$H^0(D-x)$ and $H^0(D-y)$ are both hyperplanes in $H^0(D)$ (meaning they have dim. 1 less).

So $H^0(D-x) \not\subset H^0(D-y) \iff H^0(D-x) \neq H^0(D-y)$

$\iff H^0(D-x) \cap H^0(D-y)$ has codim. 2 in $H^0(D)$

We can identify $H^0(D-x) \cap H^0(D-y)$ with $H^0(D-x-y)$

(exercise: use a sequence similar to $0 \to \mathcal{O}_X(-x) \to \mathcal{O}_X \to \mathcal{O}_x \to 0$)

So $\exists\ s$ with $s(x)=0,\ s(y)\neq 0 \iff h^0(D-x-y) = h^0(D) - 2$

Separating tangent vectors means

$\mathcal{L} = \mathcal{O}_X(D)$

$$H^0(D-x) = \{ s \in H^0(D) \mid s(x) = 0 \} \xrightarrow[\text{evaluation}]{} \mathfrak{m}_x \mathcal{L}_x / \mathfrak{m}_x^2 \mathcal{L}_x \cong k$$

Because the target is 1-dimensional, surjectivity means the map is not 0. This means $\exists\ s \in H^0(D)$ s.t. $s_x \in m_x \mathcal{O}_X(D)_x$ and $s_x \notin m_x^2 \mathcal{O}_X(D)_x$.

$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$$

$x \in \operatorname{Supp} Z(s)$

(if $D(s)$ = divisor of s)

$\iff D(s) - x \geq 0$ $\qquad\qquad$ $D(s) - 2x$ not effective.

$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$$

$s \in H^0(D - x)$ $\qquad\qquad$ $s \notin H^0(D - 2x)$

(use sequence $0 \to \mathcal{O}_X(-2x) \to \mathcal{O}_X \to \mathcal{O}_{\{2x\}} \to 0$)

This means that $H^0(D-2x) \underset{\neq}{\hookrightarrow} H^0(D-x) \hookrightarrow H^0(D)$

$$\iff h^0(D-2x) = h^0(D-x) - 1 = h^0(D) - 2 \qquad \square$$

<u>Definition</u>: The degree of a morphism $X \xrightarrow{\varphi} \mathbb{P}^n_k$ is the degree of the invertible sheaf $\varphi^* \mathcal{O}_{\mathbb{P}^n}(1)$.

This is equal to the degree of any divisor

D s.t. $\mathcal{O}_X(D) \cong \varphi^* \mathcal{O}_{\mathbb{P}^n}(1)$

which is also the degree of the divisor of zeros of any global section of $\varphi^* \mathcal{O}_{\mathbb{P}^n}(1)$,

in particular, for any $H \in H^0(\mathcal{O}_{\mathbb{P}^n}(1))$, it is the degree of $D(\varphi^* H)$. (see Chapter I Section 7 for more on degrees of morphisms)

<u>Corollary</u>: (1) $\deg D \geq 2g \Rightarrow |D|$ is base point free.

(2) $\deg D \geq 2g+1 \Rightarrow |D|$ is very ample.

(3) D is ample $\iff$ $\deg D > 0$.

<u>Proof</u>: exercise.

<u>Some other nice consequences</u>:

- We saw that $X \cong \mathbb{P}^1 \iff g=0 \iff K(X) \cong K(\mathbb{P}^1) \cong k(t)$

  In this case: ample $\iff$ very ample $\iff$ $\deg D > 0$

- If X is elliptic, i.e., $g=1$, then any divisor of degree 2 is base point free and $h^0(D)=2$ (use Riemann-Roch and $h^1(D)=0$)

so $|D|$ defines a morphism of degree 2

$$X \twoheadrightarrow \mathbb{P}^1.$$

If $\deg D = 3$, then $|D|$ is very ample and $h^0(D)=3$, and $|D|$ defines a closed embedding $X \hookrightarrow \mathbb{P}^2$ of degree 3.

The image of X is a plane cubic.

For elliptic curves, very ample $\iff \deg D \geq 3$.

<u>Corollary:</u> The canonical linear system $|K_X| = |\omega_X|$ $= |\Omega^1_{X/k}|$ is base point free iff $g > 0$.

<u>Proof:</u> $|K_X|$ is base point free iff $\forall\, x \in X$

$$h^0(K_X - x) = h^0(K_X) - 1 = g - 1.$$

$$h^0(x) \underset{\text{Serre Duality}}{=} h^1(K_X - x) \underset{\text{Riemann-Roch}}{=} h^0(K_X - x) + g - 1 - \deg(K_X - x)$$

$$= h^0(K_X - x) + g - 1 - (2g - 3) = h^0(K_X - x) - g + 2$$

$$\leq g - g + 2 = 2$$

$h^0(x)$ cannot be 2 because otherwise it would give a morphism of degree 1, $X \to \mathbb{P}^1$ which would then be an isom, but $g > 0$.

$$\Rightarrow \quad h^0(x) = 1 = h^0(K_X - x) - g + 2$$

$$\Rightarrow \quad h^0(K_X - x) = g - 1 \qquad \square$$

- <u>Curves of genus 2:</u>

If X has genus 2, then $\deg K_X = 2g - 2 = 2$ and $h^0(K_X) = g = 2$. So $|K_X|$ defines a morphism of degree 2 : $\quad X \twoheadrightarrow \mathbb{P}^1$.

Any divisor of degree ≥ 5 is very ample.

One can show that D very ample $\Leftrightarrow$ $\deg D \geq 5$.

Definition: We say that X is hyperelliptic if $\exists\ \varphi : X \twoheadrightarrow \mathbb{P}^1$ of degree 2.

We say X is trigonal if $\exists\ \varphi : X \twoheadrightarrow \mathbb{P}^1$ of degree 3.

″ ″ ″ ″ tetragonal ″ ″ ″ ″ 4.

⋮

″ n-gonal ″ ″ n.

A linear system of degree d and dim r is called (and denoted) a g^r_d. (e.g. if $\deg D = 2$ and $\dim |D| = 1$, then $|D|$ is a g^1_2).

$\overset{\shortparallel}{h^0(D)-1}$

We saw that curves of genus ≤ 2 are always hyperelliptic.

Theorem: The canonical linear system $|K_X|$ is very ample if and only if X is NOT hyperelliptic.

Proof: $|K_X|$ is very ample iff $\forall\ p, q \in X$

$$h^0(K_X - p - q) = h^0(K_X) - 2 = g - 2.$$

By Serre Duality this is equivalent to $h^1(p+q) = g-2$

and by Riemann-Roch, this is equivalent to $h^0(p+q) = 1$.

So $|K_X|$ is not very ample iff $\exists\ p, q \in X$ s.t. $h^0(p+q) \geq 2$.

Note that if $g=0$, $X \cong \mathbb{P}^1$, $\omega_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(-2) \cong \mathcal{O}(K_{\mathbb{P}^1})$ cannot be very ample.

If $g>0$ $\quad h^0(p+q) = h^1(K_X - p - q) \leq 2$.

So K_X not very ample $\iff$ $\exists\, p,q \in X$ s.t. $h^0(p+q) = 2$.

Then $|p+q|$ is a g^1_2 and has no base points because $h^0(p+q-x) \leq 1$ for any divisor of degree 1 because as before, if not we would have $X \cong \mathbb{P}^1$.

So $|p+q|$ defines a morphism of degree 2 to $\mathbb{P}^1$

$\iff$ X is hyperelliptic. $\square$

<u>Definition:</u> If X is not hyperelliptic (in particular $g \geq 3$), then the embedding of X in $\mathbb{P}^{g-1}_k$ is called the canonical embedding and the image of X by the

canonical embedding is called the canonical curve (which is isomorphic to X).

- If X has genus 3 and is not hyperelliptic, then the canonical curve $X \hookrightarrow \mathbb{P}^2_k$ is a smooth plane quartic. $\deg K_X = 2g-2 = 4$. (see I.7)
  Conversely, any smooth plane quartic is a canonical curve. $\Rightarrow$ non hyperelliptic curves of genus 3 exist.
- If X has genus 4 and is not hyperelliptic, then the canonical curve $X \hookrightarrow \mathbb{P}^3_k$ is the complete intersection of a unique irreducible quadric and an irreducible cubic in $\mathbb{P}^3$.

In other words, the homogeneous ideal of X in $\mathbb{P}^3$ is generated by an irreducible quadratic polynomial (unique up to multiplication by a scalar) and an irreducible cubic polynomial.

Sketch of proof: $I_X = \bigoplus_{d \geq 0} H^0(\mathcal{I}_X(d))$

$$0 \longrightarrow \mathcal{I}_X \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

We know that homogeneous polynomials of degree d vanishing on X are the elements of $H^0(\mathcal{I}_X(d))$.

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathcal{I}_X) \arrow[r] \arrow[d, phantom, "\parallel"] & H^0(\mathcal{O}_{\mathbb{P}^3}) \arrow[r] \arrow[r, phantom, shift right=3, "\hookrightarrow"] \arrow[d, phantom, "\cong"] & H^0(\mathcal{O}_X) \arrow[r] \arrow[d, phantom, "\cong"] & \cdots \\
& 0 & k & k &
\end{tikzcd}$$

twist by $\mathcal{O}_{\mathbb{P}^3}(1)$ and take cohomology:

$$0 \to H^0(\mathcal{I}_X(1)) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \overset{\cong}{\twoheadrightarrow} H^0(\mathcal{O}_X(1))$$

$$\mathcal{O}_X(1) \cong \mathcal{O}_X(K_X) \cong \omega_X \cong \Omega^1_X$$

morphism was given by a basis of $H^0(K_X)$: s_0, s_1, s_2, s_3

$$s_i = \varphi^* X_i \qquad X_i \in H^0(\mathcal{O}_{\mathbb{P}^3}(1))$$

$$\Rightarrow \quad H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \xrightarrow{\cong} H^0(\mathcal{O}_X(1))$$

$$X_i \longmapsto \varphi^* X_i = s_i$$

$$\Rightarrow \quad H^0(\mathcal{I}_X(1)) = 0$$

twist with $\mathcal{O}_{\mathbb{P}^3}(2)$ and take cohomology:

$$0 \longrightarrow H^0(\mathcal{I}_X(2)) \longrightarrow \underbrace{H^0(\mathcal{O}_{\mathbb{P}^3}(2))}_{\dim = 10} \longrightarrow H^0(\mathcal{O}_X(2))$$

$$\mathcal{O}_X(2) = \mathcal{O}_X(1)^{\otimes 2} \cong \omega_X^{\otimes 2}$$

$\deg = 2(2g-2) = 8 \geq 2g-1$

$\Rightarrow h^0 = d+1-g = 3g-3 = 9$

$$\Rightarrow \quad h^0(\mathcal{I}_X(2)) \geq 1$$

So there is at least one quadric containing X.
Any such quadric is irreducible because otherwise it would be a product of two (possibly proportional) linear polynomials whose zeros are planes $\Rightarrow$ X is contained in a plane $\Rightarrow$ $H^0(\mathcal{I}_X(1)) \neq 0$ we saw this is not the case.

X cannot be contained in two distinct quadrics because the degree of the proper intersection (meaning the intersection is a curve) of two quadrics is $4 < \deg X = 6$. (See Chapter I.7)

$\Rightarrow$ $h^0(\mathcal{I}_X(2)) = 1$.

Now twist with $\mathcal{O}_{\mathbb{P}^3}(3)$ and take cohomology:

$$0 \longrightarrow H^0(\mathcal{I}_X(3)) \longrightarrow \underset{\dim = 20}{H^0(\mathcal{O}_{\mathbb{P}^3}(3))} \longrightarrow H^0(\mathcal{O}_X(3))$$

$\mathcal{O}_X(3) \cong \omega_X^{\otimes 3}$

$\deg = 3(2g-2)$

$h^0(\mathcal{O}_X(3)) = 5g-5 = 15$

$$\Rightarrow \quad h^0(\mathcal{I}_X(3)) \geq 5$$

Note: we have $\quad H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \otimes \overbrace{H^0(\mathcal{I}_X(2))}^{\dim 1.} \longrightarrow H^0(\mathcal{I}_X(3))$

$$H \otimes Q \longmapsto HQ$$

$$h^0(\mathcal{I}_X(2)) = 1 \quad \Longrightarrow \underset{\dim 4}{H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \otimes H^0(\mathcal{I}_X(2))} \hookrightarrow \underset{\dim \geq 5}{H^0(\mathcal{I}_X(3))}$$

$\Rightarrow$ $\exists$ irreducible cubic vanishing on X, call it C.

(See I.7) The intersection $Q \cap C$ has degree $6 = \operatorname{degree}(X)$ and contains X.

$\Rightarrow \quad X = Q \cap C$.

$\Rightarrow \quad H^0(\mathcal{I}_X(3)) = 5$

<u>Furthermore</u>: Any (complete or proper) intersection of an irreducible quadric and an irreducible cubic in $\mathbb{P}^3$ is a canonical curve of genus 4.

$\Rightarrow$ $\exists$ nonhyperelliptic curves of genus 4.

More generally, we can prove that there exist nonhyperelliptic curves of any genus ≥ 3 (dimension count).

Proposition: X hyperelliptic of genus ≥ 2.

Then X has a unique g^1_2. Let $\varphi: X \to \mathbb{P}^1$ be the associated morphism of degree 2. The canonical morphism $X \to \mathbb{P}^{g-1}$ is the composition

$$X \xrightarrow{\varphi} \mathbb{P}^1 \underset{(g-1)\text{-uple embedding of } \mathbb{P}^1.}{\hookrightarrow} \mathbb{P}^{g-1} \qquad \left(\Rightarrow |K_X| = |(g-1)\, g^1_2|\right)$$

Any canonical divisor is the sum of $g-1$ divisors of the g^1_2.

Proof: Let $X' :=$ image of the canonical morphism.

$X' \hookrightarrow \mathbb{P}^{g-1}$ is an integral curve.

Let $\tilde{X}' := X'$ be the normalization.

Then $\tilde{X}'$ is a proper integral nonsingular curve.

By definition we have a factorization:

$$\begin{tikzcd} X \arrow[r, two heads] \arrow[rr, bend right, "\text{canonical morphism.}"'] & X' \arrow[r, hook] & \mathbb{P}^{g-1} \end{tikzcd}$$

By the universal property of normalization, because X is normal, we have a factorization

$$X \xrightarrow[\text{finite}]{} \tilde{X}' \xrightarrow[\text{finite}]{} X'$$

$$K(X) \hookleftarrow K(\tilde{X}') \overset{\hookleftarrow}{\equiv} K(X')$$

<u>Claim!</u> $\text{degree}\,(X \to \tilde{X}') \geq 2$.

recall: $\text{degree}\,(X \to \tilde{X}') = \deg\,(\text{pull-back of point of } \tilde{X}')$ as a divisor.

Let $p, q \in X$ be points s.t. $p+q \in g^1_2$.

Then, by Serre Duality, $h^0(K_X - p - q) = h^1(p+q) = h^1(g^1_2)$

By Riemann-Roch $h^1(g^1_2) = h^0(g^1_2) - d + g - 1$

$$= 2 - 2 + g - 1 = g - 1$$

$$\Rightarrow \quad h^0(K_X - p - q) = h^1(p+q) = g-1 \neq g-2.$$

$$H^0(K_X - p - q) \hookrightarrow H^0(K_X - p) \hookrightarrow H^0(K_X)$$

sections zero at p and q — dim $g-1$

sections zero at p — dim $g-1$

dim g

$\Rightarrow$ Any section vanishing at p, vanishes on $p+q$

$\Rightarrow$ Any hyperplane in $\mathbb{P}^{g-1}$ containing the image of p, also contains the image of q.

$\Rightarrow$ The divisor of zeros of any section of ω_X is the sum of divisors of the g^1_2.

5. (degree of $X \to \tilde{X}'$) ≥ 2

$$
\begin{tikzcd}
X \arrow[r, "\geq 2"] \arrow[rrr, bend right, "\text{canonical morphism}" below] & \tilde{X}' \arrow[r, "1"] & X' \arrow[r, "\leq g-1"] & \mathbb{P}^{g-1}
\end{tikzcd}
$$

degree $= 2g-2$

(degrees get multiplied when we compose morphisms)

<u>Homework</u> (IV.1.5) for any effective divisor D on X,

$\dim |D| \leq \deg D$ with equality iff ($D=0$ or $g=0$)

Consider the morphism $\tilde{X}' \to \mathbb{P}^{g-1}$: this has degree $\leq g-1$ and the dimension of the corresponding linear system $|D|$ is $g_{\tilde{X}'}-1$. Homework $\Rightarrow$ $\deg D = g_{\tilde{X}'}-1 = \dim |D|$

$\Rightarrow$ $g_{\tilde{X}'} = 0$ because $D \neq 0$.

$$\Rightarrow \quad \begin{tikzcd} \tilde{X}' \cong \mathbb{P}^1 \arrow[dr] & \\ & \mathbb{P}^{g-1} \end{tikzcd}, \quad |D| \cong |\mathcal{O}_{\mathbb{P}^1}(g-1)|$$

$$\Downarrow$$

$\tilde{X}' \longrightarrow \mathbb{P}^{g-1}$ is the $(g-1)$-uple embedding of $\mathbb{P}^1 = \tilde{X}'$

$\Rightarrow \tilde{X}' \cong X' \Rightarrow X'$ is nonsingular and is the $(g-1)$-uple embedding of $\mathbb{P}^1$. $\square$