Surfaces:

In this section, by a surface we mean a nonsingular projective surface over an algebraically closed field.

Now divisors are supported on curves.

As usual, the datum of an effective divisor is equivalent to the datum of a closed subscheme of codim 1.

We start by intersecting divisors on a surface:

We define a bilinear pairing on $Cl(X) \cong CaCl(X) \cong Pic(X)$ which is an extension of the naïve notion of intersecting transverse curves on a surface.

Recall that for two closed subschemes Y and Z of a scheme X, the intersection scheme $Y \cap Z$ is the closed subscheme with sheaf of ideals $\mathcal{I}_Y + \mathcal{I}_Z \hookrightarrow \mathcal{O}_X$.

Definition: We say that two curves (closed subschemes of codim. 1) C and D on X (a surface) intersect transversely at a (closed) point $x \in C \cap D$ if the local equations of C and D at x generate the maximal ideal $\mathfrak{m}_x$ of $\mathcal{O}_{X,x}$.

This implies, in particular, that C and D are both nonsingular at x (for instance, the local equation of D at x will generate the maximal ideal of $\mathcal{O}_{C,x}$).

We say that C and D are transverse if they are transverse at all of their points of intersection.

Theorem: There exists a unique symmetric bilinear form

$$\mathrm{Div}(X) \times \mathrm{Div}(X) \longrightarrow \mathbb{Z}$$
$$(C, D) \longmapsto C \cdot D$$

such that, if C and D are transverse curves, then
$C \cdot D = \# C \cap D$ and, if $C \sim C'$ (linear equivalence) then, for all $D \in \mathrm{Div}(X)$, $C \cdot D = C' \cdot D$.

Lemma: Suppose $C, D \subset X$ are curves meeting transversely and C is nonsingular. Then

$$\# C \cap D = \deg \mathcal{O}_C(D).$$

Proof: We have the usual exact sequence

$$0 \longrightarrow \mathcal{O}_C(-D) \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_{C \cap D} \longrightarrow 0.$$

$$\mathcal{O}_C(-D) = \mathcal{O}_C(-C \cap D)$$

Since C and D are transverse, their intersection is reduced, hence $\mathcal{O}_{C\cap D}$ has length $\# C \cap D$.

The exact sequence above now gives:

$$\deg \mathcal{O}_C(-D) = -\deg \mathcal{O}_C(D) = -\deg \mathcal{O}_{C\cap D}$$

$$\Rightarrow \quad \deg \mathcal{O}_C(D) = \# C \cap D. \qquad \square.$$

If the curves C and D are not necessarily transverse but have no common components, assuming C nonsingular, as in the lemma, we can define:

$$C \cdot D = \deg \mathcal{O}_C(D) = \sum_{x \in C \cap D} \mathrm{length}_x \, \mathcal{O}_{C \cap D, x}$$

$$= \sum_{x \in C \cap D} \mathrm{length}_x \left(\mathcal{O}_{X,x} / (f_x, g_x) \right)$$

where f_x, g_x are the local equations of C and D at x.

Proof of theorem about the existence and uniqueness of the intersection pairing:

Uniqueness: Choose a very ample divisor H on X, and let $C, D \in \mathrm{Div}(X)$ be arbitrary.

$\exists\ m > 0$ s.t. $C + mH$ and $D + mH$ are generated by global sections. By homework from last quarter (Ex. II.7.5), $C + (m+1)H$, $D + (m+1)H$ are very ample.

Replace H by $(m+1)H$ so that $C+H$ and $D+H$ are very ample.

Now apply Bertini's theorem: $\exists\ C' \in |C+H|$,

$D' \in |D+H|$, $E' \neq F' \in |H|$

all irreducible and nonsingular.

We have $C \sim C' - E'$ $\quad D \sim D' - F'$

and, putting $\ell(C' \cap D') := \text{length}(C' \cap D')$

$$C \cdot D = (C' - E') \cdot (D' - F')$$
$$= C' \cdot D' - C' \cdot F' - E' \cdot D' + E' \cdot F'$$

$$\text{A} \quad C \cdot D = \ell(C' \cap D') - \ell(C' \cap F') - \ell(E' \cap D') + \ell(E' \cap F')$$

which proves uniqueness.

To prove the existence, we take formula A as our definition for $C \cdot D$ and show that it is independent of the choices of the curves C', D', E', F' and the ample divisor H.

<u>Independence of the choice of C', D', E', F' for H fixed:</u>

For another choice of curves C'', D'', E'', F'' (similar to C', D', E', F') we need to show the intersection numbers are the same.

For instance, we show $\ell(C' \cap D') = \ell(C'' \cap D'')$: this follows from the lemma via:

$$\ell(C' \cap D') = \deg \mathcal{O}_{C'}(D') = \deg \mathcal{O}_{C'}(D'') = C' \cdot D''$$

$$= \deg \mathcal{O}_{D''}(C') = \deg \mathcal{O}_{D''}(C'') = C'' \cdot D'' = \ell(C'' \cap D'').$$

Now we show independence of the choice of H:

For another very ample divisor H', choose curves

$C'' \in |C+H'|$, $D'' \in |D+H'|$, $E'' \neq F'' \in |H'|$.

(after possibly replacing H' by a multiple)

nonsingular, irred..

We need to show:

$$\ell(C' \cap D') - \ell(C' \cap F') - \ell(E' \cap D') + \ell(E' \cap F')$$

$$= \ell(C'' \cap D'') - \ell(C'' \cap F'') - \ell(E'' \cap D'') + \ell(E'' \cap F'')$$

We have $\quad C \sim C' - E' \sim C'' - E''$

$$D \sim D' - F' \sim D'' - E''$$

via the Lemma, we obtain

$$C \cdot (D' - F') = C \cdot (D'' - F'') \text{ etc.}$$

exercise: finish the proof. $\square$.

<u>Def</u>: The self-intersection of any divisor is defined as $D^2 := D \cdot D$. If D is a curve, i.e., an effective divisor, this is equal to $\deg \mathcal{O}_D(D)$.

<u>Example</u>: $X = \mathbb{P}^2$. $L \subset \mathbb{P}^2$ a line.

Then $L^2 = \deg \mathcal{O}_L(L) = \deg \mathcal{O}_L(1) = \deg \mathcal{O}_{\mathbb{P}^1}(1) = 1$

a $\quad = \# L \cap L' \quad L'$ a line $\neq L$.

If C is a curve of degree d in $\mathbb{P}^2$.

C is the scheme of zeros of $P \in H^0(\mathcal{O}_{\mathbb{P}^2}(d))$

$C \sim dL$ because $\mathcal{O}(dL) = \mathcal{O}(d) = \mathcal{O}(1)^{\otimes d}$

So $C^2 = (dL)^2 = d^2$

If D is a curve of degree e, then

$C \cdot D = (dL)\cdot(eL) = de$ Bézout's theorem

for curves in $\mathbb{P}^2$.

Proposition: The adjunction formula:

For any nonsingular curve C on X of genus g, we have

$$2g-2 = C\cdot(C+K_X)$$

$$\text{or } g = \frac{1}{2}C\cdot(C+K_X)+1$$

Proof: We have the sequence of differentials on C:

$$0 \longrightarrow \mathcal{I}_C/\mathcal{I}_C^2 \longrightarrow \Omega^1_X|_C \longrightarrow \Omega^1_C \longrightarrow 0$$

$$\mathcal{I}_C/\mathcal{I}_C^2 = \mathcal{I}_C \otimes \mathcal{O}_X/\mathcal{I}_C = \mathcal{I}_C|_C = \mathcal{O}_X(-C)|_C$$
$$\cong \mathcal{O}_C(-C)$$

$$\Rightarrow \quad 0 \to \mathcal{O}_C(-C) \to \Omega^1_X|_C \longrightarrow \Omega^1_C \longrightarrow 0$$

Take top exterior powers:

$$\Lambda^2 \Omega^1_X|_C \cong \Omega^1_C \otimes \mathcal{O}_C(-C)$$
$$\omega_X|_C \cong \omega_C \otimes \mathcal{O}_C(-C)$$

$$\Rightarrow \quad K_X|_C \sim K_C - C|_C$$

or $$K_C \sim K_X|_C + C|_C$$

take degrees: $\quad 2g-2 = C\cdot(K_X + C)$ $\square$

__Theorem__ (Riemann-Roch): For any divisor D

$$\chi(D) = \chi(\mathcal{O}_X) + \frac{1}{2} D\cdot(D-K_X)$$

__Proof:__ As in the proof of the existence and uniqueness of the intersection pairing, we can find two nonsingular curves C, E s.t. $D \sim C-E$: $\chi(D) = \chi(C-E)$.

For E, we have the usual exact sequence: