ON INFINITESIMAL INVARIANTS OF NORMAL FUNCTIONS

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ABSTRACT. We present Griffiths' infinitesimal invariant of a normal function and some of its geometric applications.

INTRODUCTION

Griffiths [Gri83] introduced an infinitesimal invariant of normal functions which he used to obtain interesting geometric results. This invariant was refined by Green [Gre89] and used by many authors to obtain many more interesting and deep results on algebraic cycles.

1. HODGE STRUCTURES

All schemes are over the field of complex numbers. A variety is a reduced separated scheme of finite type over $\mathbb{C}$. In what follows, X will denote a smooth projective variety of dimension n. We begin with some definitions and explaining our setup.

Definition 1.1. *A rational (respectively, integral) Hodge structure of weight $k \in \mathbb{Z}$ is the datum of a $\mathbb{Q}$-vector space (respectively, free $\mathbb{Z}$-module) V such that the complexification*

$$V_{\mathbb{C}} := V \otimes \mathbb{C}$$

admits a direct sum decomposition

$$V_{\mathbb{C}} = \oplus_{p+q=k} V^{p,q} \quad \textit{with} \quad \overline{V^{p,q}} = V^{q,p}\ (p, q \in \mathbb{Z}),$$

where the bar indicates complex conjugation defined by $\overline{v \otimes z} = v \otimes \overline{z}$ for $v \in V, z \in \mathbb{C}$. A Hodge structure (of positive weight) is called effective if $V^{p,q} = 0$ when either p or q is negative.

Definition 1.2. *A polarization on the rational (respectively, integral) Hodge structure V is the datum of a bilinear map*

$$\psi : V \times V \longrightarrow \mathbb{Q} \quad (\textit{respectively, } V \times V \longrightarrow \mathbb{Z}),$$

symmetric for even k, skew-symmetric for odd k, whose complexification (i.e., linear extension to $V_{\mathbb{C}}$) satisfies

$$\psi(H^{p,q}, H^{r,s}) = 0 \quad \textit{unless} \quad p = s, q = r \tag{1.1}$$

and

$$i^{p-q}\psi(v, \overline{v}) > 0 \quad \textit{for} \quad v \in H^{p,q}, v \neq 0. \tag{1.2}$$

In particular, ψ is always nondegenerate.
Equations (1.1) *and* (1.2) *are referred to, respectively, as the first and the second Hodge-Riemann bilinear relations.*

Given X a smooth complex projective variety, the rational cohomology $V := H^k(X, \mathbb{Q})$ (respectively, integral cohomology modulo torsion) carries a natural Hodge structure:

$$V \otimes \mathbb{C} = H^k(X, \mathbb{Q}) \otimes \mathbb{C} \cong H^k(X, \mathbb{C}) \cong \oplus H^{p,q}(X) = \oplus V^{p,q}$$

where $V^{p,q} = H^{p,q}(X)$ is, as usual, the space of cohomology classes of differential forms of type (p, q). To obtain polarizations on the cohomology of algebraic varieties, we need to restrict ourselves to the primitive cohomology.

Given an ample line bundle L on X, let $\eta := c_1(L) \in H^2(X, \mathbb{Z})$ be its topological first Chern class. Left multiplication by η defines a linear map

$$\Lambda : H^k(X, \mathbb{Z}) \longrightarrow H^{k+2}(V, \mathbb{Z})$$

called the Lefschetz operator. For $0 \leq k \leq n := \dim_{\mathbb{C}} X$, the $(n-k)$-th power of Λ is injective. The primitive part $P^k(X, \mathbb{Z})$ of $H^k(X, \mathbb{Z})$ (respectively $P^k(X, \mathbb{C})$ of $H^k(X, \mathbb{C})$) is defined to be the kernel of Λ^{n-k+1}. If $k > n$, the primitive part is defined to be 0. The primitive cohomology carries a structure of polarized rational (or integral) Hodge structure. The spaces $P^{p,q}$ are defined to be the intersections

$$P^{p,q} := P^k \cap H^{p,q}$$

and the polarization is the Hodge bilinear form defined by

$$\psi(v,w) = (-1)^{\frac{k(k-1)}{2}}(\Lambda^{n-k} v \wedge w)[X] = (-1)^{\frac{k(k-1)}{2}} \int_X \eta^{n-k} \wedge v \wedge w,$$

where $[X]$ is the fundamental class of X. The Hodge-Riemann bilinear relations assert that this is indeed a polarization in the sense that we defined earlier (see, e.g., [Voi03, Vol. I, Chapter 6]).

The Hard Lefschetz Theorem is the assertion that one has the decomposition

$$H^k(X,\mathbb{Q}) = \oplus \Lambda^j P^{k-2j}(X,\mathbb{Q}), \quad j \geq max\{k-n, 0\}.$$

So the primitive cohomology determines the cohomology of an algebraic variety.

To construct a parameter space for Hodge structures, one cannot use the above definition since the $H^{p,q}$ do not vary holomorphically on a family of algebraic varieties. Fortunately, there is an equivalent definition of a Hodge structure that allows one to construct a complex analytic parameter space, called a period domain.

Definition 1.3. *A rational (respectively, integral) Hodge structure of weight k is the data of a $\mathbb{Q}$ vector space V (respectively, free $\mathbb{Z}$-module) together with a decreasing filtration F^p on its complexification $V_{\mathbb{C}}$ such that*

$$\forall p \quad V_{\mathbb{C}} = F^p \oplus \overline{F^{k-p+1}}.$$

The relation between the two definitions of Hodge structure is

$$F^p := \oplus_{i \geq p} V^{i,k-i}$$

and

$$V^{p,q} = F^p \cap \overline{F^q}.$$

In terms of the Hodge filtration, the Hodge–Riemann bilinear relations become

$$\psi(F^p, F^{k-p+1}) = 0 \quad \text{for all } p$$

and

$$\psi(Cv, \overline{v}) > 0 \text{ for } v \in V_{\mathbb{C}}, v \neq 0$$

where C is the Weil operator defined by $Cv = i^{p-q}v$ if $v \in V^{p,q}$.

Griffiths proved that the Hodge filtration varies holomorphically in families (algebraically for a family of algebraic varieties, see [Gri68b]).

Yet a third definition of a Hodge structure was given by Deligne [Del72] using representations. This point of view allowed the introduction of Mumford–Tate groups which have been very useful in the study of Hodge structures and period domains, see, e.g., [GGK12].

Denote by $\mathbb{S}$ the group scheme whose set of points $\mathbb{S}(A)$ on an algebra A is the subgroup of $\mathrm{GL}_2(A)$ of matrices

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

The group $\mathbb{S}$ naturally contains $\mathbb{G}_m$ as the group of diagonal matrices and maps to $\mathbb{G}_m$ via the norm map

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \longmapsto a^2 + b^2.$$

The group $\mathbb{S}(\mathbb{R})$ is naturally isomorphic to $\mathbb{C}^*$ via the map

$$a + ib \longmapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

Definition 1.4. *A rational (respectively, integral) Hodge structure of weight k is the data of a $\mathbb{Q}$ vector space (respectively, free $\mathbb{Z}$-module) V together with a representation*

$$\mathbb{S}(\mathbb{R}) \longrightarrow \mathrm{GL}(V \otimes \mathbb{R}).$$

Under the identification $\mathbb{S}(\mathbb{R}) = \mathbb{C}^*$, the Hodge component $V^{p,q}$ is the summand of $V \otimes \mathbb{C}$ where z acts as multiplication by $z^p\overline{z}^q$.

2. Period domains

For the results of this section, we refer to [Gri68a], [Gri68b], [GS69] and [Sch73]. Fix a finitely generated free abelian group V with complexification $V_{\mathbb{C}}$. Also fix an integer k and a collection of non-negative integers $\{h^{p,q}\}$ which satisfy $h^{p,q} = h^{q,p}$, $\sum h^{p,q} = d := \mathrm{rank}_{\mathbb{Z}} V$ and $h^{p,q} \neq 0$ only when $p + q = k$.

Let $\check{\mathcal{F}}$ be the flag variety of filtrations $F^\bullet$ of $V_{\mathbb{C}}$ of dimensions $f_p := \dim F^p = \sum_{i \geq p} h^{i,k-i}$. Then $\check{\mathcal{F}}$ can be realized as a closed subvariety of a product of Grassmannians which shows that $\check{\mathcal{F}}$ is a projective algebraic variety. Furthermore $GL(V_{\mathbb{C}})$ acts algebraically and transitively on $\check{\mathcal{F}}$ which shows, in particular, that $\check{\mathcal{F}}$ is smooth. The filtrations that satisfy

$$V_{\mathbb{C}} = F^p \oplus \overline{F^{k-p+1}}$$

form an open subset $\mathcal{F}$ of $\check{\mathcal{F}}$ in the complex analytic topology. Therefore $\mathcal{F}$ is a complex manifold that parametrizes Hodge structures of weight k on V with Hodge numbers $h^{p,q}$.

Now let ψ be a nondegenerate bilinear form on V (or $V_{\mathbb{Q}}$, if one wishes to work over the rationals), symmetric if k is even, skew-symmetric if k is odd. Let $\check{\mathcal{D}} \subset \check{\mathcal{F}}$ be the closed algebraic subvariety of those filtrations that satisfy the first Hodge-Riemann bilinear relation (1.1):

$$\psi(F^p, F^{k-p+1}) = 0 \quad \text{for all } p.$$

The orthogonal group of the bilinear form ψ acts transitively on $\check{\mathcal{D}}$, hence $\check{\mathcal{D}}$ is a smooth projective algebraic variety. In fact, if we denote $G_{\mathbb{R}}$ the group of automorphisms of $V_{\mathbb{R}}$ that preserve ψ, then $\check{\mathcal{D}} = G_{\mathbb{C}}/P$ for a parabolic subgroup P of $G_{\mathbb{C}}$ ($G_{\mathbb{C}}$ is the base change of $G_{\mathbb{R}}$ to $\mathbb{C}$). The filtrations in $\check{\mathcal{D}}$ that satisfy the second Riemann bilinear relation (1.2)

$$\psi(Cv, \overline{v}) > 0 \text{ for } v \in V_{\mathbb{C}}, v \neq 0$$

form an open subset $\mathcal{D}$ of $\check{\mathcal{D}}$ in the complex analytic topology, hence a complex manifold. One can show that the group $K := P \cap G_{\mathbb{R}}$ is compact and, under the action of $G_{\mathbb{R}}$, D is identifed with the homogeneous complex manifold $G_{\mathbb{R}}/K$.

Furthermore, one can show that the subgroup $G_{\mathbb{Z}}$ of $G_{\mathbb{R}}$ of elements that leave the lattice V globally invariant acts properly discontinuously on $\mathcal{D}$. Hence the quotient $G_{\mathbb{Z}}\backslash\mathcal{D}$ is a complex analytic variety. This quotient is called the period domain. The analogous statement about classifying spaces for weighted hodge structures without polarization fails. This is one of the reasons why one considers polarized Hodge structures.

Given a smooth projective family of algebraic varieties $f : \mathcal{X} \to S$, let $s_0 \in S$ be a base point and $X := X_{s_0}$ the fiber of f at s_0. Then, choosing $V := P^k(X, \mathbb{Z})$, the assignment $s \mapsto (P^k(X_s, \mathbb{Z}), \psi_s)$ defines a holomorphic map $S \mapsto D/\Gamma$. This map is locally liftable to D and its differential maps the space T_sS into a subspace of the tangent space of D called the horizontal tangent space [CG75], [Gri69] which we define below.

3. The horizontal tangent bundle

Under the embedding

$$\check{\mathcal{D}} \subset \check{\mathcal{F}} \subset \prod \mathrm{Grass}_p(f_p, d),$$

we have an embedding of the tangent space to $\check{D}$ at a point $F = (F^p)$ in the tangent space to the product of Grassmannians:

$$T_F\check{D} \subset \oplus_p \mathrm{Hom}(F^p, V_{\mathbb{C}}/F^p).$$

One easily checks that $T_F\check{D}$ is the set of $\xi = \oplus_p \xi_p \in \oplus_p \mathrm{Hom}(F^p, V_{\mathbb{C}}/F^p)$ satisfying the following conditions.

(1) The diagram

$$\begin{array}{ccc} F^p & \xrightarrow{\xi_p} & V_{\mathbb{C}}/F^p \\ \downarrow & & \downarrow \\ F^{p-1} & \xrightarrow{\xi_{p-1}} & V_{\mathbb{C}}/F^{p-1} \end{array}$$

is commutative

(2) and

$$\psi(\xi_p v, w) + \psi(v, \xi_{n-p+1} w) = 0, \text{ for all } v \in F^p, w \in F^{n-p+1}.$$

Definition 3.1. *The horizontal tangent space $T_{h,F}$ to $\check{D}$ is the set of vectors $\xi = (\xi_p) \in T_F\check{D}$ satisfying the infinitesimal period relations or Griffiths transversality*

$$\xi(F^p) \subset F^{p-1}$$

for all p. These tangent spaces form the horizontal tangent bundle $T_{h,\check{D}} \subset T_{\check{D}}$. The horizontal tangent bundle of D is the restriction $T_{h,D} := T_{h,\check{D}}|_D$.

It is clear that the horizontal tangent bundle is a holomorphic subbundle of $T_{\check{D}}$, invariant under the action of $G_{\mathbb{C}}$.

Note that an element $\xi = (\xi_p) \in T_{h,F}\check{D}$ induces maps

$$V^{p,q} = F^p/F^{p+1} \longrightarrow V^{p-1,q+1} = F^{p-1}/F^p$$

for all p, q. We therefore have a natural embedding

$$T_{h,F}\check{D} \subset \oplus_{1\leq p\leq k} \operatorname{Hom}(V^{p,q}, V^{p-1,q+1}).$$

4. Variation of Hodge structure

When properly defined, the datum of a morphism to a period domain is equivalent to that of a variation of Hodge structure. Before giving the definition of a variation of Hodge structure, we recall that of Gauss–Manin connections.

Given a complex manifold S with a local system $\mathbb{V}$ of finitely generated abelian groups or rational or real vector spaces, the holomorphic vector bundle $\mathcal{V} := \mathbb{V} \otimes \mathcal{O}_S$ carries a natural flat connection whose local system of flat sections is naturally identified with $\mathbb{V}$. This is the Gauss–Manin connection

$$\nabla : \mathcal{V} \longrightarrow \mathcal{V} \otimes \Omega^1_S$$

which can be locally defined as follows.

Choose a local basis $\{\sigma_1, \ldots, \sigma_d\}$ of $\mathbb{V}$. Then the Gauss–Manin connection sends a local section $\sigma = \sum \alpha_i\sigma_i$ of $\mathcal{V}$ to $\nabla(\sigma) = \sum \sigma_i \otimes d\alpha_i$. One checks that the definition is independent of the choice of local basis, hence glues to define a global connection and that its curvature form is 0 (see, e.g., [Voi03, Vol. I, Chapter 9]).

Definition 4.1. *A variation of Hodge structure* $(S, \mathbb{V}, \mathcal{F}^\bullet, \nabla)$ *on a complex manifold* S *is the data of a local system* $\mathbb{V}$ *of free abelian groups (or rational or real vector spaces) with a decreasing filtration*

$$\{0\} \subset \mathcal{F}^k \subset \ldots \subset \mathcal{F}^0 = \mathcal{V}$$

of the holomorphic bundle $\mathcal{V} := \mathbb{V} \otimes \mathcal{O}_S$ *such that if* $\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega^1_S$ *denotes the Gauss-Manin connection, then we have Griffiths transversality (or the infinitesimal period relation)*

$$\nabla(\mathcal{F}^p) \subset \mathcal{F}^{p-1} \otimes \Omega^1_S$$

and we have C^∞ *splittings*

$$\mathcal{V} = \mathcal{F}^p \oplus \overline{\mathcal{F}^{k-p+1}}$$

for all p*. The Hodge bundles are then defined as*

$$\mathcal{V}^{p,q} := \mathcal{F}^p / \mathcal{F}^{p+1}.$$

They satisfy

$$\mathcal{V}^{p,q} = \overline{\mathcal{V}^{p,q}}$$

and we have a C^∞ *splitting*

$$\mathcal{V} = \oplus_{p+q=k} \mathcal{V}^{p,q}.$$

A polarization of an integral or rational variation of Hodge structure is the datum of a flat *nondegenerate bilinear form*

$$\psi : \mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{Z}_S (resp.\ \mathbb{Q}_S)$$

which satisfies the first and second Hodge-Riemann bilinear relations (1.1) *and* (1.2)*.*

Note that the flatness of the bilinear form means that it is compatible with the Gauss-Manin connection, i.e., for all sections s, s' of $\mathcal{V}$,

$$d\psi(s, s') = \psi(\nabla s, s') + \psi(s, \nabla s').$$

A polarized variation of Hodge structure defines a monodromy representation

$$\rho : \pi_1(S, o) \longrightarrow G_{\mathbb{Z}}$$

where $o \in S$ is a base point. Let Γ be the image of the monodromy representation ρ. Sending a point of S to the Hodge structure defined by the polarized variation of Hodge structure defines a period map

$$S \longrightarrow \Gamma\backslash\mathcal{D}$$

which is holomorphic, locally liftable and whose local lifts send the tangent bundle T_S into the the horizontal tangent bundle $T_{h,\mathcal{D}}$. In fact the data of a polarized variation of Hodge structure with monodromy contained in a subgroup Γ of $G_{\mathbb{Z}}$ is equivalent to the data of a holomorphic map $S \longrightarrow \Gamma\backslash\mathcal{D}$ with the above properties (see [CG75], [Gri68b]).

Definition 4.2. *An extended variation of Hodge structure is the data of a holomorphic map*

$$f : S \longrightarrow \Gamma\backslash\mathcal{D}$$

such that the restriction of f to a dense Zariski open subset of S is a variation of Hodge structure.

Note that the difference between a variation of Hodge structure and an extended variation of Hodge structure is that for an extended variation there exists a proper analytic subset Z of S such that f fails to be locally liftable at the points of Z. If S is a Zariski open subset of a smooth projective variety $\overline{S}$, given a polarized variation of Hodge structure $f : S \to \Gamma\backslash\mathcal{D}$, there is a maximal Zariski open set $S' \subset \overline{S}$ such that f extends to an extended variation of Hodge structure $f' : S' \to \Gamma\backslash\mathcal{D}$ where f' is *proper* (see [CGGH83]).

Let $\phi : \mathcal{M} \to S$ be a holomorphic family of compact Kähler manifolds of complex dimension n. For any integer $k \in \{0, \dots, 2n\}$, we have the variation of Hodge structures $(S, \mathbb{H}^k := R^k\phi_*\mathbb{Z}_{\mathcal{M}}, \mathcal{F}^\bullet, \nabla)$, where $\mathbb{Z}_{\mathcal{M}}$ is the constant sheaf with group $\mathbb{Z}$ on $\mathcal{M}$, $\mathcal{F}^\bullet$ is the Hodge filtration on $\mathcal{H}^k := R^k\phi_*\mathbb{Z}_{\mathcal{M}} \otimes_{\mathbb{Z}} \mathcal{O}_S$ which restricts to the Hodge filtration on $H^k(M_s, \mathbb{C})$ for all s, and $\nabla : \mathcal{H}^k \to \mathcal{H}^k \otimes \Omega^1_S$ is the Gauss–Manin connection obtained by differentiating sections as above.

For a family of projective algebraic varieties $\mathcal{X} \to S$ with a relatively ample line bundle, the primitive cohomology groups of the fibers form local systems $\mathbb{V}^k$ with polarizations

$$\Psi : \mathbb{V}^k \times \mathbb{V}^k \longrightarrow \mathbb{Z}_S$$

that give polarized variations of Hodge structures $(S, \mathbb{V}^k, \mathcal{F}^\bullet, \nabla, \Psi)$.

5. Griffiths intermediate jacobians

Given an integral Hodge structure of odd weight $k = 2l - 1$, the direct sum decomposition

$$V_{\mathbb{C}} = F^l V \oplus \overline{F^l V}$$

implies that

$$F^l V \cap V_{\mathbb{R}} = 0.$$

Hence the projection

$$V_{\mathbb{R}} \longrightarrow V_{\mathbb{C}}/F^l V$$

is an isomorphism of real vector spaces and the lattice $V \subset V_{\mathbb{R}}$ maps isomorphically to a lattice in $V_{\mathbb{C}}/F^l V$.

Definition 5.1. *The intermediate jacobian $J(V)$ is the complex torus*

$$J(V) := \frac{V_{\mathbb{C}}}{F^l V \oplus V}.$$

The algebraic part $J(V)_a$ of the intermediate jacobian is the largest complex subtorus whose tangent space is contained in $V^{l-1,l}$. It is the image, in $J(V)$, of the largest Hodge substructure of V contained in $V^{l-1,l} \oplus V^{l,l-1}$.

Given a compact Kähler manifold M of dimension m, Poincaré duality gives a diagram

$$\begin{array}{ccc} H^k(M,\mathbb{Z})/_{torsion} & \xrightarrow{=} & H_{2m-k}(M,\mathbb{Z})/_{torsion} \\ \downarrow & & \downarrow \\ H^k(M,\mathbb{C}) & \xrightarrow{=} & H^{2m-k}(M,\mathbb{C})^* \end{array}$$

where the second vertical map is induced by integration of differential forms on topological cycles. This gives a natural isomorphism

$$J(V) \cong \frac{F^{m-l+1}H^{2m-k}(M,\mathbb{C})^*}{H_{2m-k}(M,\mathbb{Z})/_{torsion}}.$$

Given an algebraic cycle Z of codimension l, homologous to 0, let W be a topological chain with boundary Z. One can show (see, e.g., [Voi03, Vol. I, Chapter 12]) that integration on W gives a well-defined element

$$AJ(Z) \in J(V)$$

called the Abel-Jacobi image of Z. Griffiths proved [Gri69, Section 13] that if Z is algebraically equivalent to 0, then $AJ(Z) \in J(V)_a$. The group of homologically trivial cycles on M modulo algebraic equivalence is called the Griffiths group of M. Griffiths also proved [Gri69, Section 14] that the Griffiths groups of generic hyperplane sections of smooth hypersurfaces contain infinite cyclic groups, provided that the original hypersurface carries an algebraic cycle whose homology class is primitive and non-zero. Ceresa and Collino [CC83] proved that, for a generic quintic threefold M, there are no algebraic equivalence relations between the lines in M. In particular, the Abel-Jacobi images of the differences of two distinct lines are non-torsion (by work of Katz [Kat86], M contains finitely many lines). Griffiths also proved that $J(V)_a = 0$ [Gri69, Section 13]. Clemens [Cle83], also see [Voi92], proved that the image of the Abel-Jacobi map of a generic quintic threefold is countably generated, and, when tensored with $\mathbb{Q}$, it is not finitely generated. This implies, in particular, that the Griffiths group tensored with $\mathbb{Q}$ is not finitely generated. Nori [Nor93] defined a natural filtration on the Griffiths group of a smooth projective variety and showed that every graded piece of the filtration can be nontorsion. Using Nori's work [Nor93] and Voisin's proof [Voi92] of Clemens' theorem [Cle83], Albano and Collino [AC94] proved that the Griffiths group of a general cubic sevenfold M is not finitely generated after tensoring with $\mathbb{Q}$. They also prove the analogous result for the Griffiths group of the complete intersection Y of M with two generic hypersurfaces of sufficiently high degree. This last result is particularly interesting since, unlike the previous examples, the Griffiths intermediate jacobian of Y is trivial. The nontriviality of the Griffiths intermediate jacobian was previously used in the first proofs of the fact that the Griffiths group of a generic quintic threefold is nontorsion and non finitely generated. Later, Fakhruddin [Fak96], using Nori's connectivity theorem [Nor93], proved that the Griffiths groups of codimension 3 and 4 on generic abelian fivefolds are of infinite rank although they are in the kernel of the Abel-Jacobi map.

Given a complex manifold S, a cycle $\mathcal{Z} \subset M \times S$ of codimension l, flat over S and a point $s_0 \in S$, the map

$$\begin{array}{rcl} S & \longrightarrow & J(V) \\ s & \longmapsto & AJ(Z_s - Z_{s_0}), \end{array}$$

where Z_s is the fiber of $\mathcal{Z}$ at s, is holomorphic [Gri68b].

6. Normal functions

Given an integral variation of Hodge structure $(S, \mathbb{V}, \mathcal{F}^\bullet, \nabla)$ of odd weight $k = 2l - 1$ on the complex manifold S, one defines the associated family of intermediate jacobians

$$\mathcal{J}_S := \frac{\mathcal{V}}{\mathcal{F}^l \oplus \mathbb{V}}.$$

Since the Gauss-Manin connection

$$\nabla : \mathcal{V} \longrightarrow \mathcal{V} \otimes \Omega^1_S$$

satisfies

$$\nabla(\mathbb{V}) = 0, \quad \nabla(\mathcal{F}^m) \subset \mathcal{F}^{m-1} \otimes \Omega^1_S,$$

it induces a map

$$\overline{\nabla} : \mathcal{J}_S \longrightarrow \frac{\mathcal{V}}{\mathcal{F}^{m-1}} \otimes \Omega^1_S.$$

Definition 6.1. *The sheaf of normal functions is*

$$\mathcal{J}_{h,S} := \operatorname{Ker} \overline{\nabla}$$

and a normal function is a global section of $\mathcal{J}_{h,s}$.

When S is a quasi-projective variety, a normal function must satisfy additional conditions which govern its behavior at infinity, saying, roughly speaking, that it has at most logarithmic growth (see [EZZ84]).

7. Infinitesimal variation of Hodge structure

An infinitesimal variation of Hodge structure is an abstractification of the differential of a variation of Hodge structure. Unlike infinitesimal deformations of algebraic varieties, it contains a little bit more than just first order information.

Definition 7.1. *An infinitesimal variation of Hodge structure is the data of a polarized Hodge structure* $(V_{\mathbb{Z}}, V^{p,q}, \psi)$ *together with a vector space* T *and a linear map*

$$\delta : T \longrightarrow \oplus_{1\leq p\leq k} \operatorname{Hom}(V^{p,q}, V^{p-1,p+1})$$

satisfying the conditions

$$\delta(\xi_1) \circ \delta(\xi_2) = \delta(\xi_2) \circ \delta(\xi_1), \tag{7.1}$$

$$\psi(\delta(\xi)v, w) + \psi(v, \delta(\xi)w) = 0. \tag{7.2}$$

Condition (7.1) (which is of second order) expresses the flatness of the Gauss-Manin connection, while condition (7.2) expresses its compatibility with the polarization. In other words, the image of δ is the tangent space to the image, via the period map, of a variation of Hodge structure.

Given a variation of Hodge structure $(S, \mathbb{V}, \mathcal{F}^\bullet, \nabla)$ and a point $s \in S$, there is an associated infinitesimal variation of Hodge structure with $V_{\mathbb{Z}} = \mathbb{V}_s$, $T = T_sS$ and the map δ given by the differential of the period map of $(S, \mathbb{V}, \mathcal{F}^\bullet, \nabla)$ at s. A first interesting application of the theory of infinitesimal variations of Hodge structures was to curves: Griffiths observed [Gri83, Section 5] that if a curve C is neither hyperelliptic, nor trigonal, nor a smooth plane quintic, then it is determined by its universal infinitesimal variation of Hodge structure. Note that general curves of genus ≥ 5 satisfy the above hypothesis. For curves of genus 3, the period map is surjective hence the universal infinitesimal variation of Hodge structure only carries the information of the Hodge structure of the curve and cannot determine it. We shall see below that in the cases of genus 3 and 4 one can use infinitesimal invariants of normal functions to determine the curve.

8. The Griffiths infinitesimal invariant of a normal function

Let $(S, \mathbb{V}, \mathcal{F}^\bullet, \nabla)$ be an integral variation of Hodge structure of odd weight $k = 2l-1$ on the complex manifold S, and let ν be a normal function on its associated family of intermediate jacobians. Griffiths defined an infinitesimal invariant associated to ν whose vanishing is a necessary condition for ν to be locally constant. Here we first define this invariant and its refinement by Green [Gre89], then we discuss some of its applications.

By Griffiths transversality and flatness, for all p, r, the connection ∇ induces maps

$$\overline{\nabla} : \mathcal{V}^{l-p,l+p-1} \otimes \Omega_S^r \longrightarrow \mathcal{V}^{l-p-1,l+p} \otimes \Omega_S^{r+1}$$

such that the sequence

$$0 \longrightarrow \mathcal{V}^{l-p,l+p-1} \longrightarrow \mathcal{V}^{l-p-1,l+p} \otimes \Omega_S^1 \longrightarrow \mathcal{V}^{l-p-2,l+p+1} \otimes \Omega_S^2 \longrightarrow \dots$$

is a complex. For any local lifting $\widetilde{\nu} \in \mathcal{V}$ of ν, we have

$$\nabla(\widetilde{\nu}) \in \mathcal{F}^{l-1} \otimes \Omega_S^1.$$

One can show (see, e.g., [Voi03, Vol. II, Chapter 7]) that the projection of $\nabla(\widetilde{\nu})$ to $\mathcal{V}^{l-1,l} \otimes \Omega_S^1$ lies in the kernel of the differential $\overline{\nabla}$ and its class modulo the image $\mathrm{Im}(\overline{\nabla} : \mathcal{V}^{l,l-1} \to \mathcal{V}^{l-1,l} \otimes \Omega_S^1)$ depends only on ν and not on the choice of the lifting $\widetilde{\nu}$. We have the following

Definition 8.1. *The infinitesimal invariant $\delta_1(\nu)$ is the class of $\overline{\nabla}(\widetilde{\nu})$ in the cohomology group*

$$H^1(\mathcal{V}^{\cdot\cdot} \otimes \Omega_S^{\cdot}) = \frac{\mathrm{Ker}(\overline{\nabla} : \mathcal{V}^{l-1,l} \otimes \Omega_S^1 \to \mathcal{V}^{l-2,l+1} \otimes \Omega_S^2)}{\mathrm{Im}(\overline{\nabla} : \mathcal{V}^{l,l-1} \to \mathcal{V}^{l-1,l} \otimes \Omega_S^1)}.$$

For $s \in S$, the infinitesimal invariant $\delta_1(\nu)_s$ of ν at s is the class of $\overline{\nabla}(\widetilde{\nu})_s$ in

$$H^1(\mathcal{V}_s^{\cdot\cdot} \otimes \Omega_{S,s}^{\cdot}) = \frac{\mathrm{Ker}(\overline{\nabla}_s : \mathcal{V}_s^{l-1,l} \otimes \Omega_{S,s}^1 \to \mathcal{V}_s^{l-2,l+1} \otimes \Omega_{S,s}^2)}{\mathrm{Im}(\overline{\nabla}_s : \mathcal{V}_s^{l,l-1} \to \mathcal{V}_s^{l-1,l} \otimes \Omega_{S,s}^1)}.$$

Similarly, one has the complexes

$$0 \longrightarrow \mathcal{F}^l \longrightarrow \mathcal{F}^{l-1} \otimes \Omega_S^1 \longrightarrow \mathcal{F}^{l-2} \otimes \Omega_S^2 \longrightarrow \dots. \tag{8.1}$$

Mark Green [Gre89] (and also Robert Bryant, see [Gri83, Appendix to Section 6(a)]) observed that one could define a slightly more general invariant whose vanishing is equivalent to ν being constant.

Definition 8.2. *The infinitesimal invariant* $\delta(\nu)$ *is the class of* $\overline{\nabla}(\widetilde{\nu})$ *in the cohomology group*

$$H^1(\mathcal{F}^\cdot \otimes \Omega^\cdot_S) = \frac{\mathrm{Ker}(\overline{\nabla} : \mathcal{F}^{l-1} \otimes \Omega^1_S \to \mathcal{F}^{l-2} \otimes \Omega^2_S)}{\mathrm{Im}(\overline{\nabla} : \mathcal{F}^{l} \to \mathcal{F}^{l-1} \otimes \Omega^1_S)}.$$

For $s \in S$*, the infinitesimal invariant* $\delta(\nu)_s$ *of* ν *at* s *is the class of* $\overline{\nabla}(\widetilde{\nu})_s$ *in*

$$H^1(\mathcal{F}^\cdot_s \otimes \Omega^\cdot_{S,s}) = \frac{\mathrm{Ker}(\overline{\nabla}_s : \mathcal{F}^{l-1}_s \otimes \Omega^1_{S,s} \to \mathcal{F}^{l-2}_s \otimes \Omega^2_{S,s})}{\mathrm{Im}(\overline{\nabla}_s : \mathcal{F}^{l}_s \to \mathcal{F}^{l-1}_s \otimes \Omega^1_{S,s})}.$$

The Hodge filtration also induces filtrations on the complexes (8.1) as well as on their cohomology sheaves. The image of $\delta(\nu)$ in the first graded piece of this filtration is the invariant $\delta_1(\nu)$. Green also observed that if $\delta_1(\nu)$ is zero, then the image $\delta_2(\nu)$ of $\delta(\nu)$ in the second graded piece of the filtration is well-defined. One could continue this to define infinitesimal invariants $\delta_1(\nu), \ldots, \delta_{l+1}(\nu)$ such that $\delta_p(\nu)$ is well-defined when $\delta_1(\nu) = \ldots = \delta_{p-1}(\nu) = 0$. Then the vanishing of $\delta(\nu)$ is equivalent to the vanishing of $\delta_i(\nu)$ for $i = 1, \ldots, l+1$.

The non-vanishing of any of the infinitesimal invariants implies that the normal function and all of its multiples are not locally constant. In particular, the normal function is not torsion.

Infinitesimal invariants of normal functions have had many interesting applications to the study of cycles on algebraic varieties.

A first application by Griffiths was to show that a general curve C of genus 4 can be reconstructed from the infinitesimal invariant of the normal function with value the difference of the two g^1_3s on C. More precisely, the infinitesimal invariant can be identified with a cubic polynomial whose trace on the quadric determined by the universal infinitesimal deformation of C is the canonical image of C.

Mark Green [Gre89] used the infinitesimal invariants to prove that, for a general threefold of degree ≥ 6 in $\mathbb{P}^3$, the image of the Abel-Jacobi map from algebraic one-cycles homologically equivalent to zero is contained in the locus of torsion points of the intermediate jacobian.

Claire Voisin [Voi92] used infinitesimal invariants to give a second proof of Clemens' theorem that the Griffiths group of a generic quintic threefold is not finitely generated over $\mathbb{Q}$.

Given a smooth curve C, one has the cycle $C - C^-$ in the Jacobian of C defined up to translation. Ceresa [Cer83] proved that this cycle is not algebraically equivalent to 0 for a general curve. Using the infinitesimal invariant of the normal function of $C - C^-$, Collino and Pirola [CP95] gave a second proof of Ceresa's result. They also showed that, in genus 3, this infinitesimal invariant could be identified with the equation of the canonical curve when C is not hyperelliptic. They further showed that Ceresa's theorem also holds for general plane curves of degree ≥ 4 and also for general curves in subvarieties of codimension $< \frac{g+2}{3}$ of $\mathcal{M}_g$if $g > 3$.

Smooth cubic threefolds provided one of the first examples of non-rational unirational threefolds [CG72]. The intermediate jacobian $J(X)$ of a cubic threefold X is isomorphic to the Albanese variety of its Fano variety of lines F via the Abel-Jacobi map. Via this isomorphism, the Fano variety F admits embeddings into $J(X)$ and has, similarly to the curve case, a cycle $F - F^-$ well-defined up to translation. Collino, Naranjo and Pirola [CNP12] computed the infinitesimal invariant of the normal function of this cycle. They used it to show that, for X general, the cycle $F - F^-$ is not algebraically equivalent to 0 and that there is no divisor in $J(X)$ containing both F and F^- where F and F^- are homologically equivalent. They further showed that the infinitesimal invariant determines the curve of double lines in F which in turn determines the cubic threefold X.

9. Griffiths' formula for the infinitesimal invariant

In their computation of the infinitesimal invariant, Collino and Pirola [CP95] used a formula of Griffiths [Gri83, Section 6(e)] for the first infnitesimal invariant $\delta_1(\nu)$ of the normal function ν of a family of algebraic cycles on a family of smooth projective varieties. In this section we present Griffiths' formula.

Let $\mathcal{X} \to S$ be a family of smooth projective varieties of dimension $n = 2m+1$, choose a point $s_0 \in S$ and put $X := X_{s_0}$. Via a C^∞ trivialization $\mathcal{X} \cong X \times S$, the varying complex structures on the fibers of $\mathcal{X} \to S$ are described by a family of operators $\overline{\partial}_s : A^0(X) \to A^1(X)$ where $A^p(X)$ is the space of global C^∞ p-forms on X. Then

$$\theta_s := \overline{\partial}_s - \overline{\partial},$$

where $\overline{\partial} := \overline{\partial}_{s_0}$, is a holomorphic family of vector valued $(0,1)$-forms on X which can be locally written as

$$\theta_s(z) = \sum_{i,j} \theta^i_j(s,z) \frac{\partial}{\partial z_i} \otimes d\overline{z}_j$$

in terms of local holomorphic coordinates on X. Using the integrability condition

$$\overline{\partial}\theta_s - \frac{1}{2}[\theta_s, \theta_s] = 0,$$

one verifies that, for any $\xi \in T_{s_0}S$, the derivative $\frac{\partial \theta_s}{\partial \xi}$ of θ_s in the direction of ξ represents the image of ξ under the Kodaira–Spencer map $T_{s_0}S \to H^1(X, T_X)$.

We have the variation of Hodge structure $(S, \mathbb{H}^{2m+1} := R^{2m+1}\phi_*\mathbb{Z}_{\mathcal{M}}, \mathcal{F}^\bullet, \nabla)$ and its associated family of intermediate jacobians

$$\mathcal{J}^m_S := \frac{\mathcal{H}^{2m+1}}{\mathcal{F}^{m+1} \oplus \mathbb{H}^{2m+1}}.$$

Consider now a holomorphic family $Z_s \subset X_s$ of homologically trivial algebraic cycles of codimension $m+1$. We have the normal function

$$\begin{aligned} \nu : \ S &\longrightarrow \mathcal{J}^m_S \\ s &\longmapsto AJ_s(Z_s), \end{aligned}$$

where AJ_s is the Abel-Jacobi map for X_s (see Section 5).

For any $s \in S$, cup-product, followed by integration gives a non-degenerate pairing

$$\Psi_s : H^{m,m+1}(X_s) \otimes H^{m+1,m}(X_s) \longrightarrow \mathbb{C},$$

which is also the restriction of the polarization on $H^{2m+1}(X_s)$. Pairing $\delta_1(\nu)$ with a tangent vector $\xi \in T_{s_0}S$ gives $\nabla_\xi(\widetilde{\nu}) \in H^{m,m+1}(X)$ which is determined by the values $\Psi(\nabla_\xi(\widetilde{\nu}), \omega)$ for all $\omega \in H^{m+1,m}(X)$. Griffiths computed these values. Let η_Z be a normal vector field on Z which projects injectively to S and, for $\omega \in H^{m+1,m}(X)$, denote $i(\eta_Z)\omega$ the contraction of ω against η_Z. Let ζ be a differential form of type (m,m) on X such that $\overline{\partial}(\zeta) = \frac{\partial \theta_s}{\partial \xi} \cup \omega$.

Lemma 9.1. *(Griffiths* [Gri83, Section 6]*) With the above notation, we have*

$$\Psi(\nabla_\xi(\widetilde{\nu}), \omega) = \int_Z \zeta - \int_Z i(\eta_Z)\omega$$

10. Voisin's formula for the infinitesimal invariant

For their generalization of the results of Collino–Pirola [CP95], Pirola and Zucconi [PZ03] used a formula of Voisin [Voi88] for the computation of the first infinitesimal invariant. In this section we present Voisin's formula.

Let $\widetilde{X}$ be a smooth projective variety of dimension $n+1 = 2m+2$ and let $X \subset \widetilde{X}$ be a smooth divisor, moving in a linear system L. Let Z be an $(m+1)$-dimensional algebraic cycle on $\widetilde{X}$ whose restriction to X is homologically trivial. In other words, the cohomology class $[Z]$ is primitive, i.e.,

$$[Z] \in H^{m+1}(\widetilde{X}, \Omega^{m+1}_{\widetilde{X}})^{prim}$$

where the superscript *prim* indicates the kernel of the restriction map:

$$H^{m+1}(\widetilde{X}, \Omega^{m+1}_{\widetilde{X}})^{prim} := \mathrm{Ker}\left(H^{m+1}(\widetilde{X}, \Omega^{m+1}_{\widetilde{X}}) \xrightarrow{res} H^{m+1}(X, \Omega^{m+1}_X)\right).$$

Then, on the locus L_{reg} in L parametrizing smooth divisors, we have the family of divisors $\mathcal{X} \to L_{reg}$, the variation of Hodge structure $(L_{reg}, \mathbb{H}^{2m+1} := R^{2m+1}\phi_*\mathbb{Z}_{\mathcal{X}}, \mathcal{F}^\bullet, \nabla)$ and its associated family of intermediate jacobians

$$\mathcal{J}^m_{L_{reg}} := \frac{\mathcal{H}^{2m+1}}{\mathcal{F}^{m+1} \oplus \mathbb{H}^{2m+1}}.$$

The cycle Z restricts to a homologically trivial cycle Z_s on every fiber X_s of $\mathcal{X} \to L_{reg}$ and we have the associated normal function

$$\begin{array}{rccc} \nu : & L_{reg} & \longrightarrow & \mathcal{J}^m_{L_{reg}} \\ & s & \longmapsto & AJ_s(Z_s). \end{array}$$

For $s \in L_{reg}$, the first infinitesimal invariant $\delta_1(\nu)$ is an element of

$$H^1(\mathcal{H}^{\cdot\cdot}_s \otimes \Omega^{\cdot}_{L_{reg},s}) = \frac{\mathrm{Ker}(\overline{\nabla}_s : H^{m+1}(X_s, \Omega^m_{X_s}) \otimes \Omega^1_{L_{reg},s} \to H^{m+2}(X_s, \Omega^{m-1}_{X_s}) \otimes \Omega^2_{L_{reg},s})}{\mathrm{Im}(\overline{\nabla}_s : H^m(X_s, \Omega^{m+1}_{X_s}) \to H^{m+1}(X_s, \Omega^m_{X_s}) \otimes \Omega^1_{L_{reg},s})}$$
$$\subset \mathrm{Coker}\left(\overline{\nabla}_s : H^m(X_s, \Omega^{m+1}_{X_s}) \to H^{m+1}(X_s, \Omega^m_{X_s}) \otimes \Omega^1_{L_{reg},s}\right).$$

Dually, the infinitesimal invariant is an element of

$$\operatorname{Ker}\left({}^t\overline{\nabla} : H^m(X_s, \Omega^{m+1}_{X_s}) \otimes T_{L_{reg},s} \to H^{m+1}(X_s, \Omega^m_{X_s})\right)^\vee ,$$

where, with the notation of the previous section, we are using the pairing Ψ to identify $H^{m+1,m}(X_s) = H^m(X_s, \Omega^{m+1}_{X_s})$ with the dual of $H^{m,m+1}(X_s) = H^{m+1}(X_s, \Omega^m_{X_s})$. The map $\overline{\nabla}$ and its transpose ${}^t\overline{\nabla}$ are both given by cup-product via the Kodaira–Spencer map. This means, for instance, that for $\omega \in H^{m+1,m}(X_s)$ and $v \in T_{L_{reg},s}$, we have

$${}^t\overline{\nabla}(\omega \otimes v) = \omega \cup \xi_v$$

where $\xi_v \in H^1(X_s, T_{X_s})$ is the image of v via the Kodaira–Spencer map.

Identifying the tangent space to L at s with the quotient $H^0(X, L)/\langle f_s\rangle$, where we denote f_s an element of $H^0(X, L)$ with divisor of zeros X_s, the Kodaira–Spencer map for L_{reg} is induced by the connecting homomorphism of the usual normal bundle sequence

$$0 \longrightarrow T_{X_s} \longrightarrow T_X|_{X_s} \longrightarrow \mathcal{O}_{X_s}(L) \longrightarrow 0,$$

using the natural embedding $H^0(X, L)/\langle f_s\rangle \subset H^0(X_s, \mathcal{O}_{X_s}(L))$.

Put

$$H^{m+1}(\widetilde{X}, \Omega^{m+1}_{\widetilde{X}}|_{X_s})^{prim} := \operatorname{Ker}\left(H^{m+1}(\widetilde{X}, \Omega^{m+1}_{\widetilde{X}}|_{X_s}) \xrightarrow{res} H^{m+1}(X_s, \Omega^{m+1}_{X_s})\right).$$

It follows that we have the commutative diagram with exact rows

$$\begin{array}{ccccc}
 & & & & H^{m+1}(\widetilde{X}, \Omega^{m+1}_{\widetilde{X}})^{prim} \\
 & & & & \downarrow{\scriptstyle res} \\
H^m(X_s, \Omega^{m+1}_{X_s}) & \longrightarrow & H^{m+1}(X_s, \Omega^m_{X_s}(-L)) & \longrightarrow & H^{m+1}(\widetilde{X}, \Omega^{m+1}_{\widetilde{X}}|_{X_s})^{prim} \\
\| & & \downarrow & & \downarrow{\scriptstyle \overline{\alpha}} \\
H^m(X_s, \Omega^{m+1}_{X_s}) & \xrightarrow{\overline{\nabla}_s} & H^{m+1}(X_s, \Omega^m_{X_s}) \otimes (H^0(X, L)/\langle f_s\rangle)^\vee & \longrightarrow & \operatorname{Coker}(\overline{\nabla}_s),
\end{array}$$

where the middle row is part of the cohomology sequence of the sequence

$$0 \longrightarrow \Omega^m_{X_s}(-L) \longrightarrow \Omega^{m+1}_{\widetilde{X}}|_{X_s} \longrightarrow \Omega^{m+1}_{X_s} \longrightarrow 0.$$

Voisin proved the following [Voi88].

Proposition 10.1. *With the map $\overline{\alpha}$ defined by the above diagram, we have*

$$\delta_1(\nu_Z) = \overline{\alpha}(res([Z])).$$

When L is a base point free pencil, we have $\mathcal{O}_{X_s}(L) \cong \mathcal{O}_{X_s}$ and hence the exact sequence

$$0 \longrightarrow \Omega^{m+1}_{\widetilde{X}} \longrightarrow \Omega^{m+1}_{\widetilde{X}}(L) \longrightarrow \Omega^{m+1}_{\widetilde{X}}|_{X_s} \longrightarrow 0,$$

whose connecting homomorphism

$${}^t res : H^m(X_s, \Omega^{m+1}_{\widetilde{X}}|_{X_s}) \longrightarrow H^{m+1}(\widetilde{X}, \Omega^{m+1}_{\widetilde{X}})$$

is the transpose of res. We also have the exact sequence

$$0 \longrightarrow \Omega^m_{X_s} \longrightarrow \Omega^{m+1}_{\widetilde{X}}|_{X_s} \longrightarrow \Omega^{m+1}_{X_s} \longrightarrow 0,$$

whose connecting homomorphism can be identified with ${}^t\overline{\nabla}$ after a choice of basis for $T_sL \cong \mathbb{C}$.

Given $\omega \otimes v \in \operatorname{Ker} {}^t\overline{\nabla}$, let γ be a pre-image for $\omega \otimes v$ in $H^m(X_s, \Omega^{m+1}_{\widetilde{X}}|_{X_s})$. Voisin also proves that, dually, we have

$$\delta_1(\nu_Z)(\omega \otimes v) = {}^t res(\gamma) \cup [Z].$$

11. The Collino–Pirola–Zucconi adjunction map

Collino–Pirola [CP95] introduced an adjunction map for infinitesimal deformations of smooth varieties that they used, in conjunction with Griffiths' formula in Section 9, to compute certain values of the infinitesimal invariant of some normal functions. We present here a combination of results contained in [CP95], [CNP12], [PZ03].

Suppose given a smooth projective n-dimensional variety X with a locally free sheaf $\mathcal{F}$ of rank n and an extension

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0$$

with extension class $\xi \in \operatorname{Ext}^1(\mathcal{F}, \mathcal{O}_X) \cong H^1(X, \mathcal{F}^*)$. Denote $\partial_\xi : H^0(X, \mathcal{F}) \to H^1(X, \mathcal{O}_X)$ the first connecting homomorphism of the long exact sequence of cohomology associated to the sequence above. Put $m := h^0(X, \mathcal{F})$, assume $m > n$, and let $\mathbb{G} = G(n+1, m)$ be the Grassmannian of $n+1$ dimensional subspaces of $H^0(X, \mathcal{F})$ with universal subbundle $\mathcal{U}$. Let

$$\Gamma := \{(W, \xi) \mid W \subset \operatorname{Ker} \partial_\xi\} \subset \mathbb{G} \times \mathbb{P}\operatorname{Ext}^1(\mathcal{F}, \mathcal{O}_X)$$

be the incidence correspondence. Let $\mathcal{U}_\Gamma$ be the pull-back of $\mathcal{U}$ to Γ via the first projection and let $\mathcal{O}_\Gamma(1)$ be the pull-back of $\mathcal{O}_{\mathbb{P}\operatorname{Ext}^1(\mathcal{F},\mathcal{O}_X)}(1)$ via the second projection. The natural map $\Lambda^n H^0(X, \mathcal{F}) \to H^0(X, \Lambda^n \mathcal{F})$ gives rise to the natural map

$$\Lambda^n \mathcal{U}_\Gamma \longrightarrow H^0(X, \Lambda^n \mathcal{F}) \otimes \mathcal{O}_\Gamma$$

whose cokernel we denote $\mathcal{G}$. The adjunction map is the natural morphism

$$\alpha : \Lambda^{n+1} \mathcal{U}_\Gamma \otimes \mathcal{O}_\Gamma(-1) \longrightarrow \mathcal{G}$$

which can be concretely described as follows. Given a subspace $W \subset \operatorname{Ker} \partial_\xi$ of dimension $n+1$, choose a basis $s_1, \ldots, s_{n+1}$ of W which gives an element $s_1 \wedge \ldots \wedge s_{n+1}$ of $\Lambda^{n+1} W$. Lift the elements s_i to elements s'_i of $H^0(X, \mathcal{E})$. The class $\alpha(s_1 \wedge \ldots \wedge s_{n+1})$ is the image of $s'_1 \wedge \ldots \wedge s'_{n+1}$ via the composition

$$\Lambda^{n+1} H^0(X, \mathcal{E}) \longrightarrow H^0(X, \Lambda^{n+1} \mathcal{E}) \cong H^0(X, \Lambda^n \mathcal{F}) \twoheadrightarrow H^0(X, \Lambda^n \mathcal{F})/\Lambda^n W.$$

It is not difficult to check that this is indeed a well-defined map. It induces the above bundle map α.

12. The infinitesimal invariant and the adjunction map

Given a commutative diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\ \Phi\ } & \mathcal{A} \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \rho} & \\ & B & \end{array}$$

where $\mathcal{X}$ is a family of smooth projective varieties of dimension n, $\mathcal{A}$ is a family of abelian varieties of dimension a over the smooth analytic base B, we have the natural cycle

$$\mathcal{Z} := [\mathcal{Y}] - [\mathcal{Y}]^-$$

where $\mathcal{Y}$ is the image of $\mathcal{X}$ via Φ, $[\mathcal{Y}]$ is the associated cycle on $\mathcal{A}$ and $[\mathcal{Y}]^-$ is its image under $-\operatorname{Id}$. For $b \in B$, we denote the fibers of $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{A}$, by X_b, Y_b, Z_b, A_b respectively.

After possibly shrinking B, we can replace $\mathcal{X}$ by a desinguarization of $\mathcal{Y}$ and replace $\mathcal{A}$ by the family of its abelian subvarieties generated by the fibers of $\mathcal{Y}$ so that the following conditions are satisfied.

(1) For all $b \in B$, the restriction $\phi_b : X_b \to A_b$ is of degree 1 onto its image Y_b, and
(2) for all $b \in B$, the image Y_b generates A_b as a group.

Families with the two properties above were called "of Albanese type" in [PZ03].

The cycle $\mathcal{Z}$ is homologically trivial of codimension $a - n$ in each fiber. It therefore has an associated normal function $\nu_{\mathcal{Z}}$:

$$\begin{aligned} \nu_{\mathcal{Z}} : B &\longrightarrow \mathcal{J}^{2a-2n-1} \\ b &\longmapsto AJ(Z_b - Z_b^-) \end{aligned}$$

where

$$\mathcal{J}_B^{2a-2n+1} := \frac{\mathcal{H}^{2a-2n+1}(\mathcal{A})}{\mathcal{F}^{a-n+1} \oplus \mathbb{H}^{2a-2n+1}}$$

is the family of $(2a - 2n + 1)$-st intermediate jacobians of $\mathcal{A}$.

Suppose now that B is a disc with center b and let $v \in T_b B$ be a nonzero vector tangent to B. The usual cotangent bundle sequence

$$0 \longrightarrow \mathcal{O}_{A_b} \longrightarrow \Omega^1_{\mathcal{X}}|_{X_b} \longrightarrow \Omega^1_{X_b} \longrightarrow 0$$

has extension class $\xi_v \in H^1(X_b, T_{X_b}) = \operatorname{Ext}^1(\Omega^1_{X_b}, \mathcal{O}_{A_b})$, the Kodaira-Spencer image of v. We use the adjunction map for this extension of locally free sheaves.

Assume that there exist $n + 1$ global sections $s_1, \ldots, s_{n+1}$ of $\Omega^1_{\mathcal{A}}$ whose pull backs $\eta_1, \ldots, \eta_{n+1}$ to X lie in the kernel of the connecting homomorphism ∂_{ξ_v}. Put $\Omega := s_1 \wedge \ldots \wedge s_{n+1}$ and let ω be a form representing the class $\alpha(\eta_1 \wedge \ldots \wedge \eta_{n+1}) \in H^0(X, \Omega^n_X)/\Lambda^n W$, where W is the span of $\eta_1, \ldots, \eta_{n+1}$ in $H^0(X, \Omega^1_X)$. Let

$$\gamma : H^n(A_b, \Omega^{n+1}_{\mathcal{A}}|_{A_b}) \longrightarrow H^n(A_b, \Omega^{n+1}_{A_b})$$

be the restriction map.

Pirola and Zucconi [PZ03] prove

Proposition 12.1. *For all $\sigma \in H^0(A_b, \Omega^n_{A_b})$, we have*

(1) $\gamma(\Omega \otimes \overline{\sigma}) \otimes v \in \operatorname{Ker} {}^t\overline{\nabla}_b$,
(2) $\delta_1(\nu_{\mathcal{Z}})(\gamma(\Omega \otimes \overline{\sigma}) \otimes v) = 2 \int_X \omega \wedge \overline{\phi_b^* \sigma}$.

This was originally proved by Collino and Pirola [CP95] in the case where $\mathcal{X} \to \mathcal{A}$ is a family of curves in their jacobians, using the formula of Griffiths in Section 9. They used this calculation to show that the infinitesimal invariant of the Ceresa cycle (the cycle $[\mathcal{Y}] - [\mathcal{Y}]^-$ in the case of curves in their jacobians) does not vanish generically, hence that the Ceresa cycle is not torsion. They also obtain stronger results: for instance, that the Ceresa cycle is not algebraically equivalent to 0 if the curve Y is generic in a subvariety of codimension less than $\frac{g+2}{3}$ of $\mathcal{M}_g$, for $g \geq 4$, or if Y is a generic plane curve. Complementing Griffiths' results (see Sections 7 and 8), they also showed that the infinitesimal invariant can be identified with the equation of the canonical curve when the genus is 3. Furthermore, in genus 3, they showed that the Ceresa cycle is not algebraically equivalent to 0 on any subvariety of codimension at most 2 of $\mathcal{M}_3$ which is not contained in the hyperelliptic locus.

Pirola and Zucconi used this calculation to produce a bound for the geometric genus of subvarieties of general type of generic abelian varieties. For instance, they proved that if X has dimension n, geometric genus $p_g(X)$, and admits a generically finite morphism to a generic abelian variety of dimension a, then

$$p_q(X) \geq \binom{a-n}{2} + \binom{a}{n}.$$

When X is a surface of general type whose Albanese map is generically injective, they proved Castelnuovo's inequality:

$$m \leq p_g(X) + 2q - 3,$$

where q is the irregularity of X (i.e., the dimension of its Albanese variety) and m is the maximum variation of X, meaning the dimension of a family of deformations of X mapping to abelian varieties of dimension a whose Kodaira–Spencer map is injective. They also generalized the results of Collino–Pirola by showing that the only subvariety of dimension at least $2g-1$ of $\mathcal{M}_g$ on which the Abel-Jacobi image of the Ceresa cycle is trivial is the hyperelliptic locus.

Collino–Naranjo–Pirola [CNP12] used this calculation to prove their results on the Fano variety of lines of a smooth cubic threefold, see Section 8.

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