

$$\Rightarrow K_X|_C \sim K_C - C|_C$$

$$a \quad K_C \sim K_X|_C + C|_C$$

$$\text{take degrees: } 2g-2 = C \cdot (K_X + C) \quad \square$$

Theorem (Riemann-Roch): For any divisor D

$$\chi(D) = \chi(\mathcal{O}_X) + \frac{1}{2} D \cdot (D - K_X)$$

Proof: As in the proof of the existence and uniqueness of the intersection pairing, we can find two nonsingular curves C, E s.t. $D \sim C - E$: $\chi(D) = \chi(C - E)$.

For E , we have the usual exact sequence:

$$0 \rightarrow \mathcal{O}_X(-E) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_E \rightarrow 0$$

twist by C to get

$$0 \rightarrow \mathcal{O}_X(C-E) \rightarrow \mathcal{O}_X(C) \rightarrow \mathcal{O}_E(C) \rightarrow 0$$

and $\chi(C-E) = \chi(C) - \chi(\mathcal{O}_E(C))$

similarly $0 \rightarrow \mathcal{O}_X(-C) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_C \rightarrow 0$

twist by C : $0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(C) \rightarrow \mathcal{O}_C(C) \rightarrow 0$

and $\chi(C) = \chi(\mathcal{O}_X) + \chi(\mathcal{O}_C(C))$

and $\chi(D) = \chi(C-E) = \chi(\mathcal{O}_X) + \chi(\mathcal{O}_C(C)) - \chi(\mathcal{O}_E(C))$

R.R. for curves and the fact that $\deg \mathcal{O}_C(C) = C^2$, $\deg \mathcal{O}_E(C) = C \cdot E$

$$\Rightarrow \chi(D) = \chi(\mathcal{O}_X) + 1 - g_C + C^2 - 1 + g_E - C \cdot E$$

adjunction : $2g_C - 2 = C \cdot (C + K_X) \Rightarrow g_C = \frac{1}{2} C \cdot (C + K_X) + 1$

$$\Rightarrow \chi(D) = \chi(\mathcal{O}_X) + C^2 - C \cdot E - \frac{1}{2} C \cdot (C + K_X) + \frac{1}{2} E \cdot (E + K_X)$$

$$\chi(D) = \chi(\mathcal{O}_X) + \frac{1}{2} (C - E) \cdot (C - E + K_X)$$

$$= \chi(\mathcal{O}_X) + \frac{1}{2} D \cdot (D + K_X)$$

□

Definition: We say a divisor D is numerically equivalent to 0 , written $D \equiv 0$, if $C \cdot D = 0$ for all divisors C on X . Two divisors are numerically equivalent if their difference is numerically equivalent to 0 .

We define $\text{Num}(X) := \text{Div}(X) / \equiv$.

Remark: Linear equivalence implies numerical equivalence. $\text{Num}(X)$ is a finitely generated free abelian group (Ex. V.1.7, V.1.8).

Def: The rank of $\text{Num}(X)$ is called the Picard number of X , often denoted $\rho = \rho_X$.

Remark: The intersection pairing defines a symmetric bilinear pairing on $\text{Num}(X) \otimes \mathbb{R}$. The Hodge index theorem (below) shows that the signature of the intersection form on $\text{Num}(X) \otimes \mathbb{R}$ is $(1, \rho - 1)$.

Theorem: (The Hodge Index Theorem) Let H be an ample divisor on X and D a divisor on X s.t. $D \not\equiv 0$, $H \cdot D = 0$. Then $D^2 < 0$.

We need some preliminary results:

Lemma 1: Suppose H is an ample divisor on X . Then,
for all effective nonzero divisors D on X , $D \cdot H > 0$.

In particular, $H^2 > 0$.

Proof: By linearity, we may assume D is integral.

Some multiple mH of H is very ample. If D' is the normalization of D , then $\mathcal{O}_{D'}(H)$ defines a morphism $D' \rightarrow \mathbb{P}^n$ which is an embedding on some open subset of D' . In particular, \exists points $p, q \in D'$ s.t.

$$h^0(mH|_{D'} - p - q) = h^0(mH|_{D'}) - 2$$

In particular $h^0(mH|_{D'}) \geq 2$ and $\deg H|_{D'} > 0$.

Hence $\deg \mathcal{O}_D(H) = \deg \mathcal{O}_{D'}(H) > 0$ and $D \cdot H > 0$

To see $H^2 > 0$, note that mH is effective and $\neq 0$ for $m \gg 0$. □

Lemma 2: Suppose H is ample on X . For any divisor D , if $D \cdot H > H \cdot K_X$, then $h^0(K_X - D) = h^2(D) = 0$.

Proof: We prove the contrapositive: If $h^0(K_X - D) > 0$, then $K_X - D$ is linearly equivalent to an effective divisor and, by Lemma 1, $(K_X - D) \cdot H \geq 0$, i.e., $K_X \cdot H \geq D \cdot H$. □

Lemma 3: H ample on X . D a divisor s.t. $D^2 > 0$ and $D \cdot H > 0$. Then $h^0(nD) \rightarrow +\infty$ as $n \rightarrow +\infty$.

Proof: By Riemann-Roch:

$$\begin{aligned} \chi(nD) &= \chi(\mathcal{O}_X) + \frac{1}{2} nD \cdot (nD - K_X) \\ &= \chi(\mathcal{O}_X) + \frac{n^2}{2} D^2 - \frac{n}{2} D \cdot K_X \rightarrow \infty \text{ with } n \\ &\quad \text{because } D^2 > 0 \end{aligned}$$

$$\text{So } h^0(nD) - h^1(nD) + h^2(nD) \rightarrow \infty$$

Now, for $n \gg 0$, $nD \cdot H > H \cdot K_X$ because $D \cdot H > 0$

and, by Lemma 2, $h^2(nD) = 0$.

Hence $h^0(nD) \geq h^0(nD) - h^1(nD) = \chi(nD) \rightarrow \infty$. □

Proof of the Hodge Index Theorem:

Assume $D^2 > 0$ and obtain a contradiction.

For $n \gg 0$ $H' := D + nH$ is very ample.

Furthermore, $D \cdot H' = D^2 > 0$. So, by Lemma 3,

$h^0(mD)$ goes to ∞ as $m \rightarrow \infty$.

In particular $h^0(mD) \geq 2$ for large m and $|mD| \neq \emptyset$
($D^2 > 0$)

$\Rightarrow mD$ is effective $\Rightarrow mD \cdot H > 0$ by Lemma 1.
 $\neq 0$

$\Rightarrow D \cdot H > 0$ contradiction.

Now assume $D^2 = 0$ and look for a contradiction.

Since $D \neq 0$, $\exists E$ s.t. $D \cdot E \neq 0$

Replace E by $-E$ if necessary so that $D \cdot E > 0$.

Replace E by $(H^2)E - (E \cdot H)H$ so that $E \cdot H = 0$.

$$\forall n \quad (nD + E) \cdot H = 0$$

$$\begin{aligned} (nD + E)^2 &= n^2 D^2 + 2n D \cdot E + E^2 \\ &= 2n D \cdot E + E^2 \end{aligned}$$

So for $n \gg 0$ $(nD + E)^2 > 0$ (and $(nD + E) \cdot H = 0$)

Now apply the argument in the case $D^2 > 0$ to $nD + E$
to obtain a contradiction. $n \gg 0$ \square

Theorem: The Nakai-Moishezon criterion for ampleness. D is ample iff $D^2 > 0$ and, for all irreducible curves C on X , $C \cdot D > 0$.

Example: $\mathbb{P}^1 \times \mathbb{P}^1 =: X$ $\text{Pic}(\mathbb{P}^1) \cong \mathbb{Z}[\mathcal{O}_{\mathbb{P}^1}(1)]$
 $\begin{array}{ccc} \nearrow \pi_1 & & \searrow \pi_2 \\ \mathbb{P}^1 & & \mathbb{P}^1 \end{array}$ $\text{Pic}(\mathbb{P}^n) \cong \mathbb{Z}[\mathcal{O}_{\mathbb{P}^n}(1)]$

$$\begin{aligned} \text{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) &\cong \mathbb{Z} \oplus \mathbb{Z} \quad (\text{homework from last quarter}) \\ &= \mathbb{Z}[\pi_1^* \mathcal{O}_{\mathbb{P}^1}(1)] \oplus \mathbb{Z}[\pi_2^* \mathcal{O}_{\mathbb{P}^1}(1)] \end{aligned}$$

$F_1 =$ a fiber of π_1 $F_2 =$ fiber of π_2

$$\mathcal{O}_X(F_1) \cong \pi_1^* \mathcal{O}_{\mathbb{P}^1}(1) \quad \mathcal{O}_X(F_2) \cong \pi_2^* \mathcal{O}_{\mathbb{P}^1}(1).$$

$\forall D$ on X $D \sim aF_1 + bF_2$ for some $a, b \in \mathbb{Z}$.

D is ample (\Rightarrow) $a > 0, b > 0$.