3$. Then $\Omega(\ell)$ contains two lines from each ruling, the ones which pass through the points $Q \cap \ell$. This implies that C is a conic in the Plücker embedding. A linear complex X either intersects each conic at two points and contains two or one line from the ruling or contains C and hence contains all lines from the ruling.

Lemma 10.2.9. *Let ℓ be a line intersecting a nonsingular quadric Q in $\mathbb{P}^3$ at two different points x, y. Let $\mathbb{T}_x(Q) \cap Q = \ell_1 \cup \ell_2$ and $\mathbb{T}_y(Q) \cap Q = m_1 \cup m_2$, where ℓ_1, m_1 and ℓ_2, m_2 belong to the same ruling. Then the polar line $\ell_Q^\perp$ intersects Q at the points $x' = \ell_1 \cap m_2$ and $y' = \ell_2 \cap m_1$.*

Proof. Each line on Q is self-polar to itself. Thus $P_x(Q)$ is the tangent plane $\mathbb{T}_x(Q)$ and, similarly, $P_y(Q) = \mathbb{T}_y(Q)$. This shows that $\ell_Q^\perp = \mathbb{T}_x(Q) \cap \mathbb{T}_y(Q) = \overline{x', y'}$. □

Lemma 10.2.10. *let $\ell_1, \ell_2, \ell_3, \ell_4$ be four skew lines in $\mathbb{P}^3$. Suppose not all of them are contained in a quadric. Then there are exactly 2 lines which intersect all of them. These lines may coincide.*

Proof. This is of course well-known. It can be checked by using the Schubert calculus since $\sigma_1^4 = \# \cap_{i=1}^4 \Omega(\ell_i) = 2$. A better geometric proof can be given as follows. Let Q be the quadric containing the first 3 lines. Then ℓ_4 intersects Q at two points p, q which may coincide. The lines through these points belonging to the ruling not containing ℓ_1, ℓ_2, ℓ_3 intersect $\ell_1, \ldots, \ell_4$. Conversely, any line intersecting $\ell_1, \ldots, \ell_4$ is contained in this ruling (because it intersects Q at 3 points) and passes through the points $\ell_4 \cap Q$. □

Theorem 10.2.11. *Let $(\ell_1, \ldots, \ell_6)$ be a set of 6 lines and let $(\ell'_1, \ldots, \ell'_6)$ be the set of polar lines with respect to some nonsingular quadric Q. Assume that the first five lines are linearly independent in the Plücker space. Then $(\ell_1, \ldots, \ell_6)$ belong to a nonsingular linear complex if and only if there exists a projective transformation T such that $T(\ell_i) = \ell'_i$. This condition does not depend on the choice of Q.*

Proof. First let us check that this condition does not depend on a choice of Q. For each line ℓ let $\ell_Q^\perp$ denote the polar line with respect to Q. Suppose $A(\ell) = \ell_Q^\perp$ for some projective transformation A. Let Q' be another nonsingular quadric. We have to show that $\ell_{Q'}^\perp = B(\ell)$ for some other projective transformation B depending only on A but not on ℓ. Let us identify V with $\mathbb{C}^{n+1}$ and a quadric Q with a nonsingular symmetric matrix. Then $A(\ell) = \ell_Q^\perp$ means that $xQAy = 0$ for any vectors x, y in ℓ. We have to find a matrix B such that $xQ'By = 0$. We have

$$xQAy = xQ'(Q'^{-1}QA)y = xQ'By,$$

where $B = Q'^{-1}QA$. This checks the claim.

Suppose the set $(\ell_1, \dots, \ell_6)$ is projectively equivalent to $(\ell_1', \dots, \ell_6')$, where ℓ_i' are polar lines with respect to some quadric Q. Replacing Q with a quadric containing the first 3 lines ℓ_1, ℓ_2, ℓ_3, we may assume that $\ell_i' = \ell_i, i = 1, 2, 3$. We identify Q with $\mathbb{P}^1 \times \mathbb{P}^1$. If $\ell_j \cap Q = (a_j, b_j), (a_j', b_j')$ for $j = 4, 5, 6$, then, by Lemma 10.2.9, $\ell_j' \cap Q = (a_j, b_j'), (a_j', b_j)$. Suppose $\ell_i' = A(\ell_i)$. Then A fixes 3 lines in the first ruling hence sends Q to itself. It is also identical on the first ruling. It acts on the second ruling by switching the coordinates $(b_i, b_j'), j = 4, 5, 6$. Thus A^2 has 3 fixed points on $\mathbb{P}^1$, hence A^2 is the identity. This shows that $A = \sigma$ as in the Chasles Theorem. Hence the lines $\ell_i, \ell_i',\ i = 1, \dots, 6$, belong to the linear complex.

Conversely, assume $\ell_1, \dots, \ell_6$ belong to a nonsingular linear complex H. Applying Lemma 10.2.10, we find two lines ℓ, m intersecting $\ell_1, \ell_2, \ell_3, \ell_4$ (two transversals). By Proposition 10.2.3, the polar line $\ell' = i_H(\ell)$ intersects $\ell_1, \ell_2, \ell_3, \ell_4$. Hence it must coincide with either ℓ or m. The first case is impossible. In fact, if $\ell = \ell'$, then $\ell \in H$. The pencil of lines through $\ell \cap \ell_1$ in the plane $\overline{\ell, \ell_1}$ is contained in H. Similarly, the line $\Omega(\ell \cap \ell_2, \overline{\ell, \ell_2})$ is contained in H. Let Π be the plane of lines spanned by these two lines in G. It is contained in H. Thus Π cuts out in G a pair of lines. Thus H is singular at the point of intersections of these two lines. A contradiction.

Thus we see that $\ell, \ell' = m$ is a pair of polar lines. Now the pair of transversals $n, n' = i_H(n)$ of $\ell_1, \ell_2, \ell_3, \ell_5$ is also a pair of polar lines. Consider the quadric Q spanned by ℓ_1, ℓ_2, ℓ_3. The four transversals are the four lines from the second ruling of Q. We can always find an involution σ on Q which preserves the first ruling and such that $\sigma(\ell) = \ell', \sigma(n) = n'$. Consider the linear complex H' defined by the pair (Q, σ). Since $\ell_1, \dots, \ell_5$ belong to H, and any complex is determined by 5 linearly independent lines, we have the equality $H = H'$. Thus ℓ_6 intersects Q at a pair of lines in the second ruling which are in the involution σ. But σ is defined by the polarity with respect to H (since $\ell_1, \ell_2, \ell_3 \in H$ and the two involutions share two orbits corresponding to the pairs $(l, l'), (n, n')$). This implies $(\ell_1, \dots, \ell_6) = \sigma(\ell_1', \dots, \ell_6')$, where $\ell_i' = \ell_i^\perp$. ∎

Corollary 10.2.12. *Let $\ell_1, \dots, \ell_6$ be 6 skew lines on a nonsingular cubic surface S. Then they are linearly independent in the Plücker space.*

Proof. We first check that any 5 lines among the six lines are linearly independent. Assume that $\ell_1, \dots, \ell_5$ are linearly dependent. Then one of them, say ℓ_5, lies in the span of $\ell_1, \ell_2, \ell_3, \ell_4$. Let $(\ell_1', \dots, \ell_6')$ is the set of six skew lines which together with

$(\ell_1,\ldots,\ell_6)$ form a double-sixer. Then $\ell_1,\ell_2,\ell_3,\ell_4$ lie in the linear complex $\Omega(\ell_5')$, hence ℓ_5 lies in it too. But this is impossible because ℓ_5 is skew to ℓ_5'.

We know that there exists the unique quadric Q such that ℓ_i' are polar to Q with respect to Q (the Schur quadric). But $(\ell_1',\ldots,\ell_6')$ is not projectively equivalent to $(\ell_1,\ldots,\ell_6)$. Otherwise, S and its image S' under the projective transformation T will have 6 common skew lines. It will also have common transversals of each subset of 4. Thus the degree of the intersection curve is larger than 9. This shows that the cubic surfaces S and S' coincide and T is an automorphism of S. Its action on Pic(S) is a reflection with respect to the root corresponding to the double-sixer. It follows from Theorem 2.5.15 that S does not admit such an automorphism. □

Remark 10.2.3. The group $\mathrm{SL}(4)$ acts diagonally on the Cartesian product G^6. Consider the sheaf $\mathcal{L}$ on G^6 defined as the tensor product of the sheaves $p_i^*\mathcal{O}_G(1)$, where $p_i : G^6 \to G$ is the i-th projection. The group $\mathrm{SL}(4)$ acts naturally in the space of global sections of $\mathcal{L}$ and its tensor powers. Let

$$R = \bigoplus_{i=0}^{\infty} H^0(G^6, \mathcal{L}^i)^{\mathrm{SL}(4)}.$$

This is a graded algebra of finite type and its projective spectrum Proj(R) is denoted by $G^6//\mathrm{SL}(4)$. This is an example of a GIT-quotient. The variety G^6 has an open invariant Zariski subset U which is mapped to $G^6//\mathrm{SL}(4)$ with fibres equal to $\mathrm{SL}(4)$-orbits. This implies that $G^6//\mathrm{SL}(4)$ is an irreducible variety of dimension 9. Given 6 ordered general lines in $\mathbb{P}^3$ their Plücker coordinates make a 6×6 matrix. Its determinant can be considered as a section from the first graded piece R_1 of R. The locus of zeros of this section is a closed subvariety of G^6 whose general point is a 6-tuple of lines contained in a linear complex. The image of this locus in $G^6//\mathrm{SL}(4)$ is a hypersurface F. Now the duality of lines by means of a nondegenerate quadric defines an involution on G^6. Since it does not depend on the choice of a quadric up to projective equivalence, the involution descends to an involution of $G^6//\mathrm{SL}(4)$. The fixed points of this involution is the hypersurface F. One can show that the quotient by the duality involution is an open subset of a certain explicitly described 9-dimensional toric variety X (see [136]).

Finally, observe that a nonsingular cubic surface together with a choice of its geometric marking defines a double-sixer, which is an orbit of the duality involution in $G^6//\mathrm{SL}(4)$ and hence a unique point in X which does not belong to the branch locus of the double cover $G^6//\mathrm{SL}(4) \to X$. This embeds the 4-dimensional moduli space of marked nonsingular cubic surfaces in a 9-dimensional toric variety.

10.2.3 Linear systems of linear complexes

Let $W \subset \bigwedge^2 V^\vee$ be a linear subspace of dimension $r+1$. After projectivization and restriction to $G(2,V) \cong G_1(\mathbb{P}^{n-1})$ it defines a r-dimensional linear system of linear complexes. We have encountered already a net of linear complexes in $G(2,5)$ in Chapter 2. Let

$$\mathrm{Bs}(|W|) = \cap_{\omega\in W} X_\omega \subset G(2,V).$$

It is called the *base-locus* of $|W|$. It is a subvariety of $G(2,V)$ of dimension $2n-5-r$. Its canonical class is given by the formula

$$\omega_{\mathrm{Bs}(|W|)} \cong \mathcal{O}_{\mathrm{Bs}(|W|)}(r+1-n). \tag{10.22}$$

In particular, it is a Fano variety if $r<n-1$, a Calabi-Yau variety if $r=n-1$ and a variety of general type if $r>n-1$.

We also define the *center variety* $C_{|W|}$

$$C_{|W|} = \bigcup_{\omega\in W} C_\omega.$$

It is also called the *singular variety* of W.

For any $x=[v]\in C_{|W|}$ there exists $\omega\in W$ such that $\omega(v,v')=0$ for all $v'\in V$, or equivalently, the line $\ell=\overline{x,y}$ is contained in X_ω for all y. This implies that the codimension of $\Omega(x)\cap \mathrm{Bs}(|W|)$ in $\Omega(x)$ is $\le r$, less than expected $r+1$. Conversely, since $\Omega(x)$ is irreducible, if the codimension of the intersection $\le r$, then $\Omega(x)$ must be contained in some X_ω, and hence $x\in C_\omega$. Thus we have proved the following.

Proposition 10.2.13.

$$C_{|W|} = \{x\in|V| : \dim\Omega(x)\cap \mathrm{Bs}(|W|) > n-r-3\}$$

$$= \{x\in|V| : \Omega(x)\subset X_\omega \textit{ for some } \omega\in W\}.$$

For any linear subspace Λ in $|V|$ we can define the *polar subspace* with respect to $|W|$ by

$$i_{|W|}(\Lambda) = \bigcap_{\omega\in W} i_\omega(\Lambda).$$

Since $x\in i_\omega(x)$ for any linear complex X_ω, we obtain that, for any $x\in|V|$,

$$x\in i_{|W|}(x).$$

It is easy to see that

$$\dim i_\omega(x) = n-1-r+\dim|\{\omega\in W : x\in C_\omega\}|. \tag{10.23}$$

Now we are ready to give examples.

Example 10.2.1. A pencil $W|$ of linear complexes of lines in $\mathbb{P}^3$ is defined by a line in the Plücker space $\mathbb{P}^5$ which intersects the Klein quadric at ≤ 2. The intersection points correspond to special linear complexes of lines intersecting a given line. Thus, the base locus $|W|$ of a general pencil of line complexes consists of lines intersecting two skew lines. It is a nonsingular congruence of lines in $G_1(\mathbb{P}^3)$ of order and degree equal to 1. It is isomorphic to a nonsingular quadric in $\mathbb{P}^3$. It may degenerate to the union of an α-plane and a β-plane if the two lines are coplanar (in this case $|W|\subset G_1(3)$) or to a singular quadric if the two lines coincide.

Example 10.2.2. Assume $r = 1$ so we have a pencil of linear complexes. Assume $n = 2k + 1$ and $|W|$ does not intersect the set of linear complexes with corank > 1 (it is of codimension 3 in $|\bigwedge^2 V^\vee|$). Then we have a map $|W| \cong \mathbb{P}^1 \to \mathbb{P}^{2k}$ which assigns to $[\omega] \in |W|$ the center C_ω of X_ω. The map is given by the pfaffians of the principal minors of a skew-symmetric matrix of size $n \times n$, so the center variety $C_{|W|}$ of $|W|$ is a rational curve R_k of degree k in $\mathbb{P}^{2k}$. By Proposition 10.2.13 any secant line of R_k must be contained in Bs$(|W|)$. For example, taking $n = 3$, we obtain that the center variety is a conic in a plane, and the set of secants of the conic is contained in Bs$(|W|)$.

Now assume that $r = 2$. We obtain that $C_{|W|}$ is a projection of the Veronese surface $\nu_k(\mathbb{P}^2)$ and the variety of trisecant lines of the surface is contained in Bs$(|W|)$. We have seen it already in the case $k = 2$ (see Chapter 2, 2.1.3).

Example 10.2.3. Let $r = 3$ and $n = 5$ so we have a web $|W|$ of linear complexes in $\mathbb{P}^9 = \mathbb{P}(\bigwedge^2 V^\vee)$. We assume that $|W|$ is general enough. It intersects the Grassmann variety $G^* = G(2, V^\vee)$ in finitely many points. We know that the degree of $G(2,5)$ is equal to 5, thus $|W|$ intersects G^* at 5 points. Consider the rational map $|W| = \mathbb{P}^3 - \to C_{|W|} \subset \mathbb{P}^4$ which assigns to $[\omega] \in |W|$ the center of X_ω. As in the previous examples, the map is given by pfaffians of skew-symmetric matrices of size 4×4. They all vanish at the set of 5 points $p_1, \ldots, p_5$. The preimage of a general line in $\mathbb{P}^4$ is equal to the residual set of intersections of three quadrics, and hence consists of three points. Thus the map is birational map onto a cubic hypersurface. Any line joining two of the 5 points is blown down to a singular point of the cubic hypersurface. Thus the cubic is isomorphic to the Segre cubic primal. Observe now that the singular surface of $|W|$ is equal to the projection of the incidence variety $\{(x, \ell) \in \mathbb{P}^4 \times \text{Bs}(|W|) : x \in \ell\}$ to $\mathbb{P}^4$. It coincides with the center variety $C_{|W|}$.

One can see the center variety $C_{|W|}$ of $|W|$ as the degeneracy locus of the map of vector bundles

$$\sigma : W \otimes \mathcal{O}_{|V|} \to \Omega^1_{|V|}(2)$$

over $\mathbb{P}^{n-1}$ defined by identification of $H^0(|V|, \Omega^1_{|V|}(2))$ with $\bigwedge^2 V^\vee$ by means of the dual Euler sequence

$$0 \to \Omega^1_{|V|} \to V^\vee \otimes \mathcal{O}_{|V|}(-1) \to \mathcal{O}_{|V|} \to 0 \tag{10.24}$$

twisted by $\mathcal{O}_{|V|}(2)$. One uses that the map

$$V^\vee \otimes V^\vee = H^0(|V|, V^\vee \otimes \mathcal{O}_{|V|}(1)) \to H^0(|V|, \mathcal{O}_{|V|}(2)) = S^2 V^\vee$$

has kernel isomorphic to $\bigwedge^2 V^\vee$. Passing to the duals and using the Euler sequence we obtain that $C_{|W|}$ is equal to the degeneracy locus of the map

$$\phi : V \otimes \mathcal{O}_{|V|}(-1) \to W^\vee \otimes \mathcal{O}_{|V|}.$$

For any $x = [v] \in |V|$, the map of fibres $\phi(x)$ sends a vector v' to the linear function on W defined by $\omega \mapsto \omega(v, v')$. This linear function is equal to zero if and only if the line $\overline{[v], [v']}$ intersects Bs$(|W|)$. Applying Proposition 10.2.13, we obtain that the degeneracy locus of point $x = [v]$ for which the rank of $\phi(x)$ is smaller than $r+1$ must be equal to $C_{|W|}$.

If we choose coordinates and take a basis of W defined by $r+1$ skew-symmetric bilinear forms $\omega_k = \sum a_{ij}^{(k)} dt_i \wedge dt_j$, then the matrix is

$$\begin{pmatrix} \sum_{s=1}^n a_{1,s}^{(1)} t_s & \dots & \sum_{s=1}^n a_{n,s}^{(1)} t_s \\ \vdots & \vdots & \vdots \\ \sum_{s=1}^n a_{1,s}^{(r+1)} t_s & \dots & \sum_{s=1}^n a_{n,s}^{(r+1)} t_s \end{pmatrix},$$

where $a_{ij}^k = -a_{ji}^k$.

The expected dimension of the degeneracy locus is equal to $n-r-1$. Assume that this is the case. It follows from Example 14.3.2 in [173] that

$$\deg C_{|W|} = c_{n-r-1}(\Omega^1_{|V|}(2)) = \sum_{i=0}^{n-r-1} (-1)^i \binom{n-i-i}{r}. \tag{10.25}$$

Example 10.2.4. Assume $n = 2k$ is even. If $\omega \in W$ is nondegenerate, then $C_\omega = \emptyset$. Otherwise, $\dim C_\omega \geq 1$. Thus the varieties $C_{|W|}$ is ruled by linear subspaces. For a general W of dimension $1 < r < n-1$, the dimensions of these subspaces is equal to 1 and each point in $C_{|W|}$ is contained in a unique line C_ω. In other words, $C_{|W|}$ is a scroll with 1-dimensional generators parameterized by the subvariety B of $|W|$ parameterizing degenerate ω's. Thus B is equal to the intersection of $|W|$ with a pffafian hypersurface of degree k in $|\Lambda^2 V^\vee|$. The scrolls $C_{|W|}$ are called *Palatini scrolls*. If $n=4$, the only Palatini scroll is a quadric in $\mathbb{P}^3$. In $\mathbb{P}^5$ we get a 3-dimensional Palatini scroll of degree 7 defined by a web of linear complexes. The base B of the Palatini scroll is a cubic surface . We refer to [303] for the study of this scroll. There is also a Palatini ruled surface of degree 6 defined by a net of linear complexes. Its base is a plane cubic curve. If we take W with $\dim W = 5$, we get a quartic hypersurface in $\mathbb{P}^5$.

Remark 10.2.4. When $n = \dim V = 2k+1$ is even, the isomorphism

$$\bigwedge^2 V^\vee \cong H^0(|V|, \Omega^2_{|V|}(2))$$

from the dual Euler sequence (10.24) defines a bijections between symplectic structures on the vector space V defined by non-degenerate 2-forms $\theta \in \bigwedge^2 V^\vee$ and *contact structures* on $\mathbb{P}^{n-1} \cong |V|$ defined by surjective maps $\mathcal{T}_{|V|} \to \mathcal{O}_{|V|}(-2)$ dual to sections θ of $\Omega^2_{|V|}(2))$ satisfying $\theta \wedge \theta^k$ is a nowhere vanishing volume form.

10.3 Quadratic complexes

10.3.1 Generalities

Recall that a quadratic complex is the intersection K of the Grassmanian $G(2, V) \subset |\bigwedge^2 V|$ with a quadric hypersurface Q. Since $\omega_G \cong \mathcal{O}_G(-n)$, by the adjunction formula

$$\omega_K \cong \mathcal{O}_K(2-n).$$

If K is nonsingular, i.e. the intersection is transversal, we obtain that K is a Fano variety of index $n-1$.

Consider the incidence variety $Z \subset \mathbb{P}(\mathcal{Q}_G)$ and let Z_K be its restriction over K. We denote by $p_K : Z_K \to \mathbb{P}^{n-1}$ and $q_K : Z_K \to K$ the natural projections. For each point $x \in \mathbb{P}^{n-1}$ the fibre of p_K is isomorphic to the intersection of the Schubert variety $\Omega(x)$ with Q. We know that $\Omega(x)$ is isomorphic to $\mathbb{P}^{n-2}$ embedded in $\mathbb{P}(\bigwedge^2 V)$ as a linear subspace. Thus the fibre is isomorphic to a quadric in $\mathbb{P}^{n-2}$. This shows that K admits a structure of a *quadric bundle*, i.e. a fibration with fibres isomorphic to a quadric hypersurface. The important invariant of a quadric bundle is its *discriminant locus*. This is the set of points of the base of the fibration over which the fibre is a singular quadric or the whole space. In our case we have the following classical definition.

Definition 10.2. *The* singular variety Δ *of a quadratic complex is the set of points* $x \in \mathbb{P}^{n-1}$ *such that* $\Omega(x) \cap Q$ *is a singular quadric in* $\Omega(x) = \mathbb{P}^{n-1}$ *or* $\Omega(x) \subset Q$.

We will need the following fact from linear algebra rarely found in modern textbooks on the subject.

Lemma 10.3.1. *Let* $A = (a_{ij}), B = (b_{ij})$ *be two matrices of sizes* $k \times m$ *and* $m \times k$ *with* $k \le m$. *Let* $|A_I|, |B_I|, I = (i_1, \ldots, i_k), 1 \le i_1 < \ldots < i_k \le m$, *be maximal minors of* A *and* B. *For any* $m \times m$*-matrix* $G = (g_{ij})$

$$|A \cdot G \cdot B| = \sum_{I,J} g_{IJ} |A_I||B_J|,$$

where $g_{IJ} = g_{i_1 j_1} \cdots g_{i_k j_k}$.

Proof. Consider the product of the following block-matrices

$$\begin{pmatrix} A \cdot B & A \\ 0_{mk} & I_m \end{pmatrix} \cdot \begin{pmatrix} I_k & 0_{km} \\ -B & I_m \end{pmatrix} = \begin{pmatrix} 0_{kk} & A \\ -B & I_m \end{pmatrix},$$

where 0_{ab} is the zero matrix of size $a \times b$ and I_a is the identity matrix of size $a \times a$. The determinant of the first matrix is equal to $|A \cdot B|$, the determinant of the second matrix is equal to 1. Applying the Laplace formula, it easy to see that the determinant of the product is equal to $\sum |A_I||B_I|$. We apply this formula replacing A with $A \cdot G$. Write an j-th column of $A \cdot G$ as the sum $\sum_{i=1}^m g_{ij} A_i$. Then

$$|(A \cdot G)_{j_1, \ldots, j_k}| = \sum_{1 \le i_1 < \ldots < i_k \le m} g_{i_1 j_1} g_{i_2 j_2} \cdots g_{i_k j_k} |A_{i_1, \ldots, i_k}|.$$

This proves the assertion. □

Suppose we have a bilinear form $b : E \times E \to K$ on a vector space E over a field K with matrix $G = \big(b(e_i, e_j)\big)$ with respect to a basis $e_1, \ldots, e_m$. Let L be a subspace of E with basis $f_1, \ldots, f_k$. Then the matrix $G_L = b(f_i, f_j)$ is equal to the product ${}^t A \cdot G \cdot A$, where the $f_j = \sum a_{ij} e_i$. It follows from the previous Lemma that $|G_L| = \sum_{I,J} g_{IJ} |A_I||A_J|$. If we extend b to $\bigwedge^k E$ by the formula

$$b(v_1, \ldots, v_k; w_1, \ldots, w_k) = \det\big(b(v_i, w_j)\big),$$

then the previous formula gives an explicit expression for $b(f_1 \wedge \ldots \wedge f_k, f_1 \wedge \ldots \wedge f_k)$. If $E = \mathbb{R}^n$ and we take b to be the Euclidean inner-product, we get the well-known formula for the area of the parallelogram spanned by vectors $f_1, \ldots, f_k$ in terms of the sum of squares of maximal minors of the matrix with columns equal to f_j. If $m = 3$ this is the formula for the length of the cross-product of two vectors.

Proposition 10.3.2. *Δ is a hypersurface of degree $2(n-2)$.*

Proof. Consider the map

$$i : |V| \to |\bigwedge^2 V|, \; x \mapsto \Omega(x). \tag{10.26}$$

If $x = [v_0]$, the linear subspace of $\bigwedge^2 V$ corresponding to $\Omega(x)$ is the image of V in $\bigwedge^2 V$ under the map $v \mapsto v \wedge v_0$. This is a $(n-1)$-dimensional subspace $\Lambda(x)$ of $\bigwedge^2 V$ and hence defines a point in the Grassmann variety $G(n-1, \bigwedge^2 V)$. If we write $v_0 = \sum a_i e_i$, where we assume that $a_n \neq 0$, then $\Lambda(x)$ is spanned by the vectors $e_i \wedge v_0 = \sum_{j \neq i} a_j e_j \wedge e_i, i = 1, \ldots, n-1$. Thus the rows of the matrix of Plücker coordinates of the basis are linear functions in coordinates of v_0. Its maximal minors are polynomials of order n. Observe now that each (in)-th column contains a_n in the i-th row and has zero elsewhere. This easily implies that all maximal minors are divisible by a_n. Thus the Plücker coordinates of $\Lambda(x)$ are polynomials of degree $n-2$ in coordinates of v_0. We see now that the map i is given by a linear system of divisors of degree $n-2$. Fix a quadric Q in $|\bigwedge^2 V|$ which does not vanish on G. For any $n-2$-dimensional linear subspace L of $|\bigwedge^2 V|$ the intersection of Q with L is either a quadric or the whole L. Let us consider the locus D of L's such that this intersection is not a nonsingular quadric. We claim that this is a hypersurface of degree 2.

Let $b : E \times E$ be a nondegenerate symmetric bilinear form on a vector space E of dimension r. The restriction of b to a linear subspace $W \subset E$ with a basis $(w_1, \ldots, w_k)$ is a degenerate bilinear form if and only if the determinant of the matrix $\big(b(w_i, w_j)\big)$ is equal to zero. If we write $w_i = \sum a_{ij} e_j$ in terms of a basis in E, we see that this condition is polynomial of degree $2k$ in coefficients a_{ij}. By the previous Lemma this polynomial can be written as a quadratic polynomial in maximal minors of the matrix (a_{ij}). Applying this to our situation we interpret the maximal minors as the Plücker coordinates of L and obtain that D is a quadric hypersurface.

It remains to use that $\Delta = i^{-1}(Z)$, where i is given by polynomials of degree $n-1$. $\square$

Let

$$\Delta_k = \{x \in \Delta : \text{corank } Q \cap \Omega(x) \geq k\}.$$

These are closed subvarieties of Δ_k.

Let

$$\tilde{\Delta} = \{(x, \ell) \in Z_K : \text{rank } d_{p_K}(x, \ell) < n-1\}. \tag{10.27}$$

In other words, $\tilde{\Delta}$ is the locus of points in Z_K where the projection $p_K : Z_K \to \mathbb{P}^{n-1}$ is not smooth. This set admits a structure of a closed subscheme of Z_K defined locally by vanishing of the maximal minors of the jacobian matrix of the map p_K. Globally, we have the standard exact sequence of the sheaves of differentials

$$0 \to p_K^*\Omega^1_{\mathbb{P}^{n-1}} \xrightarrow{\delta} \Omega_{Z_K} \to \Omega^1_{Z_K/\mathbb{P}^{n-1}} \to 0, \tag{10.28}$$

and the support of $\tilde{\Delta}$ is equal to the set of points where $\Omega^1_{Z_K/\mathbb{P}^{n-1}}$ is not locally free. Locally the map δ is given by a matrix of size $(n-1)\times(2n-4)$. Thus $\tilde{\Delta}$ is given locally by $(n-1)\times(n-1)$ minors of this matrix and is of dimension $n-1$.

Tensoring (7.11) with the residue field $\kappa(p)$ at a point $p=(x,\ell)\in Z_K$, we see that $\tilde{\Delta}$ is equal to the degeneracy locus of points where the map $\delta_p : (p_K^*\Omega^1_{\mathbb{P}^{n-1}})_p \to (\Omega^1_{Z_K})_p$ is not injective. Using *Thom-Porteous formula* (see [173]), we can express the class of $\tilde{\Delta}$ in $H^*(Z_K,\mathbb{Z})$.

Definition 10.3. *Let Ω be a complex of lines of degree d in $G(2,n)$. A line ℓ in Ω is called* singular *if ℓ is a singular point of the intersection $\Omega(x)\cap\Omega$ for some $x\in\mathbb{P}^{n-1}$ or any point on $\Omega(x)$ if $\Omega(x)\subset\Omega$. The locus of singular lines is called the* singular variety *of Ω.*

Proposition 10.3.3. *Assume $n=2k$. Assume Ω is nonsingular. Then the singular variety $S(\Omega)$ of Ω is equal to the intersection of Ω with a hypersurface of degree $k(d-1)$.*

Proof. Let ℓ be a singular line of Ω and $\Omega = G\cap X$ for some smooth hypersurface of degree d. We have $\Omega(x)\subset\mathbb{T}_\ell(\Omega)=\mathbb{T}_\ell(G)\cap\mathbb{T}_\ell(X)$. Thus $\Omega(x)\subset\mathbb{T}_\ell(X)\cap G$. By Proposition 10.2.3, the linear complex $K=\mathbb{T}_\ell(X)\cap G$ consists of lines intersecting a line and its polar $(n-3)$-plane unless K is singular. Since $\Omega(x)$ is not contained in the Schubert variety of lines intersecting a codimension 2 linear subspace, we obtain that K is singular. This shows that the singular variety $S(\Omega)$ of Ω consists of lines in Ω such that $\mathbb{T}_\ell(X)$ coincides with a tangent hyperplane of G. In other words,

$$S(\Omega)=\gamma^{-1}(G^\vee),$$

where $\gamma:\Omega\to X^\vee$ is the restriction of the Gauss map $X\to X^\vee$ to Ω. Since Ω is nonsingular, X is nonsingular at any point of $X\cap G$, and hence γ is well-defined. It remains to use that γ is given by polynomials of degree $d-1$, the partials of X. ☐

Let $n=4$ and let Ω be a complex defined by a hypersurface $X=V(\Phi)$ of degree d in the Plücker space. The equation of $S(\Omega)$ in Plücker coordinates is easy to find. Let $\Phi_{ij}=\frac{\partial\Phi}{\partial p_{ij}}(l)$, where $[l]=\ell$. The tangent hyperplane to X at the point ℓ is given by the equation

$$\sum_{1\le i<j\le 4}\Phi_{ij}(l)p_{ij}=0.$$

Since the dual quadric G^* is given by the same equation as G, we obtain the equation of $S(X)$

$$\Phi_{12}\Phi_{34}-\Phi_{13}\Phi_{24}+\Phi_{14}\Phi_{23}=0.$$

10.3.2 Intersection of 2 quadrics

Let Q_1, Q_2 be two quadrics in $\mathbb{P}^{n-1}$ and $X = Q_1 \cap Q_2$. We assume that X is nonsingular. It follows from the proof of Proposition 8.5.1 that this is equivalent to the condition that the pencil $\mathcal{P}$ of quadrics spanned by Q_1, Q_2 has exactly n singular quadrics of corank 1. This set can be identified with a set of n points $p_1, \ldots, p_n$ in $\mathbb{P}^1 \cong \mathcal{P}$.

If $n = 2g+2$ is even, we get the associated nonsingular hyperelliptic curve C of genus g, the double cover of $\mathbb{P}^1$ branched at $p_1, \ldots, p_{2g+2}$.

The variety X is a Fano variety of degree 4 in $\mathbb{P}^{n-1}, n \geq 4$, of dimension $n-3$. Its canonical class is equal to $-(n-4)H$, where H is a hyperplane section. When $n = 5$ it is a quartic Del Pezzo surface.

Theorem 10.3.4. *(A. Weil). Assume $n = 2g+2$. Let $F(X)$ be the variety of $g-1$-dimensional linear subspaces contained in X. Then $F(X)$ is isomorphic to the Jacobian variety of the curve C and also to the intermediate Jacobian of X.*

Proof. We will restrict ourselves only with the case $g = 2$ leaving the general case to the reader. For each $\ell \in F(X)$ consider the projection map $p_\ell : X' = X \setminus \ell \to \mathbb{P}^3$. For any point $x \in X$ not on ℓ, the fibre over $p_\ell(x)$ is equal to the intersection of the plane $\ell_x = \langle \ell, x \rangle$ with X'. The intersection of this plane with a quadric Q from the pencil $\mathcal{P}$ is a conic containing ℓ and another line ℓ'. If we take two nonsingular generators of $\mathcal{P}$ we see that the fibre is the intersection of two lines or the whole $\ell' \in F(X)$ intersecting ℓ. In the latter case, all points on $\ell' \setminus \ell$ belong to the same fibre. Since all quadrics from the pencil intersect the plane $\langle \ell, \ell' \rangle$ along the same conic, there exists a unique quadric $Q_{\ell'}$ from the pencil which contains $\langle \ell, \ell' \rangle$. It belongs to one of the two rulings of planes on $Q_{\ell'}$ (or a unique family if the quadric is singular). Note that each quadric from the pencil contains at most one plane in each ruling which contains ℓ (two members of the same ruling intersect along a subspace of even codimension). Thus we can identify the following sets:

$$\text{pairs } (Q, r), \text{ where } Q \in \mathcal{P}, r \text{ is a ruling of planes in } Q,$$

$$B = \{\ell' \in F(X) : \ell \cap \ell' \neq \emptyset\}.$$

If we identify $\mathbb{P}^3$ with the set of planes of $\mathbb{P}^3$ containing ℓ, then the latter set is a subset of $\mathbb{P}^3$. Let D be the union of ℓ''s from B. The projection map p_ℓ maps D to B with fibres isomorphic to $\langle \ell, \ell' \rangle \setminus \{\ell\}$.

Extending p_ℓ to a morphism $f : \bar{X} \to \mathbb{P}^3$, where $\bar{X}$ is the blow-up of X with center at ℓ, we obtain that f is an isomorphism outside B and the fibres over points in B are isomorphic to $\mathbb{P}^1$. Observe that $\bar{X}$ is contained in the blow-up of $\bar{\mathbb{P}}^3$ along ℓ. The projection f is the restriction of the projection $\bar{\mathbb{P}}^5 \to \mathbb{P}^3$ which is a projective bundle of relative dimension 2. It is known how the intermediate Jacobian behaves change under blowing up of a smooth subvariety. This easily implies that $\text{Jac}(X) \cong \text{Jac}(B)$.

The crucial observation now is that B is isomorphic to our hyperelliptic curve C. In fact, consider the incidence variety

$$\mathcal{X} = \{(Q, \ell) \in \mathcal{P} \times G(2,6) : \ell \subset Q\}.$$

Its projection to $\mathcal{P}$ has fibre over Q isomorphic to the rulings of planes in Q. It consists of two connected components outside the set of singular quadrics and one connected component over the set of singular quadrics. Taking the Stein factorization we get a double cover of $\mathcal{P} = \mathbb{P}^1$ branched along 6 points. It is isomorphic to C.

Now the projection map p_ℓ maps each line ℓ' intersecting ℓ to a point in $\mathbb{P}^3$. We will identify the set of these points with the curve B. A general plane in $\mathbb{P}^3$ intersects B at $d = \deg B$ points. The preimage of the plane under the projection $p_\ell : X - \to \mathbb{P}^3$ is isomorphic to the complete intersection of 2 quadrics in $\mathbb{P}^4$. It is a Del Pezzo surface of degree 4 and hence is obtained by blowing up 5 points in $\mathbb{P}^2$. Thus $d = 5$. An easy argument using Riemann-Roch shows that B lies on a unique quadric $\mathcal{Q} \subset \mathbb{P}^3$. Its preimage under the projection $\bar{X} \to \mathbb{P}^3$ is the exceptional divisor E of the blow-up $\bar{X} \to X$. One can show that the normal bundle of ℓ in X is trivial, so $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ and hence $\mathcal{Q}$ is a nonsingular quadric. Thus (X, ℓ) defines a biregular model $B \subset \mathbb{P}^3$ of C such that B is of degree 5 and lies on a unique nonsingular quadric $\mathcal{Q}$. One can show that the latter condition is equivalent to that the invertible sheaf $\mathcal{O}_B(1) \otimes \omega_B^{-2}$ is not effective. It is easy to see that B is of bidegree $(2, 3)$.

Let us construct an isomorphism between Jac(C) and $F(X)$. Recall that Jac(C) is birationally isomorphic to the symmetric square $C^{(2)}$ of the curve C. The canonical map $C^{(2)} \to$ Pic$^2(C)$ defined by $x + y \mapsto [x + y]$ is an isomorphism over the complement of one point represented by the canonical class of C. Its fibre over K_C is the linear system $|K_C|$. Also note that Pic$^2(C)$ is canonically identified with Jac(X) by sending a divisor class ξ of degree 2 to the class $\xi - K_C$.

Each line ℓ' skew to ℓ is projected to a secant line of B. In fact, $\langle \ell, \ell' \rangle \cap X$ is a quartic curve in $\langle \ell . \ell' \rangle \cong \mathbb{P}^3$ that contains two skew line components. The residual part is the union of two skew lines m, m' intersecting both ℓ and ℓ'. Thus ℓ' is projected to the secant line joining two points on C which are the projections of the lines m, m'. If $m = m'$, then ℓ' is projected to a tangent line of B. Thus the open subset of lines in X skew to ℓ is mapped bijectively to an open subset of $C^{(2)}$ represented by "honest" secants of C, i.e. secants which are not 3-secants. Each line $\ell' \in F(X) \setminus \{\ell\}$ intersecting ℓ is projected to a point b of B. The line f of the ruling of $\mathcal{Q}$ intersecting B with multiplicity 3 and passing through a point $b \in B$ defines a positive divisor D of degree 2 such that $f \cap B = b + D$. The divisor class $[D] \in$ Pic$^2(C)$ is assigned to ℓ'. So we see that each trisecant line of B (they are necessary lie on $\mathcal{Q}$) defines three lines passing through the same point of ℓ. By taking a section of X by a hyperplane tangent to X at a point $x \in X$,we see that x is contained in 4 lines (taken with some multiplicity). Finally, the line ℓ itself corresponds to K_C. This establishes an isomorphism between Pic$^2(C)$ and $F(X)$.

□

Note that we have proved that X is a rational variety by constructing an explicit rational map from X to $\mathbb{P}^3$. This map becomes a regular map after we blow up a line ℓ on X. The image of the exceptional divisor is a quadric. This map blows down the union of lines on X that intersect ℓ to a genus 2 curve C of degree 5 lying on the quadric. The inverse map $\mathbb{P}^3 - \to X \subset \mathbb{P}^5$ is given by the linear system of cubic hypersurfaces through the curve C. It becomes a regular map after we blow-up C. Since any trisecant of C defined by one of the rulings of the quadric blows down to a point, the image of the proper transform of the quadric is the line ℓ on X. The exceptional divisor is mapped to the union of lines on X intersecting ℓ.

10.3.3 Kummer surfaces

We consider the case $n = 4$. The quadratic complex K is the intersection of two quadrics $G \cap Q$. We shall assume that K is nonsingular. Let C be the associated hyperelliptic curve of genus 2.

First let us look at the singular surface Δ of K. By Proposition 10.3.2 it is a quartic surface. For any point $x \in \Delta$ the conic $C_x = K \cap \Omega(x)$ is the union of 2 lines. A line in G is always equal to a one-dimensional Schubert variety. In fact, G is a nonsingular quadric of dimension 4, and hence contains two 3-dimensional families of planes. These are the families realized by the Schubert planes $\Omega(x)$ and $\Omega(\Pi)$. Hence a line must be a pencil in one of these planes, which shows that $C_x = \Omega(x,\Pi_1) \cup \Omega(x,\Pi_2)$ for some planes Π_1,Π_2 in $\mathbb{P}^3$. Any line in K is equal to some $\Omega(x,A)$ and hence is equal to an irreducible component of the conic C_x. Thus we see that any line in K is realized as an irreducible component of a conic $C_x, x \in K$. It follows from Theorem 10.3.4 that the variety of lines $F(K)$ in K is isomorhic to the Jacobian variety of C.

Proposition 10.3.5. *The variety A of lines in K is a double cover of the quartic surface Δ. The cover ramifies over the set Δ_1 of points such that the conic $C_x = p_K^{-1}(x)$ is a double line.*

Let $x \in \Delta$ and $C_x = \Omega(x,\Pi_1) \cup \Omega(x,\Pi_2)$. A singular point of C_x representing a line in K is called a *singular line* of K. If $x \notin \Delta_1$, then C_x has only one singular point equal to $\Omega(x,\Pi_1) \cap \Omega(x,\Pi_2)$. Otherwise, it has the whole line of them.

Let $S \subset K$ be the singular surface of K. By Proposition 10.3.3, S is a complete intersection of three quadrics.

By adjunction formula, we obtain $\omega_S \cong \mathcal{O}_S$. The assertion that S is nonsingular follows from its explicit equations (10.29) given below. Thus S is a K3-surface of degree 8.

Theorem 10.3.6. *The set of pairs (x,ℓ), where ℓ is a singular line containing x is isomorphic to the variety $\tilde{\Delta} \subset Z_K$, the locus of points where the morphism $p_K : Z_K \to \mathbb{P}^3$ is not smooth. It is a nonsingular surface with trivial canonical class. The projection $p_K : \tilde{\Delta} \to \Delta$ is a resolution of singularities. The projection $q_K : \tilde{\Delta} \to S$ is an isomorphism. The surface S is equal to $K \cap R$, where R is a quadric in $\mathbb{P}^5$.*

Proof. The first assertion is obvious since the fibres of $p_K : Z_K \to \mathbb{P}^3$ are isomorphic to the conics C_x. To see that q_k is one-to-one we have to check that a singular line ℓ cannot be a singular point of two different fibres C_x and C_y. The planes $\Omega(x)$ and $\Omega(y)$ intersect at one point $\ell = \overline{x,y}$ and hence span $\mathbb{P}^4$. If Q is tangent to both planes at the same point ℓ, then the two planes are contained in $\mathbb{T}_\ell(Q) \cap \mathbb{T}_\ell(G)$, hence $K = Q \cap G$ is singular at ℓ. This contradicts our assumption on K. Thus the projection $\tilde{\Delta} \to S$ is one-to-one. Since the fibres of $q_K : Z_K \to K$ are projective lines, this easily implies that the restriction of q_K to $\tilde{\Delta}$ is an isomorphism onto S.

□

Theorem 10.3.7. *The set Δ_1 consists of 16 points, each point is an ordinary double point of the singular surface Δ.*

Proof. Let A be the variety of lines in K. We know that it is a double cover of Δ ramified over the set Δ_1. Since Δ is isomorphic to S outside Δ_1, we see that A admits an involution with a finite set F of isolated fixed points such that the quotient is birationally isomorphic to a K3 surface. The open set $A \setminus F$ is an unramified double cover of the complement of $s = \#F$ projective lines in the K3 surface S. For any variety Z we denote by $e_c(Z)$ the topological Euler characteristic with compact support. By the additivity property of e_c, we get $e_c(A - S) = e(A) - s = 2(e_s(S) - 2s) = 48 - 4s$. Thus $e(A) = 48 - 3s$. Since $A \cong \text{Jac}(C)$, we have $e(A) = 0$. This gives $s = 16$. Thus Δ has 16 singular points. Each point is resolved by a (-2)-curve on S. This implies that each singular point is a rational double point of type A_1, i.e. an ordinary double point. □

Definition 10.4. *For any abelian variety A of dimension g the quotient of A by the involution $a \mapsto -a$ is denoted by* $\text{Kum}(A)$ *and is called the* Kummer variety *of A.*

Note that $\text{Kum}(A)$ has 2^{2g} singular point locally isomorphic to the cone over the Veronese variety $v_g(\mathbb{P}^{g-1})$. In the case $g = 2$ we have 16 ordinary double points. It is easy to see that any involution with this property must coincide with the negation involution (look at its action in the tangent space, and use that A is a complex torus). This gives

Corollary 10.3.8. *The singular surface of K is isomorphic to the Kummer surface of the Jacobian variety of the hyperelliptic curve C of genus 2.*

Proposition 10.3.9. *The surface S contains two sets of 16 disjoint lines.*

Proof. The first set is formed by the lines $q_K(p_K^{-1}(z_i))$, where $z_1, \ldots, z_{16}$ are the singular points of the singular surface. The other set comes from the dual picture. We can consider the dual incidence variety

$$\check{Z}_K = \{(\Pi, \ell) \in \check{\mathbb{P}}^3 \times K : \ell \subset \Pi\}.$$

The fibres of the projection to $\check{\mathbb{P}}^3$ are conics. Again we define the singular surface $\check{\Delta}$ as the locus of planes such that the fibre is the union of lines. A line in the fibre is a pencil of lines in the plane. These pencils form the set of lines in K. The lines are common to two pencils if lines are singular lines of K. Thus we see that the surface S can be defined in two ways using the incidence Z_K or $\check{Z}_K$. As before we prove that $\check{\Delta}$ is the quotient of the abelian surface A and is isomorphic to the Kummer surface of C. The lines in S corresponding to singular points of $\check{\Delta}$ is the second set of 16 lines. □

Choosing six mutually apolar linear complexes we write the equation of the Klein quadric as a sum of squares. The condition of non-degeneracy allows one to reduce the quadric Q to the diagonal form in these coordinates. Thus the equation of the quadratic complex can be written in the form

$$\sum_{i=0}^{5} t_i^2 = 0, \quad \sum_{i=0}^{5} a_i t_i^2 = 0. \tag{10.29}$$

Since K is nonsingular $a_i \neq a_j,\ i \neq j$. The parameters in the pencil corresponding to 6 singular quadrics are $(t_0, t_1) = (-a_0, 1), i = 0, \ldots, 5$. Thus the hyperelliptic curve C has the equation

$$y^2 = (t_1 + a_0 t_0) \cdots (t_1 + a_5 t_0),$$

which has to be considered as an equation of degree 6 in $\mathbb{P}(3,1,1)$. Since the dual of the quadric Q has the equation $\sum a_i^{-1} u_i$, and the dual of G has the equation $\sum u_i = 0$, the preimage of $\check{G}$ under the Gauss map defined by Q is the quadric $\sum a_i^{-1} t_i = 0$. This shows that the surface S, a nonsingular model, of the Kummer surface, is given by the equations

$$\sum_{i=0}^{5} t_i^2 = \sum_{i=0}^{5} a_i t_i^2 = \sum_{i=0}^{5} a_i^2 t_i^2 = 0. \tag{10.30}$$

We know that the surface given by the above equations contains 32 lines. Consider 6 lines ℓ_i in $\mathbb{P}^2$ given by the equations

$$X_0 + a_i X_1 + a_i^2 X_2 = 0, \quad i = 0, \ldots, 5. \tag{10.31}$$

Since the points $(1, a_i, a_i^2)$ lie on the conic $X_0 X_2 - X_1^2 = 0$, the lines ℓ_i are tangent to the conic.

Lemma 10.3.10. *Let $X \subset \mathbb{P}^{2k-1}$ be a variety given by complete intersection of k quadrics*

$$q_i = \sum_{j=0}^{2k-1} a_{ij} t_j^2 = 0,\ i = 1, \ldots, k.$$

Consider the group G of projective transformations of $\mathbb{P}^{2k-1}$ that consists of transformations

$$[t_0, \ldots, t_{2k-1}] \mapsto [\epsilon_0 t_0, \ldots, \epsilon_{2k-1} t_{2k-1}],$$

where $\epsilon_i = \pm 1$ and $\epsilon_0 \cdots \epsilon_{2k-1} = 1$. Then X/G is isomorphic to the double cover of $\mathbb{P}^{k-1}$ branched along the union of $2k$ hyperplanes with equations explicitly given below.

Proof. Let $R = \mathbb{C}[t_0, \ldots, t_{2k-1}]/(q_1, \ldots, q_k)$ be the ring of projective coordinates of X. Then the subring of invariants R^G is generated by the cosets of $t_0^2, \ldots, t_{2k-1}^2$ and $t_0 \cdots t_{2k-1}$. Since $(t_0 \cdots t_{2k-1})^2 = t_0^2 \cdots t_{2k-1}^2$, we obtain that

$$R^G \cong \mathbb{C}[T_0, \ldots, T_{2k-1}, T]/I,$$

where I is generated by

$$\sum_{j=0}^{2k-1} a_{ij} T_j,\ i = 1, \ldots, k, \quad T^2 - T_0 \cdots T_{2k-1}.$$

Let $A = (a_{ij})$ be the matrix of the coefficients a_{ij}. Its rank is equal to k. Choose new coordinates T_i' in $\mathbb{C}^{2k}$ such that $T'_{i+k-1} = \sum_{j=0}^{2k-1} a_{ij}T_j,\ i = 1, \ldots, k$. Write

$$T_i = \sum_{j=0}^{k-1} b_{ij}T_j' \mod (T_k', \ldots, T_{2k-1}'), i = 0, \ldots, 2k-1.$$

Then

$$X/G \cong \text{Proj } R^G \cong \text{Proj}(\mathbb{C}[T_0', \ldots, T_{k-1}', T])/(T^2 - \prod_{i=0}^{2k-1}\sum_{j=0}^{k-1} b_{ij}T_j').$$

Thus X/G is isomorphic to the double cover of $\mathbb{P}^{k-1}$ branched along the hyperplanes

$$\sum_{j=0}^{k-1} b_{ij}z_j, j = 0, \ldots, 2k-1.$$

□

Corollary 10.3.11. *Suppose the set of $2k$ points*

$$[a_{00}, \ldots, a_{k0}], \ldots, [a_{0\ 2k-1}, \ldots, a_{k\ 2k-1}]$$

in $\mathbb{P}^{k-1}$ is projectively equivalent to an ordered set of points on a Veronese curve of degree $k-1$. Then X/G is isomorphic to the double cover of $\mathbb{P}^{k-1}$ branched along the hyperplanes

$$a_{0j}z_0 + \ldots + a_{k-1j}z_{k-1} = 0, \quad i = 0, \ldots, 2k-1.$$

Proof. Choose coordinates such that the matrix $A = (a_{ij})$ has the form

$$A = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ \alpha_1 & \alpha_2 & \ldots & \alpha_{2k} \\ \vdots & \vdots & \vdots\vdots & \\ \alpha_1^{k-1} & \alpha_2^{k-1} & \ldots & \alpha_{2k}^{k-1} \end{pmatrix}.$$

Let

$$D_j = \prod_{1 \le i \le k, i \ne j} (\alpha_j - \alpha_i)$$

and

$$f(x) = (x-\alpha_1)\cdots(x-\alpha_k) = a_0 + a_1x + \ldots + a_kx^k,$$

$$f_j(x) = \frac{f(x)}{D_j(x-\alpha_j)} = a_{0j} + a_{1j}x + \ldots + a_{k-1j}x^{k-1},\ j = 1, \ldots, k.$$

We have

$$B = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ \alpha_1 & \alpha_2 & \ldots & \alpha_k \\ \vdots & \vdots & \vdots\vdots & \\ \alpha_1^{k-1} & \alpha_2^{k-1} & \ldots & \alpha_k^{k-1} \end{pmatrix}^{-1} = \begin{pmatrix} a_{01} & a_{11} & \ldots & a_{k-11} \\ a_{02} & a_{12} & \ldots & a_{k-12} \\ \vdots & \vdots & \vdots & \\ a_{0k} & a_{1k} & \ldots & a_{k-1k} \end{pmatrix}.$$

Multiplying A by B on the left we obtain

$$B\cdot A=\begin{pmatrix}1&0&0&\dots&0&f_1(\alpha_{k+1})&\dots&f_1(\alpha_{2k})\\0&1&0&\dots&0&f_2(\alpha_{k+1})&\dots&f_2(\alpha_{2k})\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\0&0&0&\dots&1&f_k(\alpha_{k+1})&\dots&f_k(\alpha_{2k})\end{pmatrix}$$

$$=\begin{pmatrix}f_1(\alpha_1)&\dots&f_1(\alpha_k)&f_1(\alpha_{k+1})&\dots&f_1(\alpha_{2k})\\f_2(\alpha_1)&\dots&f_2(\alpha_k)&f_2(\alpha_{k+1})&\dots&f_2(\alpha_{2k})\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\f_k(\alpha_1)&\dots&f_k(\alpha_k)&f_k(\alpha_{k+1})&\dots&f_k(\alpha_{2k})\end{pmatrix}.$$

The polynomials $f_1(x),\dots,f_k(x)$ form a basis in the space of polynomials of degree $\le k-1$. Thus we see that the columns of the matrix $B\cdot A$ can be taken as the projective coordinates of the images of points $[1,\alpha_1],\dots,[1,\alpha_{2k}]\in\mathbb{P}^1$ under a Veronese map. Under a projective transformation defined by the matrix B, the columns of matrix A is projectively equivalent to the set of points defined by the columns of the matrix $B\cdot A$. Write the matrix $B\cdot A$ in the block-form $[I_k\ C]$. Then the null-space of this matrix is the columns space of the matrix $[-C\ I_k]$. It defines the same set of points up to a permutation. □

The following Lemma is due to A. Verra.

Lemma 10.3.12. *Let X be the base locus of a linear system $\mathcal{N}$ of quadrics of dimension $k-1$ in $\mathbb{P}^{2k-1}$. Suppose that*

- *$\mathcal{N}$ contains a nonsingular quadric;*
- *X contains a linear subspace Λ of dimension $k-2$;*
- *X is not covered by lines intersecting Λ.*

Then X is birationally isomorphic to the double cover of $\mathcal{N}$ branched over the Hessian hypersurface of $\mathcal{N}$.

Proof. Let Λ be a linear subspace of dimension $k-2$ contained in X. Take a general point $x\in X$ and consider the span $\Pi=\langle\Lambda,x\rangle$. By our assumption x is not contained in any line. The restriction of the linear system $\mathcal{N}$ to Π is a linear system of quadrics in $\Pi\cong\mathbb{P}^{k-1}$ containing Λ and x in its base locus. The residual components of these quadrics are hyperplanes in Π containing x. The base locus of this linear system of hyperplanes consists only of x since otherwise x will be contained in a line on X intersecting Λ. By our assumption this is excluded. Thus the dimension of the restriction of $\mathcal{N}$ to Π is equal to $k-2$. This implies that there exists a unique quadric in $\mathcal{N}$ containing Π. This defines a rational map $X-\to\mathcal{N}$. A general member of $\mathcal{N}$ is a nonsingular quadric in $\mathbb{P}^{2k-1}$. It contains two rulings of $(k-1)$-planes. Our $(k-1)$-plane Π belongs to one of the rulings. The choice of a ruling to which Π belongs, defines a rational map to the double cover $Y\to\mathcal{N}$ branched along the Hessian variety of $\mathcal{N}$

parameterizing singular quadrics. The latter is constructed by considering the second projection of the incidence variety

$$\{(\Pi, Q) \in G_k(\mathbb{P}^{2k-1}) \times \mathcal{N} : \Pi \in \mathcal{N}\}$$

and applying the Stein factorization. Now we construct the inverse rational map $Y - \to X$ as follows. Take a nonsingular quadric $Q \in \mathcal{N}$ and choose a ruling of $(k-1)$-planes in Q. If $Q = V(q)$, then $\Pi = |L|$, where L is an isotropic k-dimensional linear subspace of the quadratic form q, hence it can be extended to a unique maximal isotropic subspace of q in any of the two families of such subspaces. Thus Λ is contained in a unique $(k-1)$-plane Π from the chosen ruling. The restriction of $\mathcal{N}$ to Π is a linear system of quadrics of dimension $k-2$ with Λ contained in the base locus. The free part of the linear system is a linear system of hyperplanes through a fixed point x. This point belongs to all quadrics in $\mathcal{N}$, hence belongs to X. So this point is taken to be the value of our map at the pair Q plus a ruling. $\square$

Applying this Lemma to the case when the linear system of quadrics consists of diagonal quadrics. We see that the Hessian hypersurface in $\mathcal{N}$ is the union of hyperplanes

$$\sum_{i=0}^{k} a_{ij} t_i = 0,\ j = 0, \ldots, 2k+1.$$

This shows that in the case when the hyperplanes, considered as points in the dual space, lie on a Veronese curve, the base locus X of $\mathcal{N}$ is birationally isomorphic to the quotient X/G.

This applies to our situation, and gives the following.

Theorem 10.3.13. *The surface S is birationally isomorphic to the double cover of $\mathbb{P}^2$ branched along the six lines ℓ_i. It is also birationally isomorphic to the quotient S/G, where G consists of involutions $[t_0, \ldots, t_5] \mapsto [\epsilon_0 t_0, \ldots, \epsilon_5 t_5]$ with $\epsilon_0 \cdots \epsilon_5 = 1$.*

Remark 10.3.1. Consider the double cover F of $\mathbb{P}^2$ branched over 6 lines $\ell_1, \ldots, \ell_6$ tangent to an irreducible conic C. It is isomorphic to a hypersurface in $\mathbb{P}(3,1,1,1)$ given by the equation $z^2 - f_6(x_0, x_1, x_2)$, where $V(f_6)$ is the union of 6 lines. The restriction of f_6 to the conic C is the divisor $2D$, where D is the set of points where the lines are tangent to C. Since $C \cong \mathbb{P}^1$ we can find a cubic polynomial $g(x_0, x_1, x_2)$ which cuts out D in C. Then the preimage of C in F is defined by the equation $z^2 - g_3^2 = 0$ and hence splits into the union of two curves $C_1 = V(z - g_3)$ and $C_2 = V(z + g_3)$ each isomorphic to C. These curves intersect at 6 points. The surface F has 15 ordinary double points over the points $p_{ij} = \ell_i \cap \ell_j$. Let $\bar{F}$ be a minimal resolution of F. It follows from the adjunction formula for a hypersurface in a weighted projective space that the canonical class of F is trivial. Thus $\bar{F}$ is a K3 surface. Since C does not pass through the points p_{ij} we may identify C_1, C_2 with their preimages in $\bar{F}$. Since $C_1 \cong C_2 \cong \mathbb{P}^1$, we have $C_1^2 = -2$. Consider the divisor class H on $\bar{F}$ equal to $C_1 + L$, where L is the preimage of a line in $\mathbb{P}^2$. We have

$$H^2 = C_1^2 + 2C_1 \cdot L + L^2 = C_1^2 + (C_1 + C_2) \cdot L + L^2 = -2 + 4 + 2 = 4.$$

We leave to the reader to check that the linear system $|H|$ maps $\bar{F}$ to a quartic surface in $\mathbb{P}^3$. It blows down all 15 exceptional divisors of $\bar{F} \to F$ to double points and blows down C_1 to the sixteenth double point.

Conversely, let Y be a quartic surface in $\mathbb{P}^3$ with 16 ordinary double points. Projecting the quartic from a double point q, we get a double cover of $\mathbb{P}^2$ branched along a curve of degree 6. It is the image of the intersection R of Y with the polar cubic $P_q(Y)$. Obviously, R the singular points of Y are projected to 15 singular points of the branch curve. A plane curve of degree 6 with 15 singular points must be the union of 6 lines $\ell_1, \ldots, \ell_6$. The projection of the tangent cone at q is a conic everywhere tangent to these lines.

Theorem 10.3.14. *A Kummer surface is projectively isomorphic to a quartic surface in $\mathbb{P}^3$ with equation*

$$A(x^4+y^4+z^4+w^4)+2B(x^2w^2+y^2z^2)+2C(y^2w^2+x^2z^2)$$
$$+2D(z^2w^2+x^2y^2)+4Exyzw=0, \tag{10.32}$$

where

$$A(A^2+E^2-B^2-C^2-D^2)+2BCD=0. \tag{10.33}$$

Proof. Choosing apolar linear complexes, we transform the Klein quadric to the form $t_1^2+\ldots+t_6^2=0$. Consider the Heisenberg group with nonzero elements defined by involutions associated to a pair of apolar linear complexes. The Heisenber group is induced by transformations of $\mathbb{P}^3$ listed in section 10.2.1. In these coordinates the equation of the Kummer surface must be invariant with respect to these transformations. It is immediately checked that this implies that the equation must be as in (10.32).

It remains to check the conditions on the coefficients. We know that a Kummer surface contains singular points. Taking the partial equations, we find

$$Ax^3+x(By^2+Cz^2+Dw^2)+Eyzw=0,$$
$$Ay^3+y(Bx^2+Cw^2+Dz^2)+Exzw=0,$$
$$Az^3+z(Bw^2+Cx^2+Dy^2)+Exyw=0,$$
$$Aw^3+w(Bz^2+Cz^2+Dx^2)+Exyz=0.$$

Multiplying the first equation by y and the second equation by x, and adding up the two equations, we obtain

$$(A+B)(x^2+y^2)+(C+D)(z^2+w^2)=\alpha\frac{x^2+y^2}{x^2y^2},$$

where $\alpha=-Exyzw$. Similarly, we get

$$(C+D)(x^2+y^2)+(A+B)(z^2+w^2)=\alpha\frac{z^2+w^2}{z^2w^2}.$$

Multiplying the first equation by $A+B$ and the second equation by $C+D$ and subtracting the second equation from the first one, we obtain

$$(A+B)^2-(C+D)^2+E^2=\alpha(A+B)(\frac{1}{x^2y^2}+\frac{1}{z^2w^2}).$$

Similarly, we get

$$(A-B)^2-(C-D)^2+E^2=-\alpha(A-B)(\frac{1}{x^2y^2}+\frac{1}{z^2w^2}),$$

hence,

$$\frac{(A+B)^2-(C+D)^2+E^2}{(A-B)^2-(C-D)^2+E^2}+\frac{A+B}{A-B}=0.$$

From this we easily derive (10.33). □

Equation (10.33) defines a cubic hypersurface in $\mathbb{P}^4$ isomorphic to the Segre cubic primal given by equation (9.46). The formulas making this isomorphism are

$$\begin{aligned} A &= z_0+z_3, \\ B &= z_0+z_1+2z_3+2z_4, \\ C &= z_0+z_2+2z_3+2z_4, \\ D &= -z_0-2z_1-2z_2-z_3, \\ E &= -2z_0+2z_3. \end{aligned} \tag{10.34}$$

Also it can be checked that equation (10.33) expresses the condition that the hyperplane $H: AX+BY+CZ+DW+EV=0$ is a tangent hyperplane of the Igusa-Richmond quartic primal (10.21). Thus the Segre cubic primal is projectively isomorphic to the dual hypersurface of the Igusa-Richmond quartic primal $\mathcal{I}_4$. Under the $\mathcal{H}_2$-equivariant map $\mathbb{P}^3 \to \mathbb{P}^4$ described in Proposition 10.2.5, the Kummer surface is equal to the preimage of the section of the Igusa-Richmond quartic by the hyperplane H. Since $\mathbb{P}^3/\mathcal{H}_2 \cong \mathcal{I}_4$, we obtain that the Kummer surface is isomorphic to the hyperplane section $H\cap\mathcal{I}_4$. We had already observed that $\mathcal{I}_4$ has 15 double lines. A section by a tangent hyperplane defines a quartic surface with 16 singular points confirming the fact that the Kummer surface has 16 singular points.

The sixteen singular points of the Kummer surface Y given by (10.32) form an orbit of $\mathcal{H}_2$. As we know this orbit defines a (16_6)-configuration. A plane containing a set of 6 points cuts out on Y a plane quartic curve with 6 singular points, no three lying on a line. This could happen only if the plane is tangent to the surface along a conic. This conic, or the corresponding plane, is called a *trope*. Again this confirms the fact that in any general $\mathcal{H}_2$-orbit a set of coplanar 6 points from the (16_6)-configuration lies on a conic.

On a nonsingular model of Y isomorphic to the octic surface S in $\mathbb{P}^5$ the exceptional curves (the singular lines of the quadratic complex) of the 16 singular points and the proper transforms of 16 tropes form the (16_6)-configuration of lines.

The existence of 16 tropes on the Kummer surface also follows from the following beautiful fact. Consider the Gauss map from Y to its dual surface $Y^\vee$ given by cubic partials. Obviously, it should blow down each trope to a singular point of $Y^\vee$. Thus $Y^\vee$ has at least 16 singular points. It is easy to see, as in the case of usual Plücker formulas, that each ordinary double decreases the degree of the dual surface by 2. Thus the degree of the dual surface $Y^\vee$ is expected to be equal to $36 - 32 = 4$. In fact we have the following beautiful fact.

Theorem 10.3.15. *A Kummer surface is projectively isomorphic to its dual surface.*

Proof. In the proof of Theorem 10.3.14 we had computed the partial cubics of equation (10.32). The linear system of the partial cubics is invariant with respect to the action of the Heisenberg group $\mathcal{H}_2$ and defines an isomorphism of projective representations. If we choose a basis appropriately, we will be to identify $\mathcal{H} - 2$-equivariantly the dual of the linear system with the original space $\mathbb{P}^3$. We know that the image of the surface is a quartic surface with 16 singular points. Since the tropes of the original surfaces are mapped to singular points of the dual surface, we see that the two surfaces share the same configurations of nodes and tropes. Thus they share 16 conics, and hence coincide (since the degree of intersection of two different irreducible surfaces is equal to 16). □

Remark 10.3.2. One can see the duality also from the duality of the quadratic complexes. If we identify the space $V = \mathbb{C}^4$ with its dual space by means of the standard basis e_1, e_2, e_3, e_4 and its dual basis $e_1^*, e_2^*, e_3^*, e_4^*$, then the Plücker coordinates $p_{ij} = e_i^\wedge e_j^*$ in $\bigwedge^2 V$ can be identified with the Plücker coordinates $p_{ij}^* = e_i \wedge e_j$ in $\bigwedge^2 V^\vee$. The Klein quadrics could be also identified. Now the duality isomorphism $G(2,V) \to G(2,V^\vee), \ell \mapsto \ell^\perp$, becomes compatible with the Plücker embeddings. The quadratic complex given in Klein coordinates by two diagonal quadrics (10.29) is mapped under the duality isomorphism to the quadratic complex given by two diagonal quadrics $\sum y_i^2 = 0, \sum a_i^{-1} y_i^2 = 0$, the dual quadrics. However, the intersection of these two pairs of quadrics is projectively isomorphic under the scaling transformation $y_i \mapsto \sqrt{a_i} y_i$. This shows that, under the duality isomorphism, the singular surfaces of the quadratic complex and its dual are projectively isomorphic. It follows from the definition of the duality that the tropes of the Kummer surface correspond to β-planes that intersect the quadratic complex along the union of two lines.

The Kummer surface admits an infinite group of birational automorphisms. For a general one, the generators of this group have been determined in modern works of J. Keum [245] and S. Kōndo [259]. We give only examples of some automorphisms.

- Projective automorphisms defined by the Heisenberg group. They correspond to translations by 2-torsion points on the abelian surface cover.
- Involutions defined by projections from one of 16 nodes.
- Switches defined by choosing a duality automorphism and composing it with elements of the Heisenberg group.
- Cubic transformations given in coordinates used in equation (10.32) by

$$(x, y, z, w) \mapsto (yzw, xzw, xyw, xyz)$$

  and composing them with elements of $\mathcal{H}_2$.

10.3.4 Harmonic complex

Consider a pair of quadrics Q_1 and Q_2 in $\mathbb{P}^3$. A *Harmonic complex* or a *Battaglini complex* is the closure in $G_1(\mathbb{P}^3)$ of the locus of lines which intersect Q_1 and Q_2 at two harmonically conjugate pairs. Let us see that this is a quadratic complex and find its equation.

Let $A = (a_{ij}), B = (b_{ij})$ be two symmetric matrices defining the quadrics. Let $\ell = \overline{x, y}$, where $x = [v], y = [w]$ for some $v, w \in \mathbb{C}^4$. Let $\ell = [sv + tw]$ be a parametric equation of ℓ. Then the restriction of Q_1 to ℓ is a binary form in s, t defined by $(vAw)s^2 + 2(vAw)st + (wAw)t^2$ and the restriction of Q_2 to ℓ is defined by the bilinear form $(vBw)s^2 + 2(vBw)st + (wBw)t^2$. By definition, the two roots of the binary forms are harmonically conjugate if and only if

$$(vAv)(wBw) + (wAw)(vBv) - 2(vAw)(vBw) = 0.$$

Let $[vw]$ be the matrix with two columns equal to the coordinate vectors of v and w. We can rewrite the previous expression in the form

$$\det\left({}^t[vw][AvBw]\right) + \det\left({}^t[v,w][BvAw]\right) = 0. \tag{10.35}$$

The expression is obviously a quadratic form on $\bigwedge^2(\mathbb{C}^4)$ and also a symmetric bilinear form on the space of symmetric matrices. Take the standard basis $E_{ij} + E_{ji}, E_{ii}, 1 \leq i \leq j \leq n$, of the space of symmetric matrices and compute the coefficients of the symmetric bilinear forms in terms of coordinates of v and w. We obtain

$$a_{ij;kl} = 4(x_ix_jy_ky_l + x_kx_ly_iy_j) - 2(x_ky_l + x_ly_k)(x_jy_i + x_iy_j) = 2(p'_{ik}p'_{jl} + p'_{il}p_{jk}),$$

where $p'_{ab} = p_{ab}$ if $a < b$ and $-p_{ab}$, otherwise. Thus (10.35) is equal to

$$\sum(a_{ij}b_{kl} + a_{kl}b_{ij})(p'_{ik}p'_{jl} + p'_{il}p'_{jk}) = 0.$$

This is an equation of a quadratic complex. If we assume that $a_{ij} = b_{ij} = 0$ if $i \neq j$, then the equation simplifies

$$\sum(a_{ii}b_{jj} + a_{jj}b_{ii})p_{ij}^2 = 0. \tag{10.36}$$

Note that we do not need to assume that $A \neq B$. If $A = B$, then the definition of harmonic self-conjugate pair implies that the two points in the pair coincide, i.e. the line is tangent to the quadric. This is a special case of the harmonic complex, the locus of tangent lines to a quadric.

Consider a pencil $\mathcal{P}$ of quadrics $\lambda Q_1 + \mu Q_2$. Let us assume for simplicity that the equations of the quadrics can be simultaneously diagonalized. Then a quadric from $\mathcal{P}$ touch a line ℓ if and only if

$$(\lambda a_{ii} + \mu b_{ii})(\lambda a_{jj} + \mu b_{jj})p_{ii}^2$$
$$= (\lambda^2 a_{ii}a_{jj} + \lambda\mu(a_{ii}b_{jj} + a_{jj}b_{ii}) + \mu^2 b_{ii}b_{jj})p_{ij}^2 = 0.$$

The restriction of the pencil to ℓ is a linear series g_1^1 unless ℓ has a base point in which case the line intersects the base locus of the pencil. The two quadrics which touch ℓ correspond to the points $[\lambda, \mu] \in \mathcal{P}$ which satisfy the equation in above. Denote by $A, 2B, C$ the coefficients at $\lambda^2, \lambda\mu, \mu^2$. The map

$$G(2,n) \to \mathbb{P}^2,\ \ell \mapsto [A, B, C]$$

is a rational map defined on the complement of codimension 3 subvariety of $G(2,n)$ given by the equations $A = B = C = 0$. Its general fibre is the loci of lines which touch a fixed pair of quadrics in the pencil. It is given by intersection of two quadric complexes. In case $n = 3$, we recognize a well-known fact that two conics have four common tangents. The preimage of a line $At_0 + 2Bt_1 + Ct_2 = 0$ with $AC - B^2 = 0$ is a complex of lines such that there is only one quadric in the pencil which touch the line. Hence it equals the Chow variety of the base locus, a hypersurface of degree 4 in $G(2,n)$.

Let us consider the case $n = 4$. In this case a harmonic complex is a special case of a quadratic complex given by two quadrics

$$\begin{aligned} q_1 &= p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0, \\ q_2 &= a_{12}p_{12}^2 + \cdots + a_{34}p_{34}^2 = 0. \end{aligned}$$

We assume that q_2 is a nonsingular quadric, i.e. all $a_{ij} \neq 0$. It is easy to see that the pencil $\lambda q_1 + \mu q_2 = 0$ has 6 singular quadrics corresponding to the parameters

$$[1, \pm\sqrt{a_{12}a_{34}}],\ [1, \pm\sqrt{a_{13}a_{24}}],\ [1, \pm\sqrt{a_{14}a_{23}}].$$

Thus we diagonalize both quadrics to reduce the equation of the quadratic complex to the form

$$t_0^2 + \ldots + t_5^2 = 0,$$

$$k_1(x_0^2 - x_1^2) + k_2(x_2^2 - x_3^2) + k_3(x_4^2 - x_5^2) = 0.$$

The genus 2 curve corresponding to the intersection of the two quadrics is a special one. Its branch points are $[1, \pm k_1], [1, \pm k_2], [1, \pm k_3]$. The involution of $\mathbb{P}^1$ defined by $[t_0, t_1] \mapsto [t_0, -t_1]$ leaves the set of branch points invariant and lifts to an involution of the genus 2 curve. It follows from the description of binary forms invariant under a projective automorphism of finite order given in section 8.7.4 that there is only one conjugacy class of involutions of order 2 and each binary sextic whose set of zeros is invariant with respect an involution can reduced to the form $(t_0^2 - t_1^2)(t_0^2 - \alpha t_1^2)(t_0^2 - \beta t_1^2)$. Thus we see that the harmonic complexes form a hypersurface in the moduli space of smooth complete intersections of two quadrics in $\mathbb{P}^5$. It is isomorphic to the hypersurface in $\mathcal{M}_2$ formed by isomorphism classes of genus 2 curves admitting two commuting involutions.

Proposition 10.3.16. *The singular surface of a harmonic complex is projectively isomorphic to a quartic surface given by equation* (10.32) *with coefficient E equal to 0.*

Proof. We use that in Klein coordinates our quadratic complex has additional symmetry defined by the transformation

$$(t_0, t_1, t_2, t_3, t_4, t_5) \mapsto (-it_1, it_0, -it_3, it_2, -it_5, it_4).$$

Here we may assume that $t_0 = i(p_{14} - p_{23}), t_1 = p_{14} + p_{23}$, etc. The transformation of $\mathbb{P}^3$ that induces this transformation is defined by $[x, y, z, w] \mapsto [-x, y, z, w]$. Equation (10.32) shows that in order the Kummer surface be invariant with respect to this transformation the coefficient E must be zero. $\square$

Note that under the isomorphism from the cubic (10.33) to the Segre cubic primal given by formulas (10.34), the coefficient E is equal to $-z_0 + z_3$. This agrees with a remark before Lemma 9.4.10.

Consider the Kummer surface S given by equation (10.32) with $E = 0$. Intersecting the surface with the plane $x = 0$ we obtain the plane quartic with equation $Q(x^2, y^2, z^2) = 0$ where $Q = A(s^2 + u^2 + v^2) + 2Bsu + 2Csv + 2Duv$. Its discriminant is equal to $A(A^2 - B^2 - C^2 - D^2) + 2BCD$. Comparing it with (10.33) we find that the quadratic form is degenerate. Thus the plane section of the Kummer surface is the union of two conics with equations $(ax^2 + by^2 + cz^2)(a'x^2 + b'y^2 + c'z^2) = 0$. The four intersection points of these conics are singular points of S. This easily follows from the equations of the derivatives of the quartic polynomial defining S. Thus we see that the 16 singular points of the Kummer surface lie by four in the coordinate planes $x, y, z, w = 0$. Following A. Cayley [49], a Kummer surface with this property is called a *Tetrahedroid* quartic surface.

Note the obvious symmetry of the coordinate hyperplane sections. The coordinates of 16 nodes can be put in the following symmetric matrix:

$$\begin{pmatrix} 0 & \pm a_{12} & \pm a_{13} & \pm a_{14} \\ \pm a_{21} & 0 & \pm a_{23} & \pm a_{24} \\ \pm a_{31} & \pm a_{32} & 0 & \pm a_{34} \\ a_{41} & \pm a_{42} & \pm a_{43} & 0 \end{pmatrix}.$$

The complete quadrangle formed by four nodes $p_1, \ldots, p_4$ in each coordinate plane has the property that the lines $\overline{p_i, p_j}$ and $\overline{p_k, p_l}$ with $\{i, j, \} \cap \{k, l\} = \emptyset$ intersect at the vertices of the coordinate tetrahedron. One can also find the 16 tropes. Take a vertex of the coordinate tetrahedron. There will be two pairs of nodes, not in the same coordinate plane, each pair lying on a line passing through the vertex. For example,

$$[0, a_{12}, a_{13}, a_{14}], [0, a_{12}, -a_{13}, a_{14}], [0, a_{21}, 0, a_{23}, a_{24}], [0, a_{21}, 0, -a_{23}, a_{24}].$$

The plane containing the two pairs contains the third pair. In our example, the third pair is $[a_{41}, -a_{42}, a_{43}, 0], [a_{41}, -a_{42}, -a_{43}, 0]$. This is one of the 16 tropes. Its equation is $a_{24}x + a_{14}y - a_{12}w = 0$. Similarly, we find the equations of all 16 tropes

$$\pm a_{34}y \pm a_{42}z \pm a_{23}w = 0,$$
$$\pm a_{34}x \pm a_{41}z \pm a_{13}w = 0,$$
$$\pm a_{24}x \pm a_{41}y \pm a_{12}w = 0,$$
$$\pm a_{23}x \pm a_{31}y \pm a_{12}z = 0.$$

Remark 10.3.3. For experts on K3 surfaces, let us compute the Picard lattice of a general Tetrahedroid. Let $\sigma : \tilde{S} \to S$ be a minimal resolution of S. Denote by h the class of the preimage of a plane section of S and by $e_i, i = 1, \ldots, 16$, the classes of the exceptional curves. Let c_1 and c_2 be the classes of the proper transforms of the conics C_1, C_2 cut out by one of the coordinate plane, say $x = 0$. We have

$$c_1 + c_2 = h - e_1 - e_2 - e_3 - e_4.$$

Obviously, $c_1 \cdot c_2 = 0$ and $h \cdot c_i = 2$ and $c_i^2 = -2$. Consider another coordinate plane and another pair conics. We can write

$$c_3 + c_4 = h - e_5 - e_6 - e_7 - e_8.$$

This shows that the classes of the eight conics can be expressed as linear combinations of classes h, e_i and $c = c_1$. It is known that the Picard group of a general Kummer surface is generated by the classes e_i and the classes of tropes t_i satisfying $2t_i = h - e_{i_1} - \ldots - e_{i_6}$. The Picard group of a Tetrahedroid acquires an additional class c.

The Jacobian variety of a genus 2 curve C with two commuting involutions contains an elliptic curve, the quotient of C by one of the involutions. In the symmetric product $C^{(2)}$ it represents the graph of the involution. Thus it is isogeneous to the product of two elliptic curves.

Note that the pencil of quadrics passing through the set of 8 points $(C_1 \cap C_2) \cup (C_3 \cap C_4)$ defines a pencil of elliptic curves on $\tilde{S}$ with the divisor class

$$2h - e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 - e_8 = c_1 + c_2 + c_3 + c_4.$$

Since $c_1 \cdot c_2 = c_3 \cdot c_4 = 0$, Kodaira's classification of fibres of elliptic fibrations shows that c_1, c_2, c_3, c_4 are the classes of irreducible components of a fibre of type I_4. This implies that the four intersection points $(C_1 \cup C_2) \cap (C_3 \cup C_4)$ lie on the edges of the coordinate tetrahedron.

The parameters A, B, C, D used to parameterize Tetrahedroid surfaces have be considered as points on the cubic surface

$$A(A^2 - B^2 - C^2 - D^2) + 2BCD = 0.$$

One can write an explicit rational parameterization of this surface using the formulas

$$A = 2abc,\ B = a(b^2 + c^2),\ C = b(a^2 + c^2),\ D = c(a^2 + b^2).$$

The formulas describe a rational map $\mathbb{P}^2 - \to \mathbb{P}^3$ of degree 2 given by the linear system of plane cubics with 3 base points $p_1 = [1, 0, 0], p_2 = [0, 1, 0], p_3 = [0, 0, 1]$. It extends to a degree 2 map from a Del Pezzo surface of degree 6 onto a 4-nodal cubic surface. In fact, if one considers the standard Cremona involution $[a, b, c] \mapsto [a^{-1}, b^{-1}, c^{-1}]$, then we observe that the map factors through the quotient by this involution. It has 4 singular points corresponding to the fixed points

$$[a, b, c] = [1, 1, 1],\ [-1, 1, 1],\ [1, -1, 1],\ [1, 1, -1].$$

of the Cremona involution. The corresponding singular points are the points $[1,1,1,1]$, $[1,1,-1,-1]$, $[1,-1,1,-1]$, $[1,-1,-1,1]$.

If we change the variables $X^2 = bcx^2, Y^2 = acy^2, X^2 = abx^2, W = w$, the equation

$$A(x^4+y^4+z^4+w^4)+2B(x^2w^2+y^2z^2)+2C(y^2w^2+x^2z^2)$$
$$+2D(z^2w^2+x^2y^2)=0$$

is transformed to equation

$$(X^2+Y^2+Z^2)(a^2X^2+b^2Y^2+c^2Z^2)-$$
$$[a^2(b^2+c^2)X^2W^2+b^2(c^2+a^2)Y^2W^2+c^2(a^2+b^2)Z^2W^2]+a^2b^2c^2W^4=0,$$

or, equivalently,

$$\frac{a^2x^2}{x^2+y^2+z^2-a^2w^2}+\frac{b^2y^2}{x^2+y^2+z^2-b^2w^2}+\frac{c^2z^2}{x^2+y^2+z^2-c^2w^2}=0. \quad (10.37)$$

When a,b,c are real numbers, the real points $(x,y,z,1)\in\mathbb{R}^3$ on this surface describe the propagation of light along the interface between two different media. The real surface with equation (10.37) is called a Fresnel 's *Wave surface*. It has 4 real nodes

$$(\pm c\sqrt{\frac{a^2-b^2}{a^2-c^2}},0,\pm a\sqrt{\frac{b^2-c^2}{a^2-c^2}},1),$$

where we assume that $a^2>b^2>c^2$. It has four real tropes given by planes $\alpha x+\beta y+\gamma z+w=0$, where

$$(\alpha,\beta,\gamma,1)=(\pm\frac{c}{b^2}\sqrt{\frac{a^2-b^2}{a^2-c^2}},0,\pm\frac{a}{b^2}\sqrt{\frac{b^2-c^2}{a^2-c^2}},1),$$

One of the two conics cut out on the surface by coordinate planes is a circle. On the plane $w=0$ at infinity one of the conics is the ideal conic $x^2+y^2+z^2=0$.

10.3.5 Tetrahedral complex

Consider the union of 4 planes in $\mathbb{P}^3$ which define a coordinate tetrahedron in the space. Let q_1,q_2,q_3,q_4 be its vertices, $\ell_{ij}=\overline{q_i,q_j}$ be its edges and $\pi_i=\overline{q_j,q_k,q_l}$ be its faces. Let $[A,B]\in\mathbb{P}^1$ and K be the closure of the set of lines in $\mathbb{P}^3$ intersecting the four faces at 4 distinct points with the cross ratio equal to $[A,B]$. Here we assume that the vertices of the tetrahedron are ordered in some way. It is easy to see that K is a complex line. It is called a *tetrahedral complex*.

Proposition 10.3.17. *K is a quadratic complex of lines. If p_{ij} are the Plücker coordinates with respect to the coordinates defined by the tetrahedron, then K is equal to the intersection of the Grassmannian with the quadric*

$$Ap_{12}p_{34}-Bp_{13}p_{24}=0. \quad (10.38)$$

Conversely, this equation defines a tetrahedral complex.

Proof. Let ℓ be a line spanned by the points $[a_1, a_2, a_3, a_4]$ and $[b_1, b_2, b_3, b_4]$. It intersects the face π_i at the point corresponding to the coordinates on the line $[s, t] = [b_i, -a_i], i = 1, \ldots, 4$. We assume that ℓ does not pass through one of the vertices. Then ℓ intersects the faces at four points not necessary distinct with cross ratio equal to $[p_{12}p_{34}, p_{13}p_{24}]$, where p_{ij} are the Plücker coordinates of the line. So, the equation of the tetrahedral complex containing the line is $[p_{12}p_{34}, p_{13}p_{24}] = [a, b]$ for some $[a, b] \in \mathbb{P}^1$. $\square$

Note that any tetrahedral complex T contains the set of points in $G(2, 4)$ satisfying $p_{is} = p_{it} = p_{ik} = 0$ (the lines in the coordinate plane $t_i = 0$). Also, any line containing a vertex satisfies $p_{ij} = p_{jk} = p_{ik} = 0$ and hence also is contained in K. Thus we obtain that K contains 4 planes from one ruling of the Klein quadric and 4 planes from another ruling. Each plane from one ruling intersects three planes from another ruling along a line and does not intersect the fourth plane.

Observe that the tetrahedral complex is reducible if and only if the corresponding cross ratio is equal to $0, 1, \infty$. In this case it is equal to the union of two hyperplanes representing lines intersecting one of the two opposite edges. An irreducible tetrahedral complex has 6 singular points corresponding to the edges of the coordinate tetrahedron. Their Plücker coordinates are all equal to zero except one.

Proposition 10.3.18. *The singular surface of an irreducible tetrahedral complex K defined by is equal to the tetrahedron of planes defining it.*

Proof. We know that the degree of the singular surface is equal to 4. So, it suffices to show that a general point in one of the planes of the tetrahedron belongs to the singular surface. The lines in this plane belong to the complex. So, the lines in the plane passing through a fixed point p_0 is an irreducible component of the conic $\Omega(p_0) \cap K$. This shows that p_0 belongs to the singular surface of K. $\square$

From now on we consider only irreducible tetrahedral complexes. There are different geometric ways to describe a tetrahedral complex.

First we need the following lemma, known as the *von Staudt's Theorem*.

Lemma 10.3.19. *Let ℓ be a line belonging to a tetrahedral complex T defined by the cross ratio R. Assume that ℓ does not pass through the vertices and consider the pencil of planes through ℓ. Then the cross ratio of the planes in the pencil passing through the vertices is equal to R.*

Proof. Let e_1, e_2, e_3, e_4 be a basis in $V = \mathbb{C}^4$ corresponding to the vertices of the tetrahedron. Choose the volume form $\omega = e_1 \wedge e_2 \wedge e_3 \wedge e_4$ and consider the *star-duality* in $\bigwedge^2 V$ defined by $(\alpha, \beta) = (\alpha \wedge \beta)/\omega$. Under this duality $(e_i \wedge e_j, e_k \wedge e_l) = 1(-1)$ if (i, j, k, l) is an even (odd) permutation of $(1, 2, 3, 4)$ and 0 otherwise. Let $\gamma = \sum_{1 \leq i < j \leq 4} p_{ij} e_i \wedge e_j$ be the 2-form defining the line ℓ and $\gamma^* = \sum p'_{ij} e_i \wedge e_j$ define the dual line ℓ^*, where $e_i \wedge e_j$ is replaced with $(e_i \wedge e_j, e_k \wedge e_l) e_k \wedge e_i$, where i, j, k, l are all distinct. The line ℓ (resp. ℓ^*) intersects the coordinate planes at the points represented by the columns of the matrix

$$A = \begin{pmatrix} 0 & p_{12} & p_{13} & p_{14} \\ -p_{12} & 0 & p_{23} & p_{24} \\ -p_{13} & -p_{23} & 0 & p_{34} \\ -p_{14} & -p_{24} & -p_{34} & 0 \end{pmatrix}, \text{ resp. } B = \begin{pmatrix} 0 & p_{34} & -p_{24} & p_{23} \\ -p_{34} & 0 & p_{14} & -p_{13} \\ p_{24} & -p_{14} & 0 & p_{12} \\ -p_{23} & p_{13} & -p_{12} & 0 \end{pmatrix}.$$

We have $A \cdot B = B \cdot A = 0$. It follows from the proof of the previous proposition that the cross ratio of the four points on ℓ^* is equal to $(p'_{13}p'_{24}, p'_{12}p'_{34}) = (p_{24}p_{13}, p_{24}p_{13})$. Thus ℓ and ℓ^* belong to the same tetrahedral complex . Now a plane containing ℓ can be identified with a point on ℓ^* equal to the intersection point. A plane containing e_1 and ℓ is defined by the 3-form

$$e_1 \wedge \gamma = p_{23}e_1 \wedge e_2 \wedge e_3 + p_{24}e_1 \wedge e_2 \wedge e_4 + p_{34}e_1 \wedge e_3 \wedge e_4$$

and we check that $e_1 \wedge \gamma \wedge (-p_{34}e_2 + p_{24}e_3 - p_{23}e_4) = 0$ since $B \cdot A = 0$. This means that the plane containing e_1 intersects ℓ^* at the first point on ℓ^* defined by the first column. Thus under the projective map from the pencil of planes through ℓ to the line ℓ^*, the plane containing e_1 is mapped to the intersection point of ℓ^* with the opposite face of the tetrahedron defined by $t_0 = 0$. Similarly, we check that the planes containing other vertices correspond to intersection points of ℓ^* with the opposite faces. This proves the assertion. □

Proposition 10.3.20. *A tetrahedral complex is the closure of secants of rational cubic curves in $\mathbb{P}^3$ passing through the vertices of the coordinate tetrahedron.*

Proof. Let R be one of those curves and $x \in R$. Projecting from x we get a conic C in the plane with four points, the projections of the vertices. Let $\ell = \overline{x, y}$ be a secant of R. The projection $\bar{y}$ of y is a point on the conic C and the pencil of lines through $\bar{y}$ is projectively equivalent to the pencil of planes through the secant ℓ. Under this equivalence the planes passing through the vertices of the tetrahedron correspond to the lines connecting their projection with $\bar{y}$. Applying von Staudt's Theorem, we conclude the proof. □

Consider the action of the torus $T = (\mathbb{C}^*)^4$ on $\mathbb{P}^3$ by scaling the coordinates in $V = \mathbb{C}^4$. Its action on $\bigwedge^2 V$ is defined by

$$(t_1, t_2, t_3, t_4) : (p_{12}, \ldots, p_{34}) \mapsto (t_1t_2p_{12}, \ldots, t_3t_4p_{34}), \quad (t_1, t_2, t_3, t_4) \in T.$$

It is clear that the Klein quadric is invariant with respect to this action. This defines the action of T on the Grassmannian of lines. It is also clear that the equations of a tetrahedral complex K are also invariant with respect to this action, so T acts on a tetrahedral complex. If $\ell \in K$ has nonzero Plücker coordinates (a general line), then the stabilizer of ℓ is equal to the kernel of the action of T in $\mathbb{P}^3$, i.e. equal to the diagonal group of $(z, z, z, z), z \in \mathbb{C}^\vee$. Thus the orbit of ℓ is 3-dimensional, and since K is irreducible and 3-dimensional, it is a dense Zariski subset of K. Thus we obtain that K is equal to the closure of a general line in $G(2, 4)$ under the torus action. Since any general line belongs to a tetrahedral complex we get an equivalent definition of a tetrahedral complex as the closure of a torus orbit of a line with nonzero Plücker coordinates.

Here is another description of a tetrahedral complex. Consider a projective automorphism $\phi : \mathbb{P}^3 \to \mathbb{P}^3$ with four distinct fixed points and let K be the closure of lines $\overline{x, \phi(x)}$, where x is not a fixed point of ϕ. I claim that K is an irreducible tetrahedral complex. Choose the coordinates in $\mathbb{C}^4$ such that the matrix of ϕ is a diagonal matrix with 4 distinct eigenvalues λ_i. Then K is the closure of lines defined by 2-vectors $\gamma = A \cdot v \wedge v, v \in \mathbb{C}^4$. A straightforward computation shows that the Plücker coordinates of γ are equal to $p_{ij} = t_i t_j(\lambda_i - \lambda_j)$, where $(t_1, \ldots, t_4)$ are the coordinates of the vector v. Thus if we take v with nonzero coordinates we obtain that K contains the torus orbit of the vector with nonzero Plücker coordinates $p_{ij} = \lambda_i - \lambda_j$. As we explained in above, K is an irreducible tetrahedral complex.

It is easy to see that the map which assigns to a point $x \in \mathbb{P}^3$ the line $\overline{x, \phi(x)}$ defines a birational transformation $\Phi : \mathbb{P}^3 - \to K$ with fundamental points at the fixed points of ϕ. It is given by quadrics. The linear system of quadrics through 4 general points in $\mathbb{P}^3$ is of dimension 5 and defines a rational map from $\mathbb{P}^3$ to $\mathbb{P}^5$. The preimage of a general plane is equal to the intersection of 3 general quadrics in the linear system. Since there are 4 base points, we obtain that the residual intersection consists of 4 points. This implies that the linear system defines a map of degree 2 onto a quadric in $\mathbb{P}^5$ or a degree 1 map onto a threefold of degree 4. Since a tetrahedral complex is obtained in this way and any 4 general points in $\mathbb{P}^3$ are projectively equivalent, we see that the image must be projectively isomorphic to a tetrahedral complex. Observe, that the 6 lines joing the pairs of fixed points of ϕ are blown down to singular points of the tetrahedral complex. Also, we see the appearance of 8 planes, four of these planes are the images of the exceptional divisors of the blow-up of $\mathbb{P}^3$ at the fixed points, and the other four are the images of the planes spanned by three fixed points. We see that the blow-up of $\mathbb{P}^3$ is a small resolution of the tetrahedral complex.

There is another version of the previous construction. Take a pencil of quadrics $\mathcal{Q}$ with nonsingular base curve. Consider the rational map $\mathbb{P}^3 - \to G_1(\mathbb{P}^3)$ which assigns to a point $x \in \mathbb{P}^3$ the intersection of the polar planes $P_x(Q), Q \in \mathcal{Q}$. This is a line in $\mathbb{P}^3$ unless x is a singular point of one of quadrics in $\mathcal{Q}$. Under our assumption on the pencil, there are exactly 4 such points which we can take as the points $[1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0], [0, 0, 0, 1]$. Thus we see that the rational map is of the same type as in the previous construction and its image is a tetrahedral complex.

10.4 Ruled surfaces

10.4.1 Scrolls

A *scroll* or a *ruled variety* is an irreducible subvariety S of $\mathbb{P}^N$ such that there exists an irreducible family X_0 of linear subspaces of dimension r sweeping S such that a general point of S lies in unique a r-plane from the family and no point lies in infinitely many generators. Following the classical terminology, the linear subspaces are called *generators*. Note that the condition that any point lies in finitely many generators excludes cones.

We identify X_0 with its image in the Grassmann variety $G_r(\mathbb{P}^N)$. For any $x \in X_0$ let Λ_x denote the ruling defined by the point x. The universal family

$$\{(x,p) \in X_0 \times \mathbb{P}^{n-1} : p \in \Lambda_x\}$$

is isomorphic to the restriction Z_{X_0} of the incidence variety $Z_G \to G_r(\mathbb{P}^N)$ over X_0. Let $\mathcal{S}_{X_0}$ be the restriction of the dual of the universal subbundle $\mathcal{S}_G$ to X_0. We have $Z_{X_0} \cong \mathbb{P}(\mathcal{S}_{X_0}^\vee)$. The projection $Z_{X_0} \to \mathbb{P}^{n-1}$ is a finite morphism of degree 1 which sends the fibres of the projective bundle $Z_{X_0} \to X_0$ to generators. For any finite morphism $\nu : X \to X_0$ of degree 1, the pull-back $\mathcal{E} = \nu^*(\mathcal{S}_{X_0}^\vee)$ defines a projective bundle $\mathbb{P}(\mathcal{E})$ and a finite morphism $\tilde{\nu} : \mathbb{P}(\mathcal{E}) \to Z_{X_0}$ such that the composition $f : \mathbb{P}(\mathcal{E}) \to Z_{X_0} \to S$ is a finite morphism sending the fibres to generators. Recall that the projection $Z_G \to \mathbb{P}^N = |V|$ is defined by a surjection of locally free sheave $\alpha : V^\vee \otimes \mathcal{O}_G \to \mathcal{S}_G^\vee$. Thus the morphism $f : \mathbb{P}(\mathcal{E}) \to S \subset \mathbb{P}^N$ is defined by a surjection

$$\nu^*(\alpha) : V^\vee \otimes \mathcal{O}_X \to \mathcal{E}.$$

In particular, the morphism f is a given by a linear system $|V^\vee| \subset |\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|$.

Thus we see that any scroll is obtained as the image of a birational morphism

$$f : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^N = |V|$$

defined by a linear system in $|\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|$. The linear system can be identified with the image of $V^\vee \to H^0(X, \mathcal{E})$ under the surjective map $V^\vee \otimes \mathcal{O}_X \to \mathcal{E}$. This map also gives a finite map $\nu : X \to X_0 \subset G(r+1, V)$. The base X of the projective bundle $\pi : \mathbb{P}(\mathcal{E}) \to X$ can be always assumed to be a normal variety. Then $\nu : X \to X_0$ is the normalization map.

A scroll defined by the complete linear system $|\mathbb{P}(\mathcal{E})(1)|$ is a linearly normal subvariety of $\mathbb{P}^{n-1}$. It is called a *normal scroll*. Any scroll is a projection of a normal scroll. Note that in many text-books a normal scroll is assumed to be a nonsingular variety. We have already classified smooth rational normal scrolls of dimension 2 in Chapter 8.

For any X-scheme $g : Y \to X$ an X-morphism $Y \to \mathbb{P}(\mathcal{E})$ is defined by an invertible sheaf $\mathcal{L}$ on Y and surjective map of sheaves $g^*\mathcal{E} \to \mathcal{L}$. It is equal to the composition of the canonical maps $Y = \mathbb{P}(\mathcal{L}) \to \mathbb{P}(g^*\mathcal{E}) = Y \times_X \mathbb{P}(\mathcal{E})$ and the projection to the second factor. Taking g to be the identity map $X \to X$, we obtain a bijection between sections of $\pi : \mathbb{P}(\mathcal{E}) \to X$ and surjective maps of sheaves $\mathcal{E} \to \mathcal{L}$, where $\mathcal{L}$ is an invertible sheaf on X.

A surjective map of locally free sheaves $\mathcal{E} \to \mathcal{F}$ defines a closed embedding $\mathbb{P}(\mathcal{F}) \hookrightarrow \mathbb{P}(\mathcal{E})$. If rank $\mathcal{F} = r'+1$, the image of $\mathbb{P}(\mathcal{F})$ under the map $f : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^{n-1}$ is a r'*-directrix* of the scroll, a closed subvariety intersecting each generator along a r'-plane. If $r' = 0$, we get a section of $\mathbb{P}(\mathcal{E})$. Its image is *directrix* of the scroll, a closed subvariety of the scroll that intersects each generator at one point. Note that not every directrix comes from a section, for example a generator could be a directrix.

Suppose $\mathcal{E} \to \mathcal{E}_1$ and $\mathcal{E} \to \mathcal{E}_2$ are two surjective maps of locally free sheaves sheaves on a smooth curve X. Let $\mathcal{E} \to \mathcal{E}_1 \oplus \mathcal{E}_2$ be the direct sum of the maps and $\mathcal{E}'$ be the image of this map which is locally free since X is a smooth curve. Assume that quotient sheaf $(\mathcal{E}_1 \oplus \mathcal{E}_2)/\mathcal{E}'$ is a skyscrapper sheaf. Then the surjection $\mathcal{E} \to \mathcal{E}'$ corresponds to a closed embedding $j : \mathbb{P}(\mathcal{E}') \hookrightarrow \mathbb{P}(\mathcal{E})$. We call the projective bundle $\mathbb{P}(\mathcal{E}')$ the *join* of $\mathbb{P}(\mathcal{E}_1)$ and $\mathbb{P}(\mathcal{E}_2)$ and denote it by $\langle \mathbb{P}(\mathcal{E}_1), \mathbb{P}(\mathcal{E}_2)\rangle$. The compositions $\mathcal{E} \to \mathcal{E}' \to \mathcal{E}_i$ are surjective maps, hence the projections $\mathcal{E}' \to \mathcal{E}_i$ are surjective and therefore define closed embedding $\mathbb{P}(\mathcal{E}_i) \hookrightarrow \langle \mathbb{P}(\mathcal{E}_1), \mathbb{P}(\mathcal{E}_2)\rangle$.

It follows from (10.8) that

$$\omega_{\mathbb{P}(\mathcal{E})/X} \cong \pi^*(\det \mathcal{E})(-r-1). \tag{10.39}$$

If X admits a canonical sheaf ω_X, we get

$$\omega_{\mathbb{P}(\mathcal{E})} \cong \pi^*(\omega_X \otimes \pi^* \det \mathcal{E})(-r-1). \tag{10.40}$$

Let $\xi = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$. Recall that the Chern classes $c_i(\mathcal{E})$ can be defined by using the identity in $H^*(\mathbb{P}(\mathcal{E}), \mathbb{Z})$ (see [206], Appendix A):

$$(-\xi)^{r+1} + \pi^*(c_1(\mathcal{E}))(-\xi)^r + \ldots + \pi^*(c_{r+1}(\mathcal{E})) = 0.$$

Let $d = \dim X$. Multiplying the previous identity by ξ^{d-1}, we get

$$\xi^{d+r} = \sum_{i=1}^{r+1} (-1)^i \pi^*(c_1(\mathcal{E}))\xi^{d+r-i}. \tag{10.41}$$

Assume that $d = \dim X = 1$. Then $c_i(\mathcal{E}) = 0$ for $i > 1$ and $c_1(\mathcal{E})$ can be identified with the degree of $\det \mathcal{E}$ (the degree of $\mathcal{E}$). Since ξ intersects the class of a general fibre with multiplicity 1, we obtain

$$\xi^{r+1} = \deg \mathcal{E}. \tag{10.42}$$

Since $|H|$ gives a finite map of degree 1, the degree of the scroll $S = f(\mathbb{P}(\mathcal{E}))$ is equal to $H^{r+1} = \xi^{r+1}$. Also $\mathcal{E} = \nu^*(\mathcal{S}_G^\vee)$, hence

$$\deg \mathcal{E} = \nu^*(c_1(\mathcal{S}_G^\vee)) = \nu^*(\sigma_1) = \deg \nu(X) = \deg X_0,$$

where the latter degree is taken in the Plücker embedding of G. This gives

$$\deg S = \deg X_0. \tag{10.43}$$

The formula is not true anymore if $d = \dim X > 1$. For example, when $d = 2$ we get the formula

$$\begin{aligned}\deg S = \xi^{r+2} &= \pi^*(c_1(\mathcal{E}))\xi^{r+1} - \pi^*(c_2(\mathcal{E}))\xi^r. \\ &= \pi^*(c_1(\mathcal{E})^2 - c_2(\mathcal{E}))\xi^r = c_1^2(\mathcal{E}) - c_2(\mathcal{E}) = \nu^*(\sigma_2),\end{aligned}$$

where σ_2 is the special Schubert class.

Example 10.4.1. Exercise 19.13 from [203] asks to show that the degree of S_X may not be equal to $\deg X$ if $\dim X > 1$. An example is the scroll S of lines equal to the Segre variety $s_{2,1}(\mathbb{P}^2 \times \mathbb{P}^1) \subset \mathbb{P}^5$. Its degree is equal to 3. If we identify the space $\mathbb{P}^5$ with the projective space of one-dimensional subspaces of the space of matrices of size 2×3, the Segre variety is the subvariety of matrices of rank 1. If we take homogeneous coordinates t_0, t_1, t_2 in $\mathbb{P}^2$ and homogeneous coordinates z_0, z_1 in $\mathbb{P}^1$, then S is given by

$$\text{rank}\begin{pmatrix} t_0z_0 & t_1z_0 & t_2z_0 \\ t_0z_1 & t_1z_1 & t_2z_1 \end{pmatrix} \leq 1.$$

When we fix (t_0, t_1, t_2), the parametric equation of the corresponding line in $\mathbb{P}^5$ is $z_0[t_0, t_1, t_2, 0, 0, 0] + z_1[0, 0, 0, t_0, t_1, t_2]$. The Plücker coordinates of the line are equal to $p_{i4+j} = t_it_j, 0 \leq i \leq j \leq 2$, with other coordinates equal to zero. Thus we see that the variety X parameterizing the generators of S spans a subspace of dimension 5 in $\mathbb{P}^9$ and is isomorphic to a Veronese surface embedded in this subspace by the complete linear system of quadrics. Thus the degree of X is equal to 4.

From now on we shall assume that $X = C$ is a smooth curve C so that the map $\nu : C \to C_0 \subset G_r(\mathbb{P}^{n-1})$ is the normalization map of the curve C_0 parameterizing generators.

Let S_1 and S_2 be two scrolls in $\mathbb{P}(W)$ corresponding to vector bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ of ranks r_1 and r_2 and surjections $W \otimes \mathcal{O}_X \to \mathcal{E}_1$ and $W \otimes \mathcal{O}_X \to \mathcal{E}_2$. Let $\langle \mathbb{P}(\mathcal{E}_1), \mathbb{P}(\mathcal{E}_2)\rangle$ be the join in $\mathbb{P}(W \otimes \mathcal{O}_X) = X \times \mathbb{P}(W)$ and S be the projection of the join to $\mathbb{P}(W)$. It is a scroll in $\mathbb{P}(W)$ whose generators are the joins of the corresponding generators of S_1 and S_2. Two generators corresponding to a point $x \notin \{x_1, \ldots, x_m\}$ span a linear subspace of expected dimension $r_1 + r_2 + 1$. The generators corresponding to a point x_j span a subspace of dimension $r_1 + r_2 - h_1$. The scroll S is denoted by $\langle S_1, S_2\rangle$ and is called the *join* of scrolls S_1 and S_2. Since $\deg \mathcal{E}' = \deg \mathcal{E}_1 + \deg \mathcal{E}_2$, we obtain

$$\deg\langle S_1, S_2\rangle = \deg S_1 + \deg S_2 - \sum_{i=1}^{m} h_i. \tag{10.44}$$

Let us consider some special examples.

Example 10.4.2. Let $W_i \otimes \mathcal{O}_C \to \mathcal{E}_i$ define scrolls S_i in $\mathbb{P}(W_i), i = 1, 2$. Consider the surjection $W \otimes \mathcal{O}_C = (W_1 \oplus W_2) \otimes \mathcal{O}_C \to \mathcal{E}_1 \oplus \mathcal{E}_2$. It defines the scroll equal to the join of the scroll $S_1 \subset \mathbb{P}(W_1) \subset \mathbb{P}(W)$ and the scroll $S_2 \subset \mathbb{P}(W_2) \subset \mathbb{P}(W)$. Its degree is equal to $\deg S_1 + \deg S_2$. For example, let $\mathcal{E}_i$ be an invertible sheaf on C defining a closed embedding $\tau_i : C \subset \mathbb{P}(W_i)$ so that $S_i = \tau_i(X)$ are curves of degree a_i spanning $\mathbb{P}(W_i)$. Then the join of S_1 and S_2 is a surface of degree $a_1 + a_2$ with generators parameterized by X. Specializing further, we take $X = \mathbb{P}^1$ and $\mathcal{E}_i = \mathcal{O}_{\mathbb{P}^1}(a_i)$ with $a_1 \leq a_2$. The scroll $\langle S_1, S_2\rangle$ is the rational normal scroll S_{a_1, a_1+a_2-1}. Iterating this construction we obtain rational normal scrolls $S_{a_1,\ldots,a_k,n} \subset \mathbb{P}^n$, where $n = a_1 + \ldots + a_k - k + 1$.

Example 10.4.3. Suppose we have two scrolls S_1 and S_2 in $\mathbb{P}^n$ defined by surjections $\alpha_i : W \otimes \mathcal{O}_{C_i} \to \mathcal{E}_i$, where rank $\mathcal{E}_i = r_i + 1$. Let $\Gamma_0 \subset C_1 \times C_2$ be a correspondence of bidegree (α_1, α_2) and $\mu : \Gamma \to \Gamma_0$ be its normalization map. Let $p_i : \Gamma \to C_i$ be the composition of μ and the projection maps $C_1 \times C_2 \to C_i$. Consider the surjections $p_i^*(\alpha_i) : W \otimes \mathcal{O}_\Gamma \to p_i^*\mathcal{E}_i$. Let $\langle \mathbb{P}(p_1^*\mathcal{E}_1), \mathbb{P}(p_1^*\mathcal{E}_1)\rangle$ be the corresponding join in $\mathbb{P}(W \otimes \mathcal{O}_\Gamma)$. Let S be the image of the join in $\mathbb{P}(W)$. We assume that it is a scroll whose generators are parameterized by an irreducible curve $C_0 \subset G_{r_1+r_2-1}(\mathbb{P}(W))$ equal to the closure of the image of the map $\phi : \Gamma \to G_{r_1+r_2-1}(\mathbb{P}(W))$ defined by

$\phi(z) = \overline{\nu_1(p_1(z)), \nu_2(p_1(z))}$. Let a be the degree of this map. Then

$$\deg S = \frac{1}{a}(\alpha_1 \deg S_1 + \alpha_2 \deg S_2 - h),$$

where

$$h = h^0(\text{Coker}(\mu^*(W \otimes \mathcal{O}_\Gamma \to p_1^*\mathcal{E}_1 \oplus p_2^*\mathcal{E}_2))).$$

Here are some special examples. We can take for S_1 and S_2 two curves in $\mathbb{P}^n$ of degrees d_1 and d_2 intersecting transversally at m points $x_1, \ldots, x_m$. Let Γ be the graph of an isomorphism $\sigma : S_1 \to S_2$. Let h be the number of points $x \in S_1$ such that $\sigma(x) = x$. Obviously, these points must be among the points x_i's. Assume that x_1 and $\sigma(x_1)$ do not lie on a common trisecant for a general point $x_1 \in S_1$. Then $h^0 = 1$ and the scroll S is a scroll of lines of degree $d_1 + d_2 - h$.

We could also take $S_1 = S_2$ and σ be an automorphism of S_1 with h fixed points. Then the degree of the scroll S is equal to $2d - h$ if σ^2 is not equal to the identity and $\frac{1}{2}(2d - h)$ otherwise.

10.4.2 Cayley-Zeuthen formulas

From now on until the end of this chapter we will be dealing only with scrolls with one-parameter family of generators. A two-dimensional scroll is called a *ruled surface*. This classical terminology disagrees with the modern one, where a ruled surface means a $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{E})$ over a smooth projective curve (see [206]). Our ruled surfaces are their images under a degree 1 morphism given by a linear system in $|\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|$.

We will denote the base X of $\mathbb{P}(\mathcal{E})$ by C to distinguish this case from the general case. We denote by $C_0 \subset G_1(\mathbb{P}^N)$ the irreducible curve parameterizing generators of $\mathbb{P}(\mathcal{E})$ and let $\nu : C \to C_0$ be the normalization map.

Let us remind some well-known facts about projective 1-bundles $X = \mathbb{P}(\mathcal{E})$ over smooth curves which can be found in [206], Chapter V, §2.

After tensoring $\mathcal{E}$ with an appropriate invertible sheaf we can write $\mathbb{P}(\mathcal{E}) = \mathbb{P}(\mathcal{E}_0)$, where $\mathcal{E}_0$ is normalized in the sense that $H^0(C, \mathcal{E}_0) \neq \{0\}$ but $H^0(C, \mathcal{E}_0 \otimes \mathcal{L}) = \{0\}$ for any invertible sheaf of negative degree. In this case the integer $e = -\deg \mathcal{E}_0 \geq 0$ is an invariant of the surface and the tautological invertible sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{E}_0)}(1)$ determined by $\mathcal{E}_0$ is isomorphic to $\mathcal{O}_X(E_0)$, where $E_0^2 = -e$. If $e < 0$ the curve E_0 is the unique curve on X with negative self-intersection. In fact, the cohomology group $H^2(X, \mathbb{Z})$ is freely generated by the class of a section $[E_0]$ and the class $[F]$ of a fibre. We can write for any curve E on X, $[E] = a[E_0] + b[F]$. Since we can assume that E is irreducible and different from a fibre, intersecting with F gives us $a = E \cdot F > 0$. Intersecting with E_0, we obtain that $-ae + b \geq 0$, hence $b > 0$. Now $0 > E^2 = -a^2e + 2ab = a(-ae + b) + ab > 0$ gives us a contradiction. The same argument with $E^2 = 0$ implies that $[E] = a[E_0]$. In the special case when $g = 0$ we get $E \sim aE_0$. It follows from Riemann-Roch and the formula for the canonical class that E_0 moves in a pencil, hence $a = 1$ and $E \in |E_0|$ (recall that E_0 was assumed to be irreducible).

Let $\sigma_0 : C \to \mathbb{P}(\mathcal{E})$ be the section of $\pi : \mathbb{P}(\mathcal{E}) \to C$ with the image equal to E_0. Then $\sigma_0^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(E_0) \cong \mathcal{O}_C(\mathfrak{e})$. If we identify E_0 and C by means of σ_0, then $\mathcal{O}_C(\mathfrak{e}) \cong \mathcal{O}_X(E_0) \otimes \mathcal{O}_{E_0}$ so that $\deg \mathfrak{e} = e$. A section $\sigma : C \to X$ is equivalent to a surjection of locally free sheaves $\mathcal{E} \to \mathcal{L} \cong \sigma^*\mathcal{O}_X(\sigma(C))$. In particular, $\deg \mathcal{L} = \sigma(C)^2$. The canonical class of $\mathbb{P}(\mathcal{E})$ is given by the formula

$$K_X \sim -2E_0 + \pi^*(K_C + \mathfrak{e}) \tag{10.45}$$

which is a special case of (10.8).

Let $|H|$ be a complete linear system of dimension $N > 2$ on $\mathbb{P}(\mathcal{E})$ defined by an ample section H. Since $\pi_*(\mathcal{O}_X(H)) = \mathcal{E}_0 \otimes \mathcal{L}$ for some invertible sheaf $\mathcal{L}$, we can write

$$H \sim E_0 + \pi^*(\mathfrak{a})$$

for some effective divisor class $\mathfrak{a}$ on C of degree a. Since H is irreducible, intersecting both sides with E_0 we find that $a \geq e$. Using the Moishezon-Nakai criterion of ampleness it is easy to see that H is ample if and only if $a > e$. We shall assume that H is ample. Assume also that $\mathfrak{a}$ is not special in the sense that $H^1(C, \mathcal{O}_C(\mathfrak{a})) = 0$ and $|\mathfrak{e} + \mathfrak{a}|$ has no base points on C. Then the exact sequence

$$0 \to \mathcal{O}_X(\pi^*(\mathfrak{a})) \to \mathcal{O}_X(H) \to \mathcal{O}_{E_0}(H) \to 0$$

shows that the restriction of $|H|$ to E_0 is a complete linear system without base points. It is clear that any possible base point of $|H|$ must lie on E_0, hence under the above assumptions $|H|$ has no base points. It defines a finite map $f : X \to S \subset \mathbb{P}^N$. The surface S is linearly normal surface in $\mathbb{P}^N$ swept by lines, the images of fibres. The family of lines is defined by the image of C in $G_1(\mathbb{P}^N)$. The next proposition shows that the map is of degree 1, hence S is a ruled surface.

Proposition 10.4.1. *Let $|H|$ be an ample section on $X = \mathbb{P}(\mathcal{E})$ and $|V|$ be a linear system in $|H|$ that defines a finite map $f : \mathbb{P}(\mathcal{E}) \to S \subset \mathbb{P}^N$. Then the degree of the map is equal to 1.*

Proof. Suppose $f(x) = f(y)$ for some general points $x, y \in X$. Let F_x and F_y be the fibres containing x and y. Since $|H|$ has no base points, its restriction to any fibre is a linear system of degree 1 without base points. Suppose the degree of the map is greater than 1. Take a general fibre F, then, for any general point $x \in F$, there is another fibre F_x such that $f(F_x)$ and $f(F)$ are coplanar. This implies that there exists a divisor $H(x) \in |H - F_x - F|$. We can write $H(x) = F_x + F + R(x)$ for some curve $R(x)$ such that $R(x) \cdot F_x = R(x) \cdot F = 1$. When we move x along F we get a pencil of divisors $H(x)$ contained in $|H - F|$. The divisors of this pencil look like $F_x + R(x)$ and hence all have singular point at $R(x) \cap F_x$. Since the fibre F_x moves with x, we obtain that a general member of the pencil has a singular point which is not a base point of the pencil. This contradicts Bertini's Theorem on singular points [206], Chapter 3, Corollary 10.9. $\square$

Corollary 10.4.2. *Let S be an irreducible surface in $\mathbb{P}^N$ containing a one-dimensional irreducible family of lines. Suppose S is not a cone. Then S is a ruled surface equal to the image of projective bundle $\mathbb{P}(\mathcal{E})$ over a smooth curve C under a birational morphism given by a linear subsystem in $|\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|$.*

Proof. Let $C_0 \subset G_1(\mathbb{P}^n)$ be the irreducible curve parameterizing the family of lines and $\nu : C \to C_0$ be its normalization. The preimage of the universal family $Z_{C_0} \to C_0$ is a projective bundle $\mathbb{P}(\mathcal{S}_{C_0}^\vee)$ over C. Since S is not a cone, the map $f : \mathbb{P}(\mathcal{E}) \to S$ is a finite morphism. The map is given by a linear subsystem of $|\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)|$. Since f is a finite morphism, the line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) = f^*(\mathcal{O}_{\mathbb{P}(\mathcal{S}_{C_0}^\vee)}(1))$ is ample. It remains to apply the previous proposition. $\square$

An example of a nonsingular quadric surface seems contradicts the previous statement. However, the variety of lines on a nonsingular quadric surface is not irreducible and consists of two projective lines embedded in $G_1(\mathbb{P}^3)$ as the union of two disjoint conics. So the surface has two systems of ruling, and it is a two-way scroll.

It follows from (10.43) that the degree of the ruled surface $S = f(\mathbb{P}(\mathcal{E}))$ is equal to the degree of C in the Plücker space. It is also equal to the self-intersection H^2 of the tautological line bundle on $\mathbb{P}(\mathcal{E})$. The latter is equal to $H^2 = (E_0 + aF)^2 = 2a - e$. The genus of C is called the *genus* of the ruled surface.

Proposition 10.4.3. *Let $S = f(\mathbb{P}(\mathcal{E})) \subset \mathbb{P}^n$ be a projection of a minimal ruled surface $\mathbb{P}(\mathcal{E})$ embedded in projective space by a linear system $|H|$, where $H \sim E_0 + \pi^*(\mathfrak{a})$. Let D be a directrix on S which is not contained in the singular locus of S. Then*

$$\deg \mathsf{D} \geq a - e.$$

The equality takes place if and only if the preimage of D on $\mathbb{P}(\mathcal{E})$ is in the same cohomology class as E_0.

Proof. The assumption on D implies that $\deg \mathsf{D} = H \cdot E$, where E is the preimage of D on $\mathbb{P}(\mathcal{E})$. Intersecting with H we get $H \cdot E = E \cdot E_0 + a$. If $E \neq E_0$, then $H \cdot E \geq a$, if $[E] = [E_0]$, then $E \cdot E_0 = a - e$. The equality takes place if and only if $E \cdot E_0 = 0$ and $e = 0$. Since E is a section, we can write $[E] = [E_0] + m[F]$, and intersecting with E_0, we get $m = 0$. $\square$

Now we know that $f : \mathbb{P}(\mathcal{E}) \to S$ is of degree 1, we obtain that the ruled surface is non-normal at every point over which the map is not an isomorphism.

Recall the *double-point formula* from [173], 9.3. Let $f : X \to Y$ be a morphism of nonsingular varieties of dimensions m and n, respectively. Let $X \tilde{\times} X$ be the blow-up of the diagonal of $X \times X$ and E be the exceptional divisor. We think about points in E as points in X together with a tangent direction t_x at x. Let $\tilde{D}(f)$ be the preimage of the diagonal of $Y \times Y$ under the composition of the maps $X \tilde{\times} X \to X \times X \overset{f\times f}{\to} Y \times Y$ minus E. One can view points in $\tilde{D}(f)$ as either points $x \in X$ such that there exists $x' \neq x$ with $f(x) = f(x')$, or as points (x, t_x) such that $df_x(t_x) = 0$. Let $D(f)$ be the image of $\tilde{D}(f)$ under one of the projections $X \tilde{\times} X \to X$. This is called the *double point set* of the morphism f. Define the *double point class*

$$\mathbb{D}(f) = f^* f_*[X] - (c(f^*\mathcal{T}_Y)c(\mathcal{T}_X)^{-1})_{n-m} \cap [X] \in H^{n-m}(X, \mathbb{Q}), \qquad (10.46)$$

where c denote the total Chern class $[X] + c_1 + \ldots + c_m$ of a vector bundle. In case $D(f)$ has the expected dimension equal to $2m - n$, we have

$$\mathbb{D}(f) = [D(f)] \in H^{n-m}(X, \mathbb{Z}).$$

Assume now that $f : X \to S$ is the normalization map and S is a surface in $\mathbb{P}^3$. Since S is a hypersurface, it does not have isolated non-normal points. This implies that $D(f)$ is either empty, or of expected dimension $2m - n = 1$. The double point class formula applies, and we obtain

$$[D(f)] = f^*(S) + f^*(K_Y) - K_X. \tag{10.47}$$

In fact, it follows from the theory of adjoints (see [265]) that the linear equivalence class of $D(f)$ is expressed by the same formula.

We say that a surface S in $\mathbb{P}^n$ has *ordinary singularities* if its singular locus is a double curve Γ on S. This means that the completion of the local ring of S at a general point of Γ is isomorphic to $\mathbb{C}[[z_1, z_2, z_3]]/(z_1z_2)$. The curve Γ may have also *pinch points* with completion isomorphic to $\mathbb{C}[[z_1, z_2, z_3]]/(z_1^2 - z_2z_3)$ and also triple points with completion isomorphic to $\mathbb{C}[[z_1, z_2, z_3]]/(z_1z_2z_3)$. The curve Γ is nonsingular outside triple points, the curve $D(f)$ is nonsingular outside the preimages of the triple points. It has $3t$ double points.

Under these assumptions Γ is smooth, the map $\tilde{D}(f) \to D(f) \to \Gamma$ is of degree 2. It is ramified at pinch points only, and the preimage of a triple point consists of 6 points.

Assume that S is a surface in $\mathbb{P}^3$ with ordinary singularities, $f : X \to S$ be the normalization map, and Γ be the double curve of S. The degree of any curve on X, is the degree with respect to $f^*(\mathcal{O}_{\mathbb{P}^3}(1))$. Let us introduce the following numerical invariants in their classical notation:

- $\mu_0 =$ the degree of S;
- $\mu_1 =$ the rank of S, the class of a general plane section of S;
- $\mu_2 =$ the class of S;
- $\nu_2 =$ the number of pinch-points on S;
- $t =$ the number of triple points on S;
- $\epsilon_0 = \deg \Gamma$;
- $\epsilon_1 =$ the rank of Γ, the number of tangents to Γ intersecting a general line in $\mathbb{P}^3$;
- $\rho =$ the *class of immersion* of Γ equal to the degree of the image of $D(f)$ under the Gauss map $G : X \to \check{\mathbb{P}}^3$;
- $g(\Gamma) =$ the genus of Γ;
- $c =$ the number of connected components of Γ;
- $\kappa =$ the degree of the ramification divisor $p : X \to \mathbb{P}^2$, where p is the composition of f and the general projection of S.

The following Theorem summarizes different relations between the listed invariants of S. These relations are called the *Cayley-Zeuthen formulas*.

Theorem 10.4.4. *The following relations hold:*

(i) $\mu_1 = \mu_0(\mu_0 - 1) - 2\epsilon_0$*;*

(ii) $\epsilon_0(\mu_0 - 2) = \rho + 3t$*;*

(iii) $\mu_1(\mu_0 - 2) = \kappa + \rho$*;*

(iv) $2g(\Gamma) - 2c = \epsilon_1 - 2\epsilon_0$*;*

(v) $\nu_2 = 2\epsilon_0(\mu_0 - 2) - 6t - 2\epsilon_1$*;*

(vi) $2\rho - 2\epsilon_1 = \nu_2$;

(vii) $\mu_2 = \mu_0(\mu_0 - 1)^2 + (4 - 3\mu_0)\epsilon_1 + 3t - 2\nu_2$*;*

(viii) $2\nu_2 + \mu_2 = \mu_1 + \kappa$.

Proof. (i) A general plane section is a plane curve of degree μ_0 with ϵ_0 ordinary double points as singularities. Thus (i) is just the Plücker formula. Note also that μ_1 is equal to the degree of the *contact curve*, the closure of smooth points $p \in S$ such that a general point $q \in \mathbb{P}^3$ is contained in $\mathbb{T}_p(S)$, or, equivalently, the residual curve to Γ of the intersection of S and the first polar $P_q(S)$. Taking a general plane H and a general point $q \in H$, we obtain that $\deg \Delta$ is equal to the class of $H \cap S$.

(ii) The number ρ is equal to the number of tangent planes to S at points in Γ which pass through a general point q in $\mathbb{P}^3$. Here a tangent plane to a singular point $p \in \Gamma$ means the tangent plane to one of the two branches of S at q, or equivalently, the image of a preimage of p on X under the Gauss map. Consider the intersection of the second polar P_{q^2} with the contact curve Γ. It follows from subsection 1.1.3 that $P_{q^2}(S) \cap S$ is equal to the locus of points p such that the line $\overline{p,q}$ intersects S at p with multiplicity ≥ 3. This means that $P_{q^2}(S) \cap \Gamma$ consists of t triple points points and points such that q belongs to a tangent plane of S at p. The latter number is equal to ρ. As we observed in subsection 1.1.3, P_{q^2} has a point of multiplicity 3 at p, hence each triple point enters with multiplicity 3 in the intersection of $\overline{p,q}$ with Γ. It remains to use that the degree of the second polar is equal to $\mu_0 - 2$.

(iii) Now let us consider the intersection of the second polar $P_{q^2}(S)$ with the contact curve Δ. This intersection consists of lines $\overline{q,p}$ such that p is either one of κ ramification points of the projection of the surface from q or p is one of ρ points on $\Gamma \cap \Delta$, where the tangent plane contains p. In fact, these points lie on the intersection of Δ and Γ.

(iv) - (vi) Let $\pi = h^1(\mathcal{O}_{D(f)})$ be the arithmetic genus of the curve $D(f)$ and s be the number of connected components of $D(f)$. Applying (10.47), we get

$$-2\chi(D(f), \mathcal{O}_{D(f)}) = 2\pi - 2c = (D(f) + K_X) \cdot D(f)$$

$$= (\mu_0 - 4)\deg D(f) = 2\epsilon_0(\mu_0 - 4).$$

The curve $D(f)$ has $3t$ ordinary double points and the projection from the normalization of $D(f)$ to Γ is a degree 2 cover ramified at ν_2 points. Applying the Hurwitz formula, we obtain $2\pi - 2c - 6t = 2(2g(\Gamma) - 2c) + \nu_2$. Projecting Γ from a general line defines a degree ϵ_0 map from the normalization of Γ to $\mathbb{P}^1$. The number of ramification points is equal to ϵ_1. Applying the Hurwitz formula again, we get $2g(\Gamma) - 2c = -2\epsilon_0 + \epsilon_1$. This gives (iv) and also

$$\nu_2 = 2\epsilon_0(\mu_0 - 4) - 6t - 2(2g(\Gamma) - 2c) = 2\epsilon_0(\mu_0 - 4) - 6t - 2\epsilon_1 + 4\epsilon_0$$
$$= 2\epsilon_0(\mu_0 - 2) - 6t - 2\epsilon_1.$$

This is equality (v). It remains to use (ii) to get (vi).

(vii) The formula for the class of a surface with ordinary singularities has a modern proof in [173], Example 9.3.8. In our notation it gives (vii).

$$\mu_2 = \mu_0(\mu_0 - 1)^2 + (4 - 3\mu_0)\epsilon_1 + 3t$$

Using this and (i) we get

$$\mu_2 + 2\nu_2 = (\mu_0 - 1)(\mu_1 + 2\epsilon_0) + 4\epsilon_0 - 3\mu_0\epsilon_0$$
$$= \mu_0\mu_1 - \mu_1 + 2\epsilon_0 + \rho - \epsilon_0\mu_0 + 3t.$$

It remains to use (ii) and (iii). □

Corollary 10.4.5. *Let S be a surface in $\mathbb{P}^3$ with ordinary singularities and X be its normalization. Then*

(i) $K_X^2 = \mu_0(\mu_0 - 4)^2 - (3\mu_0 - 16)\epsilon_0 + 3t - \nu_2$;

(ii) $c_2(X) = \mu_0(\mu_0^2 - 4\mu_0 + 6) - (3\mu_0 - 8)\epsilon_0 + 3t - 2\nu_2$;

(iii) $\chi(X, \mathcal{O}_X) = 1 + \binom{\mu_0 - 1}{3} - \frac{1}{2}(\mu_0 - 4)\epsilon_0 + \frac{1}{2}t - \frac{1}{4}\nu_2$.

Proof. (i) Applying (10.47), we get

$$K_X = (\mu_0 - 4)H - D(f), \tag{10.48}$$

where $H \in |f^*(\mathcal{O}_{\mathbb{P}^3}(1))|$. The first polar of S with respect to a general point cuts out on S the union of Γ and Δ. Taking the preimage on X we get

$$(\mu_0 - 1)H = D(f) + f^*(\Delta).$$

It follows from the local computation that Γ and Δ intersect simply at ν_2 pinch points and ρ additional points (see the proof of (iii) in Theorem 10.4.4). This gives

$$D(f)^2 = (\mu_0 - 1)H \cdot D(f) - \rho - \nu_2 = 2\epsilon_0(\mu_0 - 1) - \rho - \nu_2$$
$$= 2\epsilon_0(\mu_0 - 1) - \epsilon_0(\mu_0 - 2) + 3t - \nu_2 = \epsilon_0(\mu_0 - 2) + 3t - \nu_2.$$

This gives

$$K_X^2 = (\mu_0 - 4)^2\mu_0 - 4(\mu_0 - 4)\epsilon_0 + D(f)^2$$
$$= (\mu_0 - 4)^2\mu_0 - (3\mu_0 - 16)\epsilon_0 + 3t - \nu_2.$$

(ii) The preimage of a pinch point on X is a point in X such that the rank of the tangent map $\mathcal{T}_X \to f^*(\mathcal{T}_{\mathbb{P}^3})$ drops by 2. According to the modern theory of degeneracy loci (see [173]), this set is given by the relative second Chern class $c_2(f^*(\mathcal{T}_{\mathbb{P}^3})/\mathcal{T}_X)$. Computing this Chern class we find

$$\nu_2 = c_1(X)^2 - c_2(X) + 4K_X \cdot H + 6\mu_0.$$

Applying (10.48), we get

$$\nu_2 = K_X^2 - c_2(X) + 4(\mu_0 - 4)\mu_0 - 8\epsilon_0 + 6\mu_0. \tag{10.49}$$

Together with (i) we get (ii). Formula (iii) follows from the *Noether formula*

$$12\chi(X, \mathcal{O}_X) = K_X^2 + c_2(X).$$

□

Next we apply the Cayley-Zeuthen formulas to the case when S is a ruled surface in $\mathbb{P}^3$ with ordinary singularities and $X = \mathbb{P}(\mathcal{E})$. We know that μ_0 is equal to the degree d of C_0 in its Plücker embedding. The next Theorem shows that all the numerical invariants can be expressed in terms of μ_0 and g.

Theorem 10.4.6. *Let S be a ruled surface in $\mathbb{P}^3$ of degree μ_0 and genus g. Assume that S has only ordinary singularities. Then*

(i) $\epsilon_0 = \frac{1}{2}(\mu_0 - 1)(\mu_0 - 2) - g$;

(ii) $\nu_2 = 2(\mu_0 + 2g - 2)$;

(iii) $\mu_1 = 2\mu_0 - 2 + 2g$;

(iv) $\mu_2 = \mu_0 = \mu_0$;

(v) $\kappa = 3(\mu_0 + 2g - 2)$;

(vi) $\rho = (\mu_0 - 2)(2\mu_0 - 5) + 2g(\mu_0 - 5)$;

(vii) $t = \frac{1}{6}(\mu_0 - 4)[(\mu_0 - 2)(\mu_0 - 3) - 6g]$;

(viii) $\epsilon_1 = 2(\mu_0 - 2)(\mu_0 - 3) + 2g(\mu_0 - 6)$;

(ix) $2g(\Gamma) - 2s = (\mu_0 - 5)(\mu_0 + 2g - 2)$.

Proof. A general plane section of S is a plane curve of degree d with $\deg \Gamma$ ordinary singularities. This gives (i).

The canonical class formula gives

$$K_{\mathbb{P}(\mathcal{E})} = -2H + \pi^*(K_{\bar{C}} + \mathfrak{d}), \tag{10.50}$$

where $\mathcal{O}_{\bar{C}}(\mathfrak{d}) \cong \nu^*(\mathcal{O}_{C_0}(1))$ is of degree $d = \mu_0$.

Comparing it with formula (10.45), we find that

$$H \sim E_0 + \pi^*(\mathfrak{f}), \tag{10.51}$$

where $2\mathfrak{f} = \mathfrak{d} - \mathfrak{e}$. In particular, $e + d$ is always an even number.

Applying (10.50), we get $K^2_{\mathbb{P}(\mathcal{E})} = 4\mu_0 - 4(2g - 2 + \mu_0)$. Topologically, $\mathbb{P}(\mathcal{E})$ is the product of $\mathbb{P}^1$ and C. This gives $c_2(X) = 2(2 - 2g)$. Applying (10.49), we find

$$\nu_2 = 4\mu_0 - 4(2g-2+\mu_0) - 2(2-2g) + 4(\mu_0-4)\mu_0 - 4(\mu_0-1)(\mu_0-2) - 8g + 6\mu_0$$
$$= 2(\mu_0 + 2g - 2).$$

From (i) of Theorem 10.4.4, we get (iii).

To prove (iv) we have to show that the degree of S is equal to the degree of its dual surface. The image of a generator of S under the Gauss map is equal to the dual line in the dual $\mathbb{P}^3$, i.e. the set of hyperplanes containing the line. Since S has only finitely many torsor generators, the Gauss map is a birational map, this shows that S^* is a ruled surface. If S is defined by the vector bundle $\mathcal{E} = \mathcal{S}_G^\vee \otimes \mathcal{O}_{C_0}$, then the dual ruled surface is defined by the bundle $\mathcal{Q}_G \otimes \mathcal{O}_{C_0}$, where $\mathcal{O}_G$ is the universal quotient bundle. Exact sequence (10.2) shows that $\det \mathcal{Q}_G \otimes \mathcal{O}_{C_0} \cong \det \mathcal{S}_G^\vee \otimes \mathcal{O}_{C_0}$. In particular, the degrees of their inverse images under $\nu : C \to C_0$ are equal. Thus the degrees of S and S^* are equal.

Now (i) and (viii) of Theorem 10.4.4 and our formula (i) give (v). Using (iii) and (ii) of the same Theorem, we get formulas (vi) and (vii). Finally, (vi) of the Theorem gives formulas (viii) and (ix). □

The double-point formula gives

$$\mathcal{O}_{\mathbb{P}(\mathcal{E})}(D(f)) \sim \mathcal{O}_{\mathbb{P}(\mathcal{E})}(\mu_0 - 2) \otimes \pi^*(\omega_C(1)).$$

A general point of Γ is contained in two rulings and formula (10.47) implies that a general ruling intersects $\mu_0 - 2$ other rulings. Consider a symmetric correspondence on C defined by

$$T = \{(x, y) \in C \times C : |H - \ell_x - \ell_y| \neq \emptyset\}.$$

A point $(x, x) \in T$ corresponds to a generator which is called a *torsal generator*. The plane in $\mathbb{P}^3$ cutting out this generator with multiplicity ≥ 2 is tangent to the ruled surface at any smooth point of the generator. For a general point x, we have $\#T(x) = d - 2$. Since all generators $\ell_y, y \in T(x)$, intersect the same line ℓ_x the points $y \in T(x)$ lie in the tangent hyperplane of $G_1(\mathbb{P}^3)$ at the point x. This implies that the divisor $2x + T(x)$ belongs to the linear system $|\mathcal{O}_C(1)|$ and, in particular, T has valency equal to 2. Applying the Cayley-Brill formula from Corollary 5.5.2, we obtain the following

Proposition 10.4.7. *The number of torsal generators of a ruled surface in $\mathbb{P}^3$ with ordinary singularities is equal to $2(\mu_0 + 2g - 2)$.*

Comparing with Theorem 10.4.6 we find that the number of torsal generators is equal to the number ν_2 of pinch points.

When $n = 4$ we expect that a ruled surface has only finitely many singular non-normal points and for $n = 5$ we expect that it is nonsingular.

The following example is a ruled surface with a triple curve of singularities.

Example 10.4.4. Let C be a nonsingular curve of genus 3 and degree 6 embedded in $\mathbb{P}^3$ by the linear system $|K_C + \mathfrak{a}|$, where $\mathfrak{a}$ is a non-effective divisor of degree 2. By Riemann-Roch, we have $h^0(D + x) = 1$. Let $p + q + r \in |\mathfrak{a} + x|$. For each point $x \in C$ the linear system $|K_C - x|$ is of degree 3 and dimension 1. Also, we obtain that $h^0(\mathfrak{a}+K_C-p-q-r) = h^0(K_X-x) = 2$. Thus the linear system $|\mathfrak{a}+K_C-p-q-r|$ of planes through the points p, q, r is a pencil. This means that the points p, q, r are on a trisecant line of C. Let S be the union of the trisecants parameterized by the curve C. Obviously, S is not a cone. Applying Corollary 10.4.2, we obtain that S is a ruled surface.

Projecting C from any point x on it, we get a plane curve of degree 5 of genus 3. It must have 3 double points. Thus there are 3 trisecants passing through x. This shows that C is the triple curve of S but not a double curve. Let us show that the degree of S is equal to 8. The linear system of cubic surfaces containing C defines a Cremona transformation of $\mathbb{P}^3$. It blow-up the curve C and then blows down the proper transform of S to a curve in another copy of $\mathbb{P}^3$ isomorphic to C. This is an example of a *cubo-cubic space Cremona transformation*. An equation of the surface S can be obtained as the jacobian of 4 cubic polynomials defining the rational map. Its degree is equal to 8.

A general plane section is a plane curve of degree 8 of genus 3. It has 6 singular points of multiplicity 3. Applying formula (10.47) we see that the linear equivalence class of the curve $\mathbb{D}(f)$ is equal to $2H - \pi^*(K_C + \mathfrak{d})$ for some divisor $\mathfrak{d}$ of degree d. However, the curve $D(f)$ comes with multiplicity 2, so the curve C in S is the image of a curve $\tilde{C}$ on Z_C from the linear system $|H - \pi^*(\mathfrak{f})|$, where $2\mathfrak{f} \sim K_C + \mathfrak{d}$. So every generator intersects it at 3 points as expected. One can show that $\mathfrak{d} \sim K_C + 2\mathfrak{a}$ so that $\mathfrak{f} \sim K_C + \mathfrak{a}$. Note that the curve $\tilde{C}$ defines a $(3, 3)$-correspondence on the curve C with the projections p_C and q_C to C. Its genus is equal to 19 and each projection is a degree 3 cover ramified at 24 points. In the case when the divisor $\mathfrak{a}$ is an even theta characteristic, the curve $\tilde{C}$ is the Scorza correspondence which we studied in section 5.5.2.

Next example shows that the double curve of a ruled surface may be disconnected.

Example 10.4.5. Consider three nonsingular nondegenerate curves $X_i, i = 1, 2, 3$, in $\mathbb{P}^3$ with no two having common points. Let S be the set of lines intersecting each curve. Let us show that these lines sweep a ruled surface of degree $2d_1d_2d_3$, where $d_i = \deg C_i$. Recall that the set of lines intersecting a curve X of degree X is a divisor in $G_1(\mathbb{P}^3)$ of degree d. This is the *Chow form* of C (see [183]). Thus the set of lines intersecting 3 curves is a complete intersection of three hypersurfaces in $G_1(\mathbb{P}^3)$, hence a curve of degree $2d_1d_2d_3$. If we assume that the curves are general enough so that the intersection is transversal, we obtain that the ruled surface must be of degree $2d_1d_2d_3$. The set of lines intersecting two curves X_1 and X_2 is a surface W in $G_1(\mathbb{P}^3)$ of degree $2d_1d_2$. Its intersection with the Schubert variety $\Omega(\Pi)$, where Π is a general plane consists of d_1d_2 lines. It follows from the intersection theory on $G_1(\mathbb{P}^3)$ that the intersection of W with the α-plane $\Omega(p)$ is of degree d_1d_2. Thus we expect that in a general situation the number of generators of S passing through a general point on X_3 is equal to d_1d_2. This shows that a general point of X_3 is a singular point of multiplicity d_1d_2. Similarly, we show that X_1 is a singular curve of multiplicity d_2d_3

and X_2 is a singular curve of multiplicity $d_1 d_3$.

Remark 10.4.1. According to [111] the double curve Γ is always connected if $\mu_0 \geq g+4$. If it is disconnected, then it must be the union of two lines.

10.4.3 Developable ruled surfaces

A ruled surface is called *developable* if the tangent planes at nonsingular points of any ruling coincide. In other words, any generator is a torsal generator. One expects that the curve of singularities is a cuspidal curve. In this subsection we will give other characterizations of developable surfaces.

Recall the definition of the vector bundle of *principal parts* on a smooth variety X. Let Δ be the diagonal in $X \times X$ and $\mathcal{J}_\Delta$ be its sheaf of ideals. Let p and q be the first and the second projections to X from the closed subscheme Δ^m defined by the ideal sheaf $\mathcal{J}_\Delta^{m+1}$. For any invertible sheaf $\mathcal{L}$ on X one defines the sheaf of m-th principal parts $\mathcal{P}^m(\mathcal{L})$ of $\mathcal{L}$ as the sheaf $\mathcal{P}_X^m(\mathcal{L}) = p_* q^*(\mathcal{L})$ on X. Recall that the m-th tensor power of the sheaf of 1-differentials Ω_X^1 can be defined as $p_*(\mathcal{J}_\Delta^m/\mathcal{J}_\Delta^{m+1})$ (see [206]). The exact sequence

$$0 \to \mathcal{J}_\Delta^m/\mathcal{J}_\Delta^{m+1} \to \mathcal{O}_{X\times X}/\mathcal{J}_\Delta^{m+1} \to \mathcal{O}_{X\times X}/\mathcal{J}_\Delta^m \to 0$$

gives an exact sequence

$$0 \to (\Omega_X^1)^{\otimes m} \otimes \mathcal{L} \to \mathcal{P}_X^m(\mathcal{L}) \to \mathcal{P}_X^{m-1}(\mathcal{L}) \to 0. \tag{10.52}$$

We will be interested in the case when $X_0 = C_0$ is an irreducible curve of genus g and $X = C$ is its normalization. By induction, the sheaf $\mathcal{P}_C^m(\mathcal{L})$ is a locally free sheaf of rank $m+1$, and

$$\deg \mathcal{P}_C^m(\mathcal{L}) = (m+1)\deg \mathcal{L} + m(m+1)(g-1). \tag{10.53}$$

For any subspace $V \subset H^0(C,\mathcal{L})$ there is a canonical homomorphism

$$V \to H^0(\Delta^m, q^*\mathcal{L}) = H^0(C, p_*q^*\mathcal{L}) = H^0(C, \mathcal{P}_C^m(\mathcal{L}))$$

which defines a morphism of locally free sheaves

$$\alpha_m : V_C := \mathcal{O}_C \otimes V \to \mathcal{P}_C^m(\mathcal{L}). \tag{10.54}$$

Note that the fibre of $\mathcal{P}_C^m(\mathcal{L})$ at a point x can be canonically identified with $\mathcal{L}/\mathfrak{m}_{C,x}^{m+1}\mathcal{L}$ and the map α_m at a point x is given by assigning to a section $s \in V$ the element $s \mod \mathfrak{m}_{C,x}^{m+1}\mathcal{L}$. If $m = 0$, we get $\mathcal{P}_C^m(\mathcal{L}) = \mathcal{L}$ and the map is the usual map given by evaluating a section at a point x.

Suppose that $(V,\mathcal{L})$ defines a morphism $f : C \to \mathbb{P}(V)$ such that the induced morphism $f : C \to f(C) = C_0$ is the normalization map. We have $\mathcal{L} = f^*(\mathcal{O}_{\mathbb{P}}(1))$. Let $\mathcal{P}^m \subset \mathcal{P}_C^m(\mathcal{L})$ be the image of α_m. Since the composition of α_1 with the projection $\mathcal{P}_C^1 \to \mathcal{L}$ is generically surjective (because C_0 spans $\mathbb{P}(V)$), the map α_1 is generically surjective. Similarly, by induction, we show that α_m is generically surjective for all m.

Since C is a smooth curve, this implies that the sheaves $\mathcal{P}^m$ are locally free of rank $m+1$. They are called the *osculating sheaves*. Let

$$\sigma_m : C \to G(m+1, V^\vee)$$

be the morphisms defined by the surjection $\alpha_m : V_C \to \mathcal{P}^m$. The morphism σ_m can be interpreted as assigning to each point $x \in C$ the m-th *osculating plane* of $f(C)$ at the point $f(x)$. Recall that it is a m-dimensional subspace of $\mathbb{P}(V)$ such that it has the highest order contact with the branch of C_0 defined by the point $x \in C$. One can always choose a system of projective coordinates in $\mathbb{P}(V) \cong \mathbb{P}^n$ such that the branch of C_0 corresponding to x can be parameterized in the ring of formal power series by

$$x_0 = 1,\ x_i = t^{i+s_1+\ldots+s_i} + \text{ highest order terms},\ i = 1, \ldots, n, \tag{10.55}$$

where $s_i \geq 0$. Then the osculating hyperplane is given by the equation $x_n = 0$. The codimension 2 osculating subspace is given by $x_{n-1} = x_n = 0$ and so on. A point $x \in C$ (or the corresponding branch of $f(C)$) with $s_1 = \ldots = s_n = 0$ is called an *ordinary point*, other points are called *hyperosculating* or *stationary* points . It is clear that a point x is ordinary if and only if the highest order of tangency of a hyperplane at x is equal to n. For example, for a plane curve, a point is ordinary if the corresponding branch is nonsingular, or, equivalently, the differential of the map f at x is not equal to 0.

The image $\sigma_m(C)$ in $G_m(\mathbb{P}^n)$ is called the m-th *associated curve*. Locally, the map σ_m is given by assigning to a point $x \in C$ the linear subspace of $\mathbb{C}^{n+1}$ generated by $\tilde{f}(x), \tilde{f}'(x), \ldots, \tilde{f}^{(m)}(x)$, where $\tilde{f} : C \to \mathbb{C}^{n+1}$ is a local lift of the map f to a map to the affine space, and $\tilde{f}^{(k)}$ are its derivatives (see [197], Chapter II, §4).

Let $\mathbb{P}(\mathcal{P}^m) \to C \times \mathbb{P}(V)$ be the morphism corresponding to the surjection α_m. The projection of the image to $\mathbb{P}$ is called the *m-th osculating developable* of $(C, \mathcal{L}, V)$ (or of C_0). For $m = 1$ it is a ruled surface, called the *developable surface* or *tangent surface* of C_0.

Let r_m be the degree of $\mathcal{P}^m$. We have already observed that the composition of the map σ_m with the Plücker embedding is given by the sheaf $\det \mathcal{P}^m$. Thus r_m is equal to the degree of the m-th associated curve of C_0. Also, we know that the degree of a curve in the Grassmannian $G(m+1, V^\vee)$ is equal to the intersection of this curve with the Schubert variety $\Omega(A)$, where $\dim A = n - m - 1$. Thus r_m is equal to the *m-rank* of C_0, the number of osculating m-planes intersecting a general $(n-m-1)$-dimensional subspace of $\mathbb{P}$. Finally, we know that the 1-rank r_1 (called the *rank* of C_0), divided by the number of tangents through a general point on the surface, is equal to the degree of the tangent surface. More generally, r_m is equal to the degree of the m-th osculating developable (see [312]). The $(n-1)$-rank r_{n-1} is called the *class* of C_0. If we consider the $(n-1)$-th associated curve in $G(n, n+1)$ as a curve in the dual projective space $|V|$, then the class of C_0 is its degree. The $(n-1)$-th associated curve $C^\vee$ is called the *dual curve* of C_0. Note that the dual curve should not be confused with the dual variety of C_0. The latter coincides with the $(n-2)$-th osculating developable of the dual curve.

Proposition 10.4.8. *Let $r_0 = \deg \mathcal{L} = \deg f(C)$. For any point $x \in C$ let $s_i(x) = s_i$, where the s_i's are defined in (10.55), and $k_i = \sum_{x\in C} s_i(x)$. Then*

$$r_m = (m+1)(r_0 + m(g-1)) - \sum_{i=1}^{m}(m-i+1)k_i$$

and

$$\sum_{i=1}^{n}(n-i+1)k_i = (n+1)(r_0 + n(g-1)).$$

In particular,

$$r_1 = 2(r_0 + g - 1) - k_1.$$

Proof. Formula (10.53) gives the degree of the sheaf of principal parts $\mathcal{P}^m_C(\mathcal{L})$. We have an exact sequence

$$0 \to \mathcal{P}^m \to \mathcal{P}^m_C(\mathcal{L}) \to \mathcal{F} \to 0,$$

where $\mathcal{F}$ is a skyscraper sheaf whose fibre at $x \in C$ is equal to the cokernel of the map $\alpha^m(x) : V \to \mathcal{L}/\mathfrak{m}_{C,x}\mathcal{L}$. It follows from formula (10.55) that $\dim \mathcal{F}(x)$ is equal to $s_1 + (s_1+s_2) + \ldots + (s_1 + \ldots + s_m) = \sum_{i=1}^{m}(m-i+1)s_i$. The standard properties of Chern classes give

$$\deg \mathcal{P}^m = \deg \mathcal{P}^m_C(\mathcal{L}) - h^0(\mathcal{F}) = (m+1)(r_0 + m(g-1)) - \sum_{i=1}^{m}(m-i+1)k_i.$$

The second formula follows from the first one by taking $m = n$ in which case $r_n = 0$ (the surjection of bundles of the same rank $V_C \to \mathcal{P}^n$ must be an isomorphism). $\square$

Adding up r_{m-1} and r_{m+1} and subtracting $2r_m$, we get

Corollary 10.4.9.

$$r_{m-1} - 2r_m + r_{m+1} = 2g - 2 - k_{m+1}, \quad m = 0, \ldots, n,$$

where $r_{-1} = r_n = r_{n+1} = 0$.

The previous formulas can be viewed as Plücker formulas for space curves. If $n = 2$ and C is a plane curve with δ ordinary nodes and κ ordinary cusps, we have for the degree $d^\vee$ of the dual curve

$$d^\vee = d(d-1) - 2\delta - 3\kappa = 2d + (d(d-3) - 2\delta - 2\kappa) - \kappa = 2d + 2g - 2 - \kappa.$$

In this case $d^\vee = r_1, d = r_0$ and $\kappa = k_1$, so the formulas agree.

Example 10.4.6. If R_n is a rational normal curve in $\mathbb{P}^n$, then it has no hyperosculating hyperplanes (since no hyperplane intersects it with multiplicity $> n$). So $r_m = (m+1)(n-m) = r_{n-m-1}$. Its dual curve is a rational normal curve in the dual space. Its tangent surface is of degree $r_1 = 2(n-1)$ and the $(n-1)$-th osculating developable is the discriminant hypersurface for binary forms of degree n. For example, for $n = 3$, the tangent surface of R_3 is a quartic surface with equation $Q_0Q_1 + Q_2^2 = 0$, where Q_0, Q_1, Q_2 are some quadrics containing R_3. To see this one considers a rational map $\mathbb{P}^3 - \to \mathcal{N}^\vee \cong \mathbb{P}^2$ defined by the net $\mathcal{N}$ of quadrics containing R_3. After we blow-up R_3 we get a regular map $Y \to \mathbb{P}^2$ which blows down the proper transform of the tangent surface to a conic in $\mathbb{P}^2$. Its equation can be chosen in the form $t_0t_1 + t_2^2 = 0$. The preimage of this conic is the quartic surface $Q_0Q_1 + Q_2^2 = 0$. It contains R_3 as its double curve. Also it is isomorphic to the discriminant hypersurface for binary forms of degree 3.

Conversely, assume that C has no hyperosculating hyperplanes. Then all $k_i = 0$, and we get

$$\sum_{m=0}^{n-1}(n-m)(r_{m-1}-2r_m+r_{m+1}) = -(n+1)r_0 = \sum_{m=0}^{n-1}(2g-2) = n(n+1)(g-1). \tag{10.56}$$

This implies $g = 0$ and $r_0 = n$.

Example 10.4.7. Let E be an elliptic curve embedded in $\mathbb{P}^n$ by a complete linear system $|(n+1)x_0|$, where x_0 is a point on E. Then E has $(n+1)^2$ hyperosculating hyperplanes corresponding to $n+1$-torsion divisors $\mathfrak{d}$ such that $(n+1)\mathfrak{d} \in |(n+1)x_0|$. Thus $k_{n-1} = (n+1)^2$. The second formula in Proposition 10.4.8 shows that $k_i = 0, i \neq n-1$. Thus we obtain $r_m = (m+1)(n+1), m = 0, \ldots, n-1$. For example, the tangent surface of an elliptic quartic curve in $\mathbb{P}^3$ is of degree 8. Also, the dual of E is a curve of degree $r_2 = 12$. It has 16 singular points corresponding to 16 hyperosculating planes.

Example 10.4.8. Assume C is a canonical curve in $\mathbb{P}^{g-1}$. Recall that a *Weierstrass point* of a smooth curve of genus $g > 1$ is a point x such that

$$W(x) = \sum_{i=1}^{g}(h^0(x) + \ldots + h^0(ix) - i) > 0.$$

Let $a_i = h^0(x)+\ldots+h^0(ix)$. We have $a_1 = 1$ and $a_i = i$ if and only if $h^0(x) = \ldots = h^0(ix) = 1$. By Riemann-Roch, this is equivalent to that $h^0(K_C - ix) = g - i$, i.e. the point x imposes expected number of conditions for a hyperplane to have a contact with C of order i at x. A point x is a Weierstrass point if and only if there exists $i \leq g$ such that the number of such conditions is less than expected by the amount equal to $a_i - i$. With notation (10.55), this shows that

$$s_1 + \ldots + s_{i-1} = a_i - i, \ i = 2, \ldots, g.$$

In particular, the point x is hyperosculating if and only if it is a Weierstrass point. We have

$$W(x) = \sum_{i=1}^{g-1}(s_1 + \ldots + s_i) = \sum_{i=1}^{g-1}(g-i-1)s_i.$$

Applying the Plücker formulas with $r_0 = 2g-2, n = g-1$ and computation from (10.56), we obtain

$$-2g(g-1) = \sum_{m=0}^{g-1}(g-m-1)(r_{m-1}-2r_m+r_{m+1}) = \sum_{m=0}^{g-1}(g-1-m)(2g-2-k_m)$$

$$= (2g-2)\sum_{m=0}^{g-1}(g-1-m) + W(x).$$

This gives a formula for $W = \sum_{x\in C} W(x)$

$$W = g(g-1)(g+1). \tag{10.57}$$

Since $h^0(2x) = 1$ for all points on C (because C is not hyperelliptic), we get $k_1 = 0$. Applying Proposition 6.38, we obtain that the rank r_1 of C is equal to $6(g-1)$. If $g = 3$, this agrees with the formula for the degree of the dual quartic. In this case the tangent surface is the plane. If $g = 4$, we get $r_1 = 18$. For a general curve of genus 4 all Weierstrass points are ordinary (i.e. $W(x) = 1$), hence we have 60 hyperosculating planes at Weierstrass points. The linear system of cubics through C defines a birational map from $\mathbb{P}^3$ to a cubic hypersurface in $\mathbb{P}^4$ with an ordinary double point. The image of the tangent surface is the enveloping cone at the node, the intersection of the cubic with its first polar with respect to the node. Its degree is equal to 6, so the tangent surface is the proper inverse image of the cone under the rational map.

We refer for the proof of the following proposition to [312].

Proposition 10.4.10. *Let $f^\vee : C \to \check{\mathbb{P}}^n$ be the normalization of the $n-1$-th associated curve of $f : C \to \mathbb{P}^n$, the dual curve of $f(C)$. Then*

(i) $r_m(f^\vee(C)) = r_{n-m-1}(f(C))$;

(ii) $(f^\vee)^\vee = f$;

(iii) $k_i(f^\vee) = k_i(f)$.

Recall from Chapter 1 that the dual variety of C_0 is the closure in $\check{\mathbb{P}}^n$ of the set of tangent hyperplanes to smooth points of C_0. If $x_0 = f(x)$ is a smooth point, the set of tangent hyperplanes at x is a codimension 2 subspace of the dual space equal to $(n-2)$-th developable scroll of the dual curve. By the duality, we obtain that the dual of the $(n-2)$-th developable scroll of a curve C_0 is the dual curve of C_0. In particular, if $n = 3$, we obtain that the dual of the tangent surface to a nondegenerate curve C_0 in $\mathbb{P}^3$ is the dual curve of C_0, and the dual of a nondegenerate curve C_0 in $\mathbb{P}^3$ is the tangent surface of its dual curve.

Proposition 10.4.11. *Let S be a ruled surface in $\mathbb{P}^3$. The following properties are equivalent:*

(i) all tangent planes to S at smooth points of a fixed ruling coincide;

(ii) S is a tangent surface corresponding to some curve C_0 lying on S;

(iii) the tangent lines of the curve $C_0 \subset G_1(\mathbb{P}^3)$ parameterizing the rulings are contained in $G_1(\mathbb{P}^3)$.

Proof. (i) $\Rightarrow$ (ii). Consider the Gauss map $S \to \check{\mathbb{P}}^3$ which assigns to a smooth point on S the tangent plane. Obviously, it blows down generators of S, hence the image of S is a curve $\check{C}_0$ in the dual space. This curve is the dual variety of S. Its dual variety is our surface S, and hence coincides with the tangent surface of the dual curve C_0 of $\check{C}_0$.

(ii) $\Rightarrow$ (iii) Let $q_C : Z_C \to C$ be the projection from the incidence variety and $D \in |\mathcal{O}_{Z_C}(1)|$. The tangent plane at points of a ruling ℓ_x cuts out the ruling with multiplicity 2. Thus the linear system $|D - 2\ell_x)|$ is non-empty (as always we identify a ruling with a fibre of q_C). The exact sequence

$$0 \to \mathcal{O}_{Z_C}(D - 2\ell_x) \to \mathcal{O}_{Z_C}(D - \ell_x) \to \mathcal{O}_{\ell_x}(D - \ell_x) \to 0$$

shows that $h^0(\mathcal{O}_{l_x}(D-\ell_x)) = 1$, i.e. $|D-\ell_x|$ has a base-point $y(x)$ on ℓ_x. This means that all plane sections of S containing ℓ_x have residual curves passing through the same point $y(x)$ on ℓ_x. Obviously, this implies that the point $y(x)$ is a singular point of S and the differential of the projection $p_C : Z_C \to S$ at $y(x)$ is not surjective. Applying Proposition 10.1.11, we obtain that the tangent line $\mathbb{T}_x(C)$ is contained in the α-plane $\Omega(y(x)) \subset G_1(\mathbb{P}^3)$.

(iii) $\Rightarrow$ (i) Applying Proposition 10.1.11, we obtain that each ℓ_x has a point $y(x)$ such that its image in S is a singular point and the differential of p_C at $y(x)$ is not surjective. This implies that $y(x)$ is a base point of the linear system $|D - \ell_x|$ on ℓ_x. As above, this shows that $|D-2\ell_x|$ is not empty and hence there exists a plane tangent to S at all points of the ruling ℓ_x. □

The set of points $y(x) \in \ell_x, x \in C$ is a curve C_0 on S such that each ruling ℓ_x is tangent to a smooth point on C_0. So S is the tangent surface of C_0. The curve C_0 is called the *cuspidal edge* of the tangent surface. It is a curve on S such that at its general point s the formal completion of $\mathcal{O}_{S,s}$ is isomorphic to $\mathbb{C}[[z_1, z_2, z_3]]/(z_1^2 + z_2^3)$.

10.4.4 Quartic ruled surfaces in $\mathbb{P}^3$

Here we will discuss the classification of quartic ruled surfaces in $\mathbb{P}^3$ due to A. Cayley and L. Cremona. Note that we have already classified ruled surfaces of degree 3 in Chapter 9. They are non-normal cubic surfaces and there are two kinds of them. The double curve Γ is a line and the map $D(f) \to \Gamma$ is an irreducible (reducible) degree 2 cover. The surface Z_C is isomorphic to $\mathbf{F}_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$. The linear system $|H|$ which gives the map $f : \mathbf{F}_1 \to \mathbb{P}^3$ is equal to $|E+2F|$, where E is the exceptional section and F is a fibre. The curve $D(f) \in |H - F| = |E + F|$. In the first case the surface S has ordinary singularities and $D(f)$ is an irreducible curve. In the second case $D(f) \in |H|$ and consists of the exceptional section and a fibre.

Now let us deal with quartic surfaces. We do not assume that the surface has only ordinary singularities. We start with the following.

Proposition 10.4.12. *The genus of a ruled quartic surface is equal to* 0 *or* 1.

Proof. A general plane section H of S is a plane quartic. Its geometric genus g is the genus of S. If $g = 3$, the curve H is nonsingular, hence S is normal and therefore nonsingular. Since $K_S = 0$, it is not ruled. If $g = 2$, the singular curve of S is a line. The plane sections through the line define a pencil of cubic curves on S. Its preimage on the normalization X of S is a pencil of elliptic curves. Since X is a $\mathbb{P}^1$-bundle over a curve of genus 2, a general member of the pencil cannot map surjectively to the base. This contradiction proves the assertion. □

So we have two classes of quartic ruled surfaces: rational surfaces ($g = 0$) and *elliptic ruled surfaces* ($g = 1$). Each surface S is defined by some curve C_0 of degree 4 in $G_1(\mathbb{P}^3)$. We denote by X the minimal ruled surface $\mathbb{P}(\mathcal{E})$ obtained from the universal family Z_{C_0} by the base change $\nu : C \to C_0$, where ν is the normalization map. We will denote by E_0 an exceptional section of X defined by choosing a normalized vector bundle $\mathcal{E}_0$ with $\mathbb{P}(\mathcal{E}_0)$ isomorphic to X.

We begin with classification of rational quartic ruled surfaces.

Proposition 10.4.13. *A rational quartic ruled surface is a projection of a rational normal scroll $S_{1,5}$ or $S_{2,5}$ of degree 4 in $\mathbb{P}^5$.*

Proof. It follows from (10.50) that the surface $X = \mathbb{P}(\mathcal{E})$ is isomorphic to $\mathbf{F}_e$, where $e = 0$ or 2. In both cases the complete linear system $|H|$ maps the surface to a surface of degree 4 in $\mathbb{P}^5$. □

Let $\mathbb{D}(f)$ be the double-point class. We know that the singular curve Γ on S is the image of a curve $D(f)$ from $\mathbb{D}(f)$. Applying (10.47), this gives

$$D(f) \sim 2H - 2F = \begin{cases} 2E_0 + 2F & \text{if } X \cong \mathbf{F}_0, \\ 2E_0 + 4F & \text{if } X \cong \mathbf{F}_2. \end{cases}$$

Since a general plane section of S is a rational curve, $D(f)$ and Γ consist of at most three irreducible components. The linear system

$$|H| = \begin{cases} E_0 + 2F & \text{if } X \cong \mathbf{F}_0, \\ E_0 + 3F & \text{if } X \cong \mathbf{F}_2, \end{cases}$$

maps a component D_i of $D(f)$ to an irreducible component Γ_i of Γ of degree $d_i = \frac{1}{m_i} H \cdot D_i$, where m_i is the degree of the map $D_i \to \Gamma_i$. The number m_i is equal to the multiplicity of a general point on Γ_i as a singular point of the surface unless Γ_i is a curve of cusps. In the latter case $m_i = 1$ but D_i enters in D with multiplicity 2. A fibre $F_x = \pi^{-1}(x)$ could be also a part of D. In this case Γ has a singular point at $\nu(x)$. If it is an ordinary double point, the fibre component enters with multiplicity 1, if it is a cusp, it enters with multiplicity 2. Other cases will not occur. Finally, we use that $\dim |H - D_i| > 0$ if Γ_i is contained in a plane, i.e. a line or a conic.

This gives us the following cases making a “rough classification” according to possible singular locus of the surface.

1. $X \cong \mathbf{F}_0$:

   (i) $D(f) = D_1, d_1 = 3$;

   (ii) $D(f) = D_1 + D_2, D_1 \in |E_0|, D_2 \in |E_0 + 2F|, d_1 = 1, d_2 = 2$;

   (iii) $D(f) = D_1 + D_2 + F_1 + F_2, D_1, D_2 \in |E_0|, d_1 = d_2 = 1$;

   (iv) $D(f) = 2D_1, D_1 \in |E_0 + F|, d_1 = 1$;

(iv)' $D(f) = 2D_1, D_1 \in |E_0 + F|, d_1 = 3$;

(v) $D(f) = 2D_1 + 2F_1, D_1 \in |E_0|, d_1 = 1$;

(vi) $D(f) = 2D_1 + F_1 + F_2, D_1 \in |E_0|, d_1 = 2$;

(vi)' $D(f) = 2D_1 + 2F_1, D_1 \in |E_0|, d_1 = 2$.

2. $X \cong \mathbf{F}_2$:

   (i) $D(f) = D_1, d_1 = 3$;

   (ii) $D(f) = E_0 + D_1 + F, D_1 \in |E_0 + 3F|, d_1 = 1, d_2 = 2$;

   (iii) $D(f) = 2E_0 + 2F_1 + 2F_2, d_1 = 1$;

   (iv) $D(f) = 2D_1, D_1 \in |E_0 + 2F|, d_1 = 1$.

Theorem 10.4.14. *There are 12 different types of rational quartic ruled surfaces corresponding to 12 possible cases from above.*

Proof. It suffices to realize all possible cases from above. We know that any quartic ruled surface in $\mathbb{P}^3$ must be a projection of a rational normal scroll of degree 4 in $\mathbb{P}^5$. If $X = \mathbf{F}_0$ it is the scroll $S_{1,5}$ and, if $X = \mathbf{F}_2$, it is the scroll $S_{2,5}$. So the different types must correspond to different choices of the center of the projection.

Let us introduce some special loci in $\mathbb{P}^5$ which will play a role for choosing the center of the projection.

Let $X = \mathbb{F}_0$. We will identify curves on X with their images in $S_{1,5}$. A conic directrix is a curve $E \in |E_0|$. Consider the union of planes spanning the E's. It is a scroll Σ_1 of dimension 3 parameterized by $|E_0| \cong \mathbb{P}^1$. Let us compute its degree. Fix two generators corresponding to fibres F_1 and F_2 of $\mathbf{F}_0$. Then $|H - F_1 - F_2| = |E_0|$. If we fix three pairs of generators $F_1^{(i)}, F_2^{(i)}, i = 1, 2, 3$, each spanning a $\mathbb{P}^3$, then we can establish a correspondence Γ of tri-degree $(1, 1, 1)$ on $|E_0| \times |E_0| \times |E_0|$ such that the point $(x, y, z) \in \Gamma$ corresponds to three hyperplanes from each linear system $|H - F_1^{(i)} - F_2^{(i)}|$ which cut out the same curve $E \in |E_0|$. The three hyperplanes intersect along the plane spanning E. This shows that our scroll is the join of three disjoint lines in the dual $\mathbb{P}^5$ which can be identified with the same $\mathbb{P}^1$. Applying formula (10.44), we obtain that the degree of Σ_1 is equal to 3.

The next scroll we consider is the union Σ_2 of 3-dimensional spaces spanned by tangent planes of $S_{1,5}$ along points on a fixed generator. This 3-dimensional space is spanned by the tangent lines of two fixed conic directrices at the points where they intersect the generator. Thus our scroll is the join of the tangent scrolls of the two directrices with respect to the correspondence between the directrices defined by the generators. The degree of this scroll is given by the formula in Example 10.4.3. Since the tangent lines of a conic are parameterized by the conic, and the two conics are disjoint, the degree of Σ_2 is equal to 4. Obviously, Σ_1 is a 2-directrix of Σ_2. Since the tangent plane to $S_{1,5}$ at a point x is spanned by the generator passing through this point and the tangent line of the conic directrix passing through this point, we obtain that Σ_2 coincides with the tangent scroll of $S_{1,5}$.

One more scroll we need is constructed as follows. Consider directrices of $S_{1,5}$ defined by the images of curves $\Gamma_3 \in |E_0 + F|$. We identify them with the images.

These are directrices of degree 3. Let Σ_3 be the union of tangent planes to $S_{1,5}$ at the points of Γ_3. These tangent planes can be obtained as joins of tangent lines of Γ_3 at points $x \in \Gamma_3$ and the points x' on a conic directrix E such that the points x, x' lie on the same generator. Thus Σ_3 is obtained by construction from Example 10.4.3 as the join of the tangent scroll of Γ_3 and the conic. The degree of the tangent scroll has been computed there in and it is equal to 4. Thus the degree of Σ_3 is equal to $4+2-1=5$, where we subtracted 1 because the conic and Γ_3 meet at one point dropping the dimension of the join by 1.

Let $p_\ell : S_{1,5} \to S$ be the projection map from a line ℓ. We will use that any two points x_1, x_2 in the double locus $D(f)$ which are projected to the same point must lie on a secant of $D(f)$ passing through these points and intersecting ℓ. The secant becomes a tangent line if $x_1 = x_2$ is a critical point of p_ℓ.

- Type 1 (i).

Take a line ℓ in $\mathbb{P}^5$ which intersects the quartic scroll Σ_2 at four distinct points and is not contained in any 3-dimensional space spanned by a cubic directrix $\Gamma_3 \in |E_0 + F|$. Let D be an irreducible component of $D(f)$ and x be a general point of D. We know from the classification of all possible components of $D(f)$ that the degree of the projection map must be 2 or 3. If the degree is equal to 3, then $D \in |E_0 + F|$ is a cubic directrix and its projection is a line. This implies that ℓ belongs to the linear span of D. By assumption on ℓ this does not happen. So the degree is equal to 2. The map which assigns to a point $x \in D$ the intersection point of ℓ and the secant passing through x is a degree 2 map $D \to \ell$. The intersection points of ℓ with Σ_2 are the branch points of this map. By Hurwitz's formula, the normalization of D is a genus 1 curve, hence the arithmetic genus is ≥ 1. The classification of possible D's shows that this could happen only if D is a nonsingular curve from $|2E_0 + 2F|$. So this realizes Type 1(i).

The quartic scroll S can be described as follows. Consider a Veronese cubic R_3 in $\mathbb{P}^3$ and let S be the set of its secants contained in a non-special linear line complex. The set of secants of R_3 is a surface in $G = G_1(\mathbb{P}^3)$ of degree 4 in its Plücker embedding. This can be seen by computing its cohomology class in G. A general α-plane $\Omega(p)$ contains only one secant. A general β-plane $\Omega(\Pi)$ contains three secants. This shows that the degree of the surface of secants is equal to 4. The surface must be a Veronese surface in $\mathbb{P}^5$ because it does not contain lines. The intersection of the surface with a general linear line complex is a curve C of degree 4. It defines a quartic ruled surface S_C. Take a point $p \in R_3$ and consider the set of secants $\ell_x, x \in C$ such that $p \in \ell_x$. The intersection of the Schubert plane $\Omega(p, \mathbb{P}^3)$ with the Veronese surface is a conic. Its intersection with the linear line complex must consist of 2 lines. Thus through each point of R_3 passes two generators of the surface S_C. The curve R_3 is the double curve of S.

- Type 1 (ii).

In this case we take ℓ intersecting Σ_1 at some point x_0 in the plane spanned by some conic directrix $E \in |E_0|$. The projection of E is a line and the map is 2:1. Note that in this case the point x_0 is contained in two tangents to E so two of the four intersection points of ℓ and Σ_2 coincide. It also shows that Σ_1 is contained in the singular locus of Σ_2. The remaining two points in $\ell \cap \Sigma_2$ are the branch points of the double cover $E' \to \ell$, where $E' \in |E_0 + 2F|$ is the residual component of $D(f)$. Arguing as above we see that it is a smooth rational curve of degree 4. Its projection is a conic.

- Type 1 (iii).

This time we take ℓ intersecting Σ_1 at two points p_1, p_2. These points lie in planes Π_1 and Π_2 spanned by directrix conics E_1 and E_2. The projection from ℓ maps these conics to disjoint double lines of S. Let us now find two generators F_1 and F_2 which are projected to the third double line. Consider the pencil $\mathcal{P}_i$ of lines in the plane Π_i with base point p_i. By intersecting the lines of the pencil with the conic E_i, we define an involution on E_i and hence an involution γ_i on the pencil $|F| \cong \mathbb{P}^1$ (interchanging the generators intersecting E_i at two points in the involution). Now we have two involutions on $|F|$ whose graphs are curves of type $(1,1)$. They have two common pairs in the involution which give us two generators on $S_{1,5}$ intersecting E_i at two points on a line ℓ_i through p_i. The 3-dimensional subspace spanned by ℓ, ℓ_1 and ℓ_2 contains the two generators. They are projected to a double line of S.

- Type 1 (iv).

The image of D_1 on $S_{1,5}$ is a rational normal cubic R_3 spanning a 3-plane M of $\mathbb{P}^5$. We project from a general line in this subspace. The restriction of the projection to D_1 is a degree 3 map. So the projection of D_1 is a triple line of S.

Another possibility here is to project from a line directrix ℓ of the tangent scroll Σ_3. Each point on ℓ lies in a tangent plane to a cubic directrix $\Gamma_3 \in |E_0 + F|$. So the projection from ℓ maps Γ_3 to a rational curve R_3 of degree 3 and maps the tangent lines to Γ_3 to tangent lines to R_3. Thus the scroll S is a developable quartic surface considered in Example 10.4.6. Let us find a line directrix on Σ_3. We know that Σ_3 is equal to the image of a projective bundle $\mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is a vector bundle over $\mathbb{P}^1$ of rank 3 and degree 5. Thus $\deg \mathcal{E}^\vee(1) = -5 + 3 = -2$, and applying Riemann-Roch, we obtain $h^0(\mathcal{E}^\vee(1)) \geq \deg \mathcal{E}^\vee(1) + 3 > 0$. This implies that there is a non-trivial map of sheaves $\mathcal{E} \to \mathcal{O}_{\mathbb{P}^1}(1)$. Let $\mathcal{L}$ be the image of this map. It defines a section $\sigma : \mathbb{P}^1 \to \mathbb{P}(\mathcal{E})$ such that $\sigma^*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \cong \mathcal{L}$. Thus the restriction of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ to $\mathsf{D} = \sigma(\mathbb{P}^1)$ is of degree ≤ 1. Since Σ_3 is a scroll in our definition, the sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is ample, hence the degree must be equal to 1. So, the image of D in Σ_3 is a line directrix.

- Type 1 (v).

This is a degeneration of the previous case. The rational normal cubic degenerates into the union of a directrix conic and a generator. The projection is a degree 2 map on the conic and degree 1 on the line. The double curve Γ is a triple line. It is a generator and a directrix at the same time. Through each point on Γ passes two generators other than itself. As in the previous case a plane containing Γ contains only one of other generators.

- Type 1 (vi).

Consider a hyperplane section $L \cap \Sigma_1$, where L contains two generators F_1 and F_2 of $S_{1,5}$. The quartic curve $L \cap S_{1,5}$ consists of the two generators and a directrix conic D from $|E_0|$. Thus the cubic surface $L \cap \Sigma_1$ contains a plane, and the residual surface is a quadric Q containing D. Take a line ℓ in the 3-dimensional subspace M spanned by F_1 and F_2 which is tangent to the quadric $M \cap Q$. The projection from ℓ maps $S_{1,5}$ to a quartic ruled surface with double line equal to the image of the two generators F_1 and F_2 and the cuspidal conic equal to the image of the directrix conic D.

- Type 1 (vi)'.

The same as in the previous case but we choose L to be tangent along a generator F_1. The double locus is a reducible cuspidal cubic.

- Type 2 (i).

Type 2 corresponds to projection of the rational normal quartic scroll $S_{2,5} \cong \mathbf{F}_2$ in $\mathbb{P}^5$. The exceptional section E_0 is a line directrix on $S_{2,5}$. The curves from the linear system $|E_0 + 2F|$ are cubic directrices. The analog of the tangent scroll Σ_2 here is the join Σ_2' of the tangent surface of a cubic directrix D with the line E_0. It is the union of 3-dimensional spaces spanned by a tangent line to D and E_0. We know that the degree of the tangent scroll of rational normal cubic is of degree 4. Thus the degree of Σ_2' is equal to 4. The rest of the argument is the same as in case 1 (i). We take ℓ intersecting Σ_2' at 4 distinct points and not contained in a 3-space spanned by a cubic directrix. The double curve is a smooth elliptic curve of degree 6 from $|2E_0 + 4F|$.

- Type 2 (ii).

This time we take ℓ intersecting the plane Π spanned by E_0 and a generator F. We also do not take it in any 3-plane spanned by a cubic directrix. Then E_0 and F will project to the same line on F, double line. The residual part of the double locus must be a curve E from $|E_0 + 3F|$. Since no cubic directrix is a part of the double locus, we see that E is an irreducible quartic curve. Its image is a double conic on F.

- Type 2 (iii).

We choose a line ℓ intersecting two planes as in the previous case. Since the two planes have a common line E_0, they span a 3-dimensional subspace. It contains 3 lines which are projected to the same line on F, a triple line of F.

- Type 2 (iv).

Take a cubic curve from $|E_0 + 2F|$ and a line in the 3-dimensional space spanned by the cubic. The cubic is projected to a triple line.

□

Remark 10.4.2. We have seen that a developable quartic surface occurs in case 1 (iv). Let us see that this is the only case when it may occur.

The vector bundle of principal parts $\mathcal{P}^1_C(\mathcal{L})$ must be given by an extension

$$0 \to \Omega^1_C \otimes \mathcal{L} \to \mathcal{P}^1_C(\mathcal{L}) \to \mathcal{L} \to 0, \tag{10.58}$$

where C is a rational cubic in $\mathbb{P}^3$ and $\mathcal{L} = \mathcal{O}_C(1) \cong \mathcal{O}_{\mathbb{P}^1}(3)$. It is known that the extension

$$0 \to \Omega^1_C \to \mathcal{P}^1_C \to \mathcal{O}_C \to 0,$$

from which the previous extension is obtained by twisting with $\mathcal{L}$, does not split. Its extension class is defined by a non-zero element in $\mathrm{Ext}^1(\mathcal{O}_C, \Omega^1_C) \cong H^1(C, \Omega^1_C) \cong \mathbb{C}$ (this is the first Chern class of the sheaf $\mathcal{O}_{\mathbb{P}^1}(1)$). After tensoring (10.58) with $\mathcal{O}_{\mathbb{P}^1}(-2)$ we get an extension

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{P}^1_{\mathbb{P}^1}(\mathcal{L})(-2) \to \mathcal{O}_{\mathbb{P}^1}(1) \to 0.$$

The locally free sheaf $\mathcal{E} = \mathcal{P}^1_C(\mathcal{L})(-2)$ has 2-dimensional space of globals sections. Tensoring with $\mathcal{O}_{\mathbb{P}^1}(-1))$ and using that the coboundary homomorphism

$$H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \to H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2))$$

is non-trivial, we obtain that $\mathcal{E}(-1)$ has no non-zero sections, hence $\mathcal{E}$ is a normalized vector bundle of degree 0 defining the ruled surface $\mathbb{P}(\mathcal{E})$. There is only one such bundle over $\mathbb{P}^1$, the trivial bundle $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$. Untwisting $\mathcal{E}$, we obtain that the sheaf $\mathcal{P}^1_R(\mathcal{L})$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$, so $\mathbb{P}(\mathcal{P}^1_R(\mathcal{L})) \cong \mathbf{F}_0$ and the complete linear system defined by the tautological invertible sheaf corresponding to $\mathcal{P}^1_R(\mathcal{L})$ embeds $\mathbf{F}_0$ in $\mathbb{P}^5$ as the rational normal scroll $S_{1,5}$. The divisor $D(f)$ must be divisible by 2 and the only case when it happens is case 1 (iv)'.

We can also distinguish the previous cases by a possible embedding of the quartic curve C_0 parameterizing generators of S in $G = G_1(\mathbb{P}^3)$. Since $\deg C_0 = 4$ in the Plücker embedding, the curve is always contained in a hyperplane L on $\mathbb{P}^5$. If furthermore, C_0 lies in a codimension 2 subspace, then this subspace is either contained in one tangent hyperplane of G or is equal to the intersection of two tangent hyperplanes (because the dual variety of G is a quadric). So we have the following possibilities:

- I C_0 is a rational normal quartic contained in a hyperplane L which is not tangent to G;
- II C_0 is a rational normal quartic contained in a hyperplane L which is tangent to G at a point O not contained in C_0;
- III C_0 is a rational normal quartic contained in a hyperplane L which is tangent to G at a point O contained in C_0;
- IV C_0 is a rational quartic curve contained in the intersection of two different tangent hyperplanes of G;
- V C_0 is a rational quartic curve contained in a 3-dimensional subspace through which passes only one tangent hyperplane of $G_1(\mathbb{P}^3)$. The tangency point is an ordinary node of C_0.

A quartic surface of type 1 (i) or 1 (iv)' from Theorem 10.4.14 belongs to type I. Following W. Edge [150] we re-denote types 1 (i) and 1 (iv)' with I.

In type 1 (ii) the line component of the double curve is a directrix, so all generators belong to a line complex tangent to $G_1(\mathbb{P}^3)$ at the point O representing this directrix. This is Edge's type II. Through any point p on the directrix passes two generators, the point O belongs to a secant of C_0 formed by the line $\Omega(p,\Pi)$, where Π is the plane spanned by the two generators. It is a nonsingular point of C_0. We have Edge's type II (C).

In case 1(iii) we have two directrices which are not generators. This means that C is contained in the intersection of two special linear complexes tangent to $G_1(\mathbb{P}^3)$ at two points. This is type IV (B). The tangency points correspond to the line directrices on S. The curve C_0 is contained in the intersection of two special linear complexes which is a nonsingular quadric. The curve C_0 has an ordinary node at the point corresponding to two generators mapped to a double line on F.

In case 1 (iv), the triple line is a directrix of S, so we are again in case II but in this case the point O intersects the α-plane $\Omega(p)$ at three non-collinear points and intersects the β-plane $\Omega(\Pi)$ at one point. This is Edge's type II (A).

In case 1 (v) the double curve is a triple line. One of the generators F is contained in $D(f)$ with multiplicity 2 and is mapped to the triple line. Thus S is contained in a unique special line complex which is tangent to G at a cusp of C_0. Since C_0 is singular, it is contained in a 3-dimensional space. So C is contained in a quadric cone equal to the intersection of $G_1(\mathbb{P}^3)$ with two line complexes. The singular point of this cone is the singular point of C_0. This is Edge's Type III (A).

In case 1 (vi) two generators on $S_{1,5}$ are projected to a double generator of S. The curve C_0 has an ordinary double point, hence it lies in two line complexes. The double generator is the only line directrix on S. Thus there is only one special line complex containing S and its tangency point is an ordinary double point of C_0. This is Edge's Type V (A). In case 1 (vi)', we also have Type V (A), only this time the singular point of C_0 is a cusp.

In case 2 (i) the line directrix ℓ corresponding to E_0 defines a line complex containing C. Thus we are in case II. The Schubert plane $\Omega(p,\mathbb{P}^3), p\in\ell$, contains only one point, the α-plane $\Omega(\pi), \ell\subset\pi$, contains three points. This is Edge's type II (B).

In case 2 (ii) we have a line directrix which is at the same time is a generator g . This shows that we are in case III. The curve C has a cuspidal singularity at the point O corresponding to the generator g. The curve C intersects any plane $\Omega(p,\mathbb{P}^3), p\in g$, in one point and every plane $\Omega(\pi,\mathbb{P}^3), g\subset\pi$, at two points. This is Edge's type III (B).

In case 2 (iii) we have a triple line on S formed by the projection of the line directrix E_0 of $S_{2,5}$ and its two generators. We are in case V, where the singular point of C is the singular point of the quadric cone. This is Edge's type V (B).

In case 2 (iv) we have a triple line projected from a rational cubic curve. We have two line directrices of S, one is a triple line. The curve C is nonsingular. This is type IV (A) of Edge.

Finally, we have to classify elliptic ruled quartic surfaces in $\mathbb{P}^3$. Let $\pi : X \to C$ be a minimal ruled surface with a base C. We write X in the form $X = \mathbb{P}(\mathcal{E}_0)$, where

$\mathcal{E}_0$ is a normalized rank 2 locally free sheaf. Since $K_C = 0$ in our case, the canonical class formula (10.8) gives

$$K_X = -2E_0 + \pi^*(\mathfrak{e}), \tag{10.59}$$

where $e = -\deg \mathfrak{e} \geq 0$.

Let $|H|$ be a linear system on X which defines the normalization map $f : X \to S$. We can write $H \equiv E_0 + mF$. Since H is ample, intersecting both sides with E_0, we get $m > e$. We also have $H^2 = 2m - e = 4$. This gives two possibilities $e = 0, m = 2$ and $e = 2$ and $m = 3$. In the second case $H \cdot E_0 = 1$, hence $|H|$ has a fixed point on E_0. This case is not realized (it leads to the case when S is a cubic cone). The formula for the double-point locus gives $\mathbb{D}(f) \equiv 2H - \pi^*(\mathfrak{d})$, where $d = \deg \mathfrak{d} = 4$. Thus we obtain

$$H \equiv E_0 + 2F, E_0^2 = 0, D(f) \equiv 2E_0.$$

By Riemann-Roch, $\dim |H| = 3$. Since $\dim |H - E_0| = \dim |2F| = 1$, we obtain that the image of E_0 is a line. Since the restriction of $|H|$ to E_0 is a linear series of degree 2, the image of E_0 is a double line. We have two possibilities: $D(f)$ consists of two curves $E_0 + E_0'$ or $D(f)$ is an irreducible curve D with $H \cdot D = 4$. Since $|H - D| = \emptyset$, we obtain that the image of D is a space quartic, so it cannot be the double locus. This leaves us with two possible cases $D(f)$ is the union of two disjoint curves $E_0 + E_0'$ or $D(f) = 2E_0$.

In the first case $H \cdot E_0 = H \cdot E_0' = 2$ and $\dim |H - E_0| = \dim |H - E_0'| = \dim |2F| = 1$. This shows that the images of E_0 and E_0' are two skew double lines on S. The curve C is a nonsingular elliptic curve in $G_1(\mathbb{P}^3)$. It spans a 3-dimensional subspace equal to the intersection of two special line complexes.

Since $X = \mathbb{P}(\mathcal{E})$ has two disjoint sections with self-intersection 0, the sheaf $\mathcal{E}$ splits into the direct sum $\mathcal{L}_1 \oplus \mathcal{L}_2$ of invertible sheaves of degree 0. This easily follows from [206], Chapter V, Proposition 2.9. One of them must have a nonzero section, i.e. must be isomorphic to $\mathcal{O}_C$. So we obtain

$$X \cong \mathbb{P}(\mathcal{O}_C \oplus \mathcal{O}_C(\mathfrak{a})),$$

where $\deg \mathfrak{a} = 0$. Note that X cannot be the direct product $C \times \mathbb{P}^1$ because in this case the image of any $C \times \{x\}$ must be a double line, in other words, in this case $|H|$ defines a degree 2 map. So, we have $\mathfrak{a} \nsim 0$.

In the second case two double lines come together forming the curve of tacnodes. In this case the curve C lies only in one special linear line complex. The pencil of hyperplanes containing C intersects the dual Klein quadric at one point.

Let $\sigma : \mathcal{E} \to \mathcal{O}_C(\mathfrak{e})$ be the surjective map of sheaves corresponding to the section E_0. Since $\deg \mathcal{E} = \deg \mathfrak{e} = 0$, we have $\deg \ker(\sigma) = 0$. Thus $\mathcal{E}$ can be given as an extension of invertible sheaves

$$0 \to \mathcal{O}_C(\mathfrak{b}) \to \mathcal{E} \to \mathcal{O}_C(\mathfrak{e}) \to 0,$$

where $\deg \mathfrak{b} = 0$. Suppose this extension splits, then X has two disjoint sections with self-intersection zero. By the above, we see that the map defined by the linear system $|H|$ maps each section to a double line of S. This leads to the first case. So in our case, there are no disjoint sections, and hence the extension does not split. This implies that

$\mathrm{Ext}^1(\mathcal{O}_C(\mathfrak{e}), \mathcal{O}_C(\mathfrak{b})) = H^1(C, \mathcal{O}_C(\mathfrak{e}-\mathfrak{b})) \neq \{0\}$. This is possible only if $\mathfrak{b} \sim \mathfrak{e}$. Since $\mathcal{E}$ has a non-zero section, we also have $H^0(C, \mathcal{O}_C(\mathfrak{e})) \neq \{0\}$, i.e. $\mathfrak{e} \sim 0$. Thus we obtain that $\mathcal{E}$ is given by a non-split extension

$$0 \to \mathcal{O}_C \to \mathcal{E} \to \mathcal{O}_C \to 0.$$

In fact, it is known that any elliptic ruled surface with $e = 0$ which corresponds to a non-split vector bundle, must be isomorphic to the ruled surface $\mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is defined by the above extension (see [206], Chapter V, Theorem 2.15).

Let us summarize our classification with Table 10.1.

Type	Double cuve	Edge	Cremona	Cayley	Sturm
I (i),(iv)'	R_3	I	1	10	III
I (ii)	L+K	II (C)	2	7	V
I (iii)	L+L'+G	IV (B)	5	2	VII
I (v)	3L	II (A)	8	9	IX
I (iv)	3L	III (A)	3	-	XI
I (vi),(vi)'	2L+G	V (A)	6	5	VIII
II (i)	R_3	II (B)	7	8	IV
II (ii)	L+K	III (B)	4	-	VI
II (iii)	3L	V (B)	10	6	XII
II (iv)	3L	IV (A)	9	3	X
$g=1$	L+L'	VI(A)	11	1	I
$g=1$	2L	VI(B)	12	4	II

Table 10.1: Quartic ruled surfaces

Here R_3 denotes a curve of degree 3, L denotes a line, K is a conic and G is a generator.

A finer classification of quartic ruled surfaces requires to describe the projective equivalence classes. We refer to [323] for a modern work on this. Here we only explain, following [35], the fine classification assuming that the double curve is a Veronese cubic R_3. First, by projective transformation we can fix R_3 which will leave us only with 3-dimensional subgroup G of $\mathrm{PGL}(4)$ leaving R_3 invariant. It is isomorphic to PSL_2.

Let $\mathcal{N}$ be the net of quadrics in $\mathbb{P}^3$ that contains R_3. It defines rational map $\alpha : \mathbb{P}^3 - \to \mathcal{N}^\vee$. The preimage of a point s in $\mathcal{N}^\vee$, i.e. a pencil in $\mathcal{N}$, is the base locus of the pencil. It consists of the curve R_3 plus a line intersecting R_3 at two points. This makes an identification between points in $\mathcal{N}^\vee \cong \mathbb{P}^2$ and secants of R_3. The preimage of a conic K in $\mathcal{N}^\vee$ is a quartic surface which is the union of secants of R_3. It is a quartic ruled surface. Conversely, every quartic ruled surface S containing R_3 as its double curve is obtained in this way. In fact, we know that S is the union of secants of R_3 and hence the linear system of quadrics containing R_3 should blow down each secant to a point in $\mathcal{N}^\vee$. The preimage of a general line in the plane is a quadric that cuts out on S a curve of degree 8 that consists of the curve R_3 taken with multiplicity 2 and two lines. This shows that the image of S is a conic. Thus we find a bijection between quartic surfaces with double curve R_3 and conic in the plane. The group G is naturally isomorphic to the group of projective transformations of $\mathcal{N}^\vee$. It is well-known that the projective representation of PSL_2 in $\mathbb{P}^2$ leaves a nonsingular conic K_0 invariant. The quartic surface corresponding to K_0 is the only quartic surface invariant under G. This is of course the developable quartic ruled surface (see Example 10.4.6). In this way our classification is reduced to the classification of orbits in the space of nonsingular conics $\mathbb{P}^5$ under the action of the group PSL_2 of projective automorphisms leaving K_0 invariant. The orbit space is of dimension 2. Let K be a conic different from K_0. There are 5 possible cases for the intersection $K \cap K_0$: 4 distinct points, one double coincidence, two double coincidences, one triple coincidence and one quartuple coincidence. Together with K_0 it gives 6 different types. The first type has two parameters, the cross ratio of 4 points and a point in the pencil of conics with the same cross ratio. The second type is a one-parameter family. All other types have finitely many orbits. We refer to [323] and [35] for explicit equations.

There are many direct geometric constructions of quartic ruled surfaces. The first historical one uses Cayley's construction of a ruled surface as the union of lines intersecting three space curves (see Example 10.4.5). For example, taking $(d_1, d_2, d_3) = (2, 2, 1)$ and $(a_{12}, a_{13}, a_{23}) = (2, 0, 1)$ gives a quartic ruled surface with a double conic and a double line which intersect at one point. Another construction is due to L. Cremona. It is a special case of the construction from Example 10.4.3, where we take the curves C_1 and C_2 of degree 2. If the two conics are disjoint, a correspondence of bidegree $(1, 1)$ gives a quartic ruled surface. We can also take two conics intersecting at 2 points and a correspondence of the next section we will discuss a more general construction due to B. Wong [427].

Finally, we reproduce equations of quartic ruled surfaces (see [150], p. 69).

$$
\begin{aligned}
I &: \quad Q(xz - y^2, xw - yz, yw - z^2) = 0, \\
&\quad\ \text{where } Q = \sum_{1 \le i \le j \le 3} a_{ij} T_i T_j \text{ is a nondegenerate quadratic form;} \\
II(A) &: \quad zy^2(ay + bx) + wx^2(cy + dx) - ex^2y^2 = 0; \\
II(B) &: \quad \text{same as in (I) with } a_{22}^2 + a_{22}a_{13} - 4a_{12}a_{23} + a_{11}a_{33} = 0; \\
II(C) &: \quad (cyz + bxz + axy + zw - wx)^2 - xz(ax - by + cz)^2 = 0; \\
III(A) &: \quad ax^2y^2 - (x + y)(x^2w + y^2z) = 0; \\
III(B) &: \quad (xw + yz + azw)^2 - zw(x + y)^2 = 0; \\
IV(A) &: \quad x(az + bw)w^2 - y(cz + dw)z^2 = 0; \\
IV(B) &: \quad y^2z^2 + axyzw + w^2(bz + cx)x = 0, \\
&\quad\ (yz - xy + awx)^2 - xz(x - z + bw)^2 = 0; \\
V(A): &: \quad (yz - xy + axw)^2 - xz(x - z)^2 = 0; \\
V(B) &: \quad (az^2 + bzw + cw^2)(yz - xw) - z^2w^2 = 0; \\
VI(A) &: \quad ax^2w^2 + xy(bz^2 + czw + dw^2) + ey^2z^2 = 0; \\
VI(B) &: \quad (xw - yz)^2 + (ax^2 + bxy + cy^2)(xw - yz) + \\
&\quad\ (dx^3 + ex^2y + fxy^2 + gy^3)x = 0.
\end{aligned}
$$

10.4.5 Ruled surfaces in $\mathbb{P}^3$ and tetrahedral line complexes

Fix a pencil $\mathcal{Q}$ of quadrics in $\mathbb{P}^3$ with nonsingular base curve. The pencil contains exactly four singular quadrics of corank 1. We can fix coordinate systems to transform the equations of the quadrics to the diagonal forms

$$\sum_{i=0}^{3} a_i t_i = 0, \quad \sum_{i=0}^{3} b_i t_i = 0.$$

The singular points of four singular quadrics in the pencil are the reference points $A_1 = [1,0,0,0],\ A_2 = [0,1,0,0],\ A_3 = [0,0,1,0],\ A_4 = [0,0,0,1]$. For any point not equal to one of these points the intersection of polar planes $P_x(Q), Q \in \mathcal{Q}$, is a line in $\mathbb{P}^3$. This defines a rational map $T : \mathbb{P}^3 - \to G_1(\mathbb{P}^3) \subset \mathbb{P}^5$ whose image is a tetrahedral line complex (see the end of section 10.3.5). In coordinates, the Plücker coordinates p_{ij} of the line $T([x_0, x_1, x_2, x_3])$ are given by

$$p_{ij} = (a_i b_j - a_j b_i) x_i x_j.$$

For any space curve C of degree m not passing through the points $A_1, \ldots, A_4$, its image under the map T is a curve of degree $d = 2m$ in the tetrahedral complex. It defines a ruled surface in $\mathbb{P}^3$ of degree $2d$, the union of lines $T(x), x \in C$. If we consider the graph $G_T \subset \mathbb{P}^3 \times G_1(\mathbb{P}^3)$ of T, its projection to $G_1(\mathbb{P}^3)$ is the universal family Z_C. Its projection to $\mathbb{P}^3$ is our ruled surface.

Let Π be a plane in $\mathbb{P}^3$ not containing the points A_i. For any $Q \in \mathcal{Q}$, the intersection of polars $P_x(Q), x \in \Pi$, is a point in $\mathbb{P}^3$. Varying Q in the pencil we get a cubic curve. Explicitly, if we choose three non-collinear points $[y_0^{(j)}, y_1^{(j)}, y_2^{(j)}], j = 1,2,3$, in the plane Π, the cubic curve is the image of the map $\mathcal{Q} \to \mathbb{P}^3$ given by the solution line of the system of linear equations

$$\sum_{i=0}^{3} (u_0 a_i + u_1 b_i) y_i^{(j)} t_i = 0,\ j = 1,2,3.$$

In coordinate-free approach, we let $\mathcal{Q} = |U|$, $\Pi = |W|$ and $\mathbb{P}^3 = |V|$ for some linear spaces, then the polar map defines a bilinear map $P : W \otimes U \to V^\vee$, or, equivalently, a linear map

$$U \to \mathrm{Hom}(W, V^\vee). \tag{10.60}$$

Choose a volume form on V to identify $\bigwedge^3 V^\vee$ with V and a volume form on W to identify $\bigwedge^3 W$ with $\mathbb{C}$. The composition of the previous map with the map

$$\mathrm{Hom}(W, V^\vee) \to \bigwedge^3 \mathrm{Hom}(W, V^\vee) = \mathrm{Hom}(\bigwedge^3 W, \bigwedge^3 V^\vee) \cong V$$

is a map $U \to V$ given by polynomials of degree 3, the corresponding map of projective spaces does not depend on the choice of volume forms, and defines a map of degree 3

$$f_\Pi : \mathcal{Q} = |U| \to \mathbb{P}^3 = |V|. \tag{10.61}$$

Its image is a cubic curve $R_3(\Pi)$ in $\mathbb{P}^3$. Each line ℓ in Π defines a ruled surface of degree 2, i.e. a quadric, which must contain $R_3(\Pi)$. So one can identify the net of quadrics containing $R_3(\Pi)$ with the dual plane Π^*. More generally, the ruled surface S corresponding to any curve C of degree m in Π is a surface of degree $2m$ containing $R_3(\Pi)$. Consider a point $x \in \Pi$ as the intersection of two lines in Π. Then the line $T(x)$ is contained in the intersection of the two quadrics corresponding to the lines, hence it is a secant of the curve $R_3(\Pi)$. Thus we obtain that generators of S are secants of $R_3(\Pi)$. If $m = 2$, this gives that $T(C)$ is the intersection of a Veronese surface by a linear line complex, a general choice of Π gives us quartic surfaces of type I (i).

Take a line ℓ in Π. It defines a quadric containing $R_3(\Pi)$ It comes with a ruling on the quadrics whose generators are secants of $R_3(\Pi)$. The set of lines in Π which parametrizes singular quadrics containing $R_3(\Pi)$ is a conic in $\Pi^\vee$. The dual conic K_0 in Π parametrizes pencil of quadrics containing $R_3(\Pi)$ with residual line tangent to $R_3(\Pi)$. The corresponding ruled quartic surface is the developable quartic surface, a special case of type I (iii).

Take again a line ℓ on Π. Recall that it corresponds to a quadric Q_ℓ containing $R_3(\Pi)$. The points on the line are pencils of quadrics containing Q_ℓ. If ℓ is tangent to K_0, then the tangency point is a pencil of quadrics which all tangent to $R_3(\Pi)$ at one point. The point is the singular point of a unique singular quadric in the pencil.

The lines $T(x), x \in \ell$, are generators of the quadric Q_ℓ which intersect $R_3(\Pi)$ at two points. If ℓ is tangent to K_0 then Q_ℓ is a singular quadric and all the lines $T(x), x \in \ell$, pass through the singular point of Q_ℓ. The curve $R_3(\Pi)$ also passes through this point. In this case, the line ℓ intersects a curve C of degree m in Π at m-points different from K_0, all the generators of S corresponding to these points must pass through one point on $R_3(\Pi)$. The converse is also true, if the generators $T(x), x \in C$, all pass through the same point on $R_3(\Pi)$, then these points lie on a line tangent to K_0. Thus we obtain that $R_3(\Pi)$ is m-multiple curve on S. This agrees with case $1(i)$ of quartic ruled surfaces. Also note that C intersects at K_0 at $2m$ points corresponding to generators tangent to $R_3(\Pi)$. If $m = 2$, we get four torsal generators.

Now let us see what happens if we choose special plane Π. For example, let us take Π passing through one of the points $A_1, \ldots, A_4$, say A_1. Then the map f_Π defined in (10.61) is not anymore of degree 3. Under the map (10.60), the quadric Q which has A_1 as its singular point is mapped to a linear map $W \to V^\vee$ of rank 2. Composing with the determinant map, it goes to zero. Thus the map (10.61) has a base point at Q, and hence extends to a map of degree 2. Thus the cubic $R_3(\Pi)$ degenerates to a conic $R_2(\Pi)$. The lines in Π correspond to quadrics containing the conic $R_2(\Pi)$ and some line intersecting the conic. This is a degeneration of the singular curve to the union of a conic and a line.

Finally, let us see how elliptic quartic surfaces arise. Take Π passing through the points A_1 and A_2. Take a nonsingular cubic C in the plane which passes through A_1 and A_2. The linear system of quadrics defining the rational map T has two of its base points on C. Thus its image in $G_1(\mathbb{P}^3)$ is a quartic elliptic curve. We see that a ruled surface o degree 6 which corresponds to a general cubic degenerates in this case to the union of a quartic surface and two planes (the images of the blow-ups of A_1 and A_2). The cubic $R_3(\Pi)$ degenerates to a line, one of the two double lines of S. A quadric corresponding to a line through A_1 or A_2 degenerates to a plane with a choice of a pencil of lines in it. This plane does not depend on the line, only the pencil of lines in the plane does. The line passing through A_1 and A_2 is blown down under T to a point in $G_1(\mathbb{P}^3)$ defining the second double line of S. This is the intersection line of the planes corresponding to A_1 and A_2.

Exercises

10.1 Let $P_n \subset \mathbb{C}[t]$ be the space of polynomials of degree $\le n$. Let $f_0, \dots, f_m$ be a basis of a subspace L of P_n of dimension $m+1$. Consider the *Wronskian* of the set $(f_0, \dots, f_m)$

$$W(f_0, \dots, f_m) = \det \begin{pmatrix} f_0 & f_1 & \dots & f_m \\ f_0'(t) & f_1'(t) & \dots & f_m'(t) \\ \vdots & \vdots & \vdots & \vdots \\ f_0^{(m)}(t) & f_1^{(m)}(t) & \dots & f_m^{(m)}(t) \end{pmatrix}.$$

Show that the map

$$G(m+1, n+1) \to \mathbb{P}^{(m+1)(n-m)}, \quad L \mapsto [W((f_0, \dots, f_m)]$$

is well defined and is a finite map of degree equal to the degree of the Grassmannian in its Plücker embedding.

10.2 Show that any $\binom{n}{2} - 1$ lines in $G(2,n)$ lie in a linear line complex. Using this, prove that one can choose coordinates in $\mathbb{P}^{n-1}$ so that any linear line complex can be given by Plücker equations $p_{12} + \lambda p_{34} = 0$, where $\lambda = 0$ if and only if the complex is special.

10.3 Show that the tangent lines of any smooth curve of genus g and degree d in $\mathbb{P}^{n-1}$ is contained in a linear complex if $2(d+g-1) < \binom{n}{2}$.

10.4 Show that any linear k-plane Λ of $G_r(\mathbb{P}^n)$ coincides with the locus of m-planes containing a fixed $(m-1)$-plane and contained in a fixed $(m+k)$-plane or with the locus of m-planes contained in a fixed $(k+1)$-plane and containing a fixed $(k-m)$-plane. Identify these loci with appropriate Schubert varieties.

10.5 Using the previous exercise, show that any automorphism of $G(m,n)$ arises from a unique projective automorphism of $\mathbb{P}^{n-1}$ unless $n = 2m$ in which case PGL(n) is isomorphic to a subgroup of index 2 of Aut$(G_r(\mathbb{P}^{n-1}))$.

10.6 How many lines intersect a set of m general k-planes in $\mathbb{P}^n$?

10.7 Show that Sec$_k(G(2,n))$ is equal to the set of singular points of Sec$_{k+1}(G(d,n))$ for all $k = 0, \dots, [\frac{n-4}{2}]$. If $n = 2m$ show that $G(2,n)$ is the locus of $(m-1)$-fold points of the pfaffian hypersurface of degree m.

10.8 Using the Schubert calculus prove that the projective plane embedded in $G(2,n)$ as the surface of secants of a Veronese curve of degree d in $\mathbb{P}^d$ has degree 2^{d-1} and is isomorphic to a Veronese surface in its linear span.

10.9 Show that tangent lines of a nonsingular quadric in $\mathbb{P}^3$ belong to a quadratic line complex.

10.10 Let $Q_1$and Q_2 be two nonsingular quadrics in $\mathbb{P}^3$ with a choice of a ruling of lines on each of them. Any general line ℓ intersects$Q_1 \cup Q_2$ at four lines, two from each ruling. Together with ℓ, these lines span four planes in the pencil of planes through ℓ. Show that the closure of the locus of lines ℓ such that the four planes is projective equivalent to the four intersection points of ℓ with Q_1 and Q_2 form a Battaglini line complex. Also show that any general Battaglini line complex can be obtained in this way [379].

10.11 Show that the linear system of quadrics in $\mathbb{P}^4$ passing through a Veronese curve Γ of degree 4 defines a rational maps $\Phi : \mathbb{P}^4 - \to \mathbb{P}^5$ whose image is nonsingular quadric in $\mathbb{P}^5$ identified with $G(2,4)$. Show that:

(i) the secant variety $S_1(\Gamma)$ is mapped to a Veronese surface;

(ii) the map Φ extends to a regular map of the blow-up of $\mathbb{P}^4$ along Γ which maps the exceptional divisor to a ruled hypersurface of degree 6 which is singular along the Veronese surface;

(iii) the image of a hyperplane in $\mathbb{P}^4$ is a tetrahedral complex of lines;

(iv) the image of a plane in $\mathbb{P}^4$ not intersecting Γ is a Veronese surface;

(v) the image of a trisecant plane of Γ is a plane in $G(2,4)$. Show that planes from other family are the images of a cubic ruled surface singular along Γ.

10.12 Show that four general lines in $\mathbb{P}^4$ determine the unique fifth one such that the corresponding points in $G(2,5) \subset \mathbb{P}^9$ lie in the same three-dimensional subspace. Any plane which meets four lines meets the fifth line.

10.13 Show that two linear complexes X_ω and $X_{\omega'}$ in $G_1(\mathbb{P}^3)$ are apolar to each other if and only if $i_\omega(X_{\omega'}) = X_\omega$.

10.14 Show that a general web of linear complexes in $G(2,5)$ contains five special complexes.

10.15 Show that the projection of a tetrahedral quadratic from its nonsingular point is isomorphic to the Segre cubic primal .

10.16 Show that the projection of the Segre cubic primal from its nonsingular point is a double cover with branch locus isomorphic to a Kummer surface.

10.17 Show that a general 5-plane in $\mathbb{P}^9$ intersects $G(2,5)$ along a quintic Del Pezzo surface

10.18 Show that tangent lines to a nonsingular quadric in $\mathbb{P}^3$ form a quadratic complex. If the quadric has equation $\sum a_i t_i^2 = 0$, then the equation of the complex is $\sum_{0\le i<j\le 3} a_i a_j p_{ij}^2 = 0$. Show that the singular surface of the complex is equal to the quadric taken with multiplicity 2.

10.19 Let $\mathcal{N}$ be a general 2-dimensional linear system of quadrics in $\mathbb{P}^3$. Show that the set of lines contained on a quadric from $\mathcal{N}$ is parameterized by a cubic line complex (called a *Montesano complex*) [285].

10.20 Consider a smooth curve C of degree d and genus g in $\mathbb{P}^3$ and choose two general lines ℓ and ℓ'. Find the degree of the scroll of lines that intersect C, ℓ and ℓ'.

10.21 Let F be a surface of degree 6 in $\mathbb{P}^3$ which has the edges of the coordinate tetrahedron as its double lines. Find an equation of F and show that its normalization is an Enriques surface.

10.22 Show that the Hessian of a developable quartic ruled surface is equal to the surface itself taken with multiplicity 2. The Steinerian in this case is the whole space [420].

10.23 Show that a generator intersecting the double curve of a ruled surface at a pinch point is a torsal generator.

10.24 Classify all ruled surfaces in $\mathbb{P}^3$ which have two line directrices.

10.25 For each type of a quartic ruled surface find the type of its dual quartic ruled surface.

10.26 Find projective equivalence classes of quartic ruled surfaces with a triple line.

Historical Notes

The main sources for these notes are [268], [271], [380], [308], and [432]. Line Geometry originates from J. Plücker who was the first to consider lines in the 3-space as elements of a new four-dimensional space. These ideas had appeared first in [316] and the details were published much later in [318]. The study of lines in $\mathbb{P}^3$ was very much motivated by mechanics and optics. An early differential geometrical treatment of line geometry can be found in works of E. Kummer [261] and [262]. The six Plücker coordinates p_{ij} of a line were first introduced by H. Grassmann in 1844 [193] in a rather obscure notation. Unaware of the work of Grassmann, in 1859 A. Cayley introduces the coordinates in its modern form as six determinants of a 2×4-matrix and exhibits the quadric equation satisfied by the coordinates [54]. In the subsequent paper, under the same title, he introduced, what is now called, the Chow form of a space curve. The notions of a linear complex of lines and a congruence of lines (the intersection of two linear complexes) are due to Plücker and the first proofs of some of his results were given by G. Battaglini [20]. Among other earlier contributers to theory of general line complexes we cite M. Pash [309].

Plücker began the study of quadratic line complexes by introducing its singular quartic surface with 16 nodes. Although in a special case, many Plïcker's results about quadratic complexes were independently obtained by Battaglini. In his dissertation and later published paper [247], Klein introduced the coordinate system determined by six mutually apolar linear complexes and showed that the singular surface can be identified with a Kummer surface. The notion of the singular surface of a quadratic complex is due to Klein. We refer to [224] and [235] for the history of Kummer surfaces and their relationship with Line Geometry.

Plücker defined a linear complex as we understand it now, i.e. as a set of lines whose coordinates satisfy a linear equation. The set of lines in a linear complex passing through a point x lie in a plane $\Pi(x)$, this defines a linear correlation from the space to the dual space. The correlations arising in this way satisfy the property $x \in \Pi(x)$. They were first considered by Giorgini [185] and Möbius [284] and were called Nullsystems by von Staudt ([392], p. 191). The notions of a null-line and a null-plane belong to Möbius. Chasles' Theorem 10.2.8 gives a purely geometric definition of a Nullsystem [66]. Linear systems of linear complexes were extensively studied in Sturm's book [401].

In 1868, in his Inauguraldissertation at Bonn published later in [247], [250], F. Klein pointed out that Weierstrass's theory of canonical forms for a pair of quadratic forms can be successfully used for the classification of quadratic complexes. This was accomplished later by A. Weiler (see also [419], [378]). The classification consists of 49 different types of complexes corresponding to different Segre symbols of the pencil of quadrics. As we have already noticed earlier, the Segre symbol was first introduced by A. Weiler [425] and Segre acknowledges this himself in [378]. In each case the singular surface is described. For example some of ruled quartic surfaces can be obtained as singular surfaces of a degenerate quadratic complex. A full account of the classification and the table can be found in Jessop's book [235]. Many special quadratic complexes were introduced earlier by purely geometric means. Among them are the tetrahedral complexes and Battaglini's harmonic complexes [21] considered in the present chapter. A complete historical account of tetrahedral complexes can be found in Lie's book [268]. Its general theory is attributed to T. Reye [330] and even they are often called Reye complexes. However, in different disguises, tetrahedral complexes appear in much earlier works, for example, as the locus of normals to two confocal surfaces of degree 2 [29] (see a modern exposition in [385], p. 376), or as the locus of lines spanned by an argument and the value of a projective transformation [68], or as the locus of secants of twisted cubics passing through the vertices of a tetrahedron [292]. We refer to [345] and [207] for the role of tetrahedral complexes in Lie's theory of differential equations and transformation groups.

The modern multi-linear algebra originates in Grassmann's work [192], [193]. We refer to [36] for the history of multi-linear algebra. The editorial notes for the English translation of [193] are very helpful to understand Grassmann's work. As a part of Grassmann's theory, a linear k-dimensional subspace of a linear space of dimension n corresponds to a decomposable k-vector. Its coordinates can be taken as the coordinates of the linear subspace and of the associated projective subspace of $\mathbb{P}^{n-1}$. In this way Grassmann was the first to give a higher-dimensional generalization of the Cayley-Plücker coordinates of lines in $\mathbb{P}^3$. The equations (10.1) of Grassmann varieties could not be found in his book. The fact that any relation between the Plücker coordinates follow from these relations was first proven by G. Antonelli [8] and much later by W. Young [428]. In [365] and [366] H. Schubert defines, what we now call, Schubert varieties, computes their dimensions and degrees in the Plücker embedding. In particular, he finds the formula for the degree of a Grassmann variety. A modern account of Schubert's theory can be found in Hodge-Pedoe's book [218], v. II and Fulton's book [173].

The study of linear complexes in arbitrary $[n]$ (the classical notation $[n]$ for $\mathbb{P}^n$ was introduced by Schubert in [365]) was initiated in the work of S. Kantor [243], F. Palatini [306] and G. Castelnuovo [46] (in case $n = 4$). Palatini scroll was first studied in [307] and appears often in modern literature on vector bundles (see, for example, [305]). Quadratic complexes of lines in $\mathbb{P}^4$ were extensively studied by B. Segre [382].

Although ruled surfaces were studied earlier (more from differential point of view), A. Cayley was the fist who laid the foundations of the algebraic theory of ruled surface [51], [57], [58]. The term scroll belongs to Cayley. The study of non-normal surfaces in $\mathbb{P}^3$, and, in particular, ruled surfaces, began by G. Salmon [351], [352]. Salmon's work was extended by A. Cayley [63], [64]. The formulas of Cayley and Salmon were revised in a long memoir of H. Zeuthen [430]. The fact that the class of a ruled surface is equal to its degree is due to Cayley. The degree of a ruled surface defined by three directrices from Examples 10.4.5 was first determined by G. Salmon [350]. Cubic ruled surfaces were classified by A. Cayley in [58], Part II, and, independently, by L. Cremona [102]. The classification of quartic ruled surfaces were started by A. Cayley [58], Parts II and III. However he had missed two types. A complete classification was given later by L. Cremona [107]. An earlier attempt for this classification was made by M. Chasles [68]. The classification based on the theory of tetrahedral complexes was given by B. Wong [427]. Ruled surfaces of degree 5 were classified by H. Schwarz [373]. Much later this classification was extended to surfaces of degree 6 by W. Edge [150]. Edge's book and Sturm's book [400], vol. 1, give a detailed exposition of the theory of ruled surfaces. The third volume of Sturm's book contains an extensive account of the theory of quadratic line complexes.

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