Let x be a closed point of X and put $y = f(x)$.
The stalks of the exact sequence from the proposition are:

$$
\begin{tikzcd}
0 \arrow[r] & \Omega^1_{Y,y} \otimes_{\mathcal{O}_{Y}} \mathcal{O}_{X,x} \arrow[r, "f^*"] & \Omega^1_{X,x} \arrow[r] & \Omega^1_{X/Y,x} \arrow[r] & 0 \\
& \Omega^1_{Y,y}. \arrow[u] & &
\end{tikzcd}
$$

$$m \otimes 1 \longmapsfrom m$$

$f^*: \Omega^1_{Y,y} \longrightarrow \Omega^1_{X,x}$ is the differential of the morphism of local rings $f^*: \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$.
Let t_x be a uniformizer at x (or a local parameter), and t_y a " " y. Thinking of $\mathcal{O}_{Y,y}$ as a subring of $\mathcal{O}_{X,x}$, we can write $t_y = u\, t_x^{e_x}$ where u is a unit in $\mathcal{O}_{X,x}$.

Then $\quad dt_y = t_x^{e_x} du + e_x u\, t_x^{e_x - 1} dt_x .$

Therefore, if the ramification is tame at x, then the dimension of $\Omega^1_{X/Y,x}$ is $e_x - 1$. If the ramification is wild at x, then $\dim_k(\Omega^1_{X/Y,x})$ is at least e_x.

<u>Definition</u>: The ramification divisor of f is

$$R := \sum_{x \in X} \dim_k(\Omega^1_{X/Y,x})[x]$$

The Hurwitz formula is

$$2g_X - 2 = (\deg f)(2g_Y - 2) + \deg R$$

Ampleness and very ampleness on curves: (nonsingular, complete)

<u>Def</u>: If D is a divisor on X, we say D is ample, respectively very ample, if $\mathcal{O}_X(D)$ is.

We say a point $x \in X$ is a base point of $|D|$

(recall $|D| = \mathbb{P}H^0(\mathcal{O}_X(D)) = \{\text{lines in } H^0(\mathcal{O}_X(D))\}$

$$= \{Z(s) \mid s \in H^0(\mathcal{O}_X(D))\}$$

$$= \{\text{effective Weil divisors lin. eq. to } D\}).$$

if for all $D' \in |D|$, x belongs to the support of D',

i.e., $\forall\ s \in H^0(\mathcal{O}_X(D))$, $s(x) = 0$ ($s_x \in m_x \mathcal{O}_X(D)_x$)

$$s(x) \in \mathcal{O}_X(D)_x / m_x \mathcal{O}_X(D)_x \cong k.$$

We say $|D|$ or $\mathcal{O}_X(D)$ is base point free if it has no base points.

If we choose a basis $s_0, \dots, s_n \in H^0(\mathcal{O}_X(D))$, the associated rational map $X \dashrightarrow \mathbb{P}^n_k \cong |D|^* = \mathbb{P}H^0(\mathcal{O}_X(D))^*$ is a morphism if and only if $|D|$ is base point free $\Leftrightarrow$ $\mathcal{O}_X(D)$ is globally generated.

<u>Proposition:</u> $|D|$ is base point free iff $\forall\ x \in X$,

$$h^0(D-x) = h^0(D) - 1,$$

or $\dim |D-x| = \dim |D| - 1$.

In fact:

<u>Proposition</u>: $x \in X$ is a base point of $|D|$ iff

$$h^0(D-x) = h^0(D) \quad \text{or}$$

$$\dim |D-x| = \dim |D| .$$

<u>Proof</u>: Consider the exact sequence:

$$0 \to \mathcal{O}_X(-x) \to \mathcal{O}_X \to \mathcal{O}_{\{x\}} \to 0$$

$$\mathcal{O}_X(-x)_x = m_x$$

twist by $\mathcal{O}_X(D)$:

$$0 \to \mathcal{O}_X(D-x) \to \mathcal{O}_X(D) \to \mathcal{O}_x \to 0$$

and write the cohomology sequence:

$$0 \to H^0(D-x) \to H^0(D) \xrightarrow{ev_x} k \longrightarrow H^1(D-x) \to H^1(D) \to 0.$$

$$s \longmapsto s(x) \in \mathcal{O}_X(D)_x / m_x \mathcal{O}_X(D)_x \cong k$$

So there exists $s \in H^0(D)$ not vanishing at x iff ev_x is surjective. So x is a base point of $|D|$

iff $ev_x = 0 \iff H^0(D-x) = H^0(D)$

and $ev_x \neq 0 \iff h^0(D-x) = h^0(D) - 1$

$$\iff |D-x| = |D| - 1 \qquad \square$$

<u>Note</u>: It also follows from the proof that

x is <u>not</u> a base point $\iff h^0(D-x) = h^0(D) - 1$.

<u>Very ampleness</u>: Last quarter we had some criteria for a morphism $X \to \mathbb{P}^n_k$ to be a closed embedding:

Proposition: X scheme / A noetherian ring

$\varphi : X \to \mathbb{P}^n_A$ morphism associated to an invertible sheaf $\mathcal{L}$ with global sections $s_0, \dots, s_n$. Then φ is a closed embedding iff

(1) All open sets $X_{s_i} := \{x \in X \mid (s_i)_x \notin m_x \mathcal{L}_x\}$ are affine

(2) $\forall i$ the morphism of rings

$$A\left[\frac{X_j}{X_i}, 0 \le j \le n\right] \longrightarrow \mathcal{O}_X(X_{s_i})$$

$$\frac{X_j}{X_i} \longmapsto \frac{s_j}{s_i}$$

is surjective.

Theorem: X projective $/k$ alg. closed, $\varphi : X \to \mathbb{P}^n_k$ a morphism associated to an invertible sheaf $\mathcal{L}$ and global sections $s_0, \dots, s_n$. Let $V \subset H^0(\mathcal{L})$ be the k-span of $s_0, \dots, s_n$. Then φ is a closed embedding iff

(1) elements of V separate points, i.e., $\forall\ x \neq y \in X$ closed points, $\exists\ s \in V$ s.t. $\underset{ev_x(s)}{\underset{\shortparallel}{s(x)}} = 0$ and $s(y) \neq 0$

(2) elements of V separate tangent vectors, i.e., $\forall\ x \in X$ closed point, the map

$$\{ s \in V \mid s(x) = 0 \} \longrightarrow m_x \mathcal{L}_x / m_x^2 \mathcal{L}_x \cong \text{Zariski cotangent space at } x$$

is surjective.

In the case of curves, the theorem becomes:

Theorem: D divisor on the smooth (nonsingular) projective curve X over k. Then D is very ample iff $\forall$ $x, y \in X$ (closed, distinct or not),

$$h^0(D-x-y) = h^0(D) - 2, \text{ or}$$

$$\dim|D-x-y| = \dim|D| - 2.$$

Proof: We are looking at the rational map

$$\varphi : X \dashrightarrow \mathbb{P}^n_k = |D|^* = \mathbb{P}H^0(D)^*$$

associated to sections $s_0, \dots, s_n \in H^0(D)$ which form a basis.

φ is a morphism iff $\forall x \in X$ $h^0(D-x) = h^0(D) - 1$.

The morphism is a closed embedding if it separates points and tangent vectors. Choose $x \in X$

Recall the cohomology sequence:

$$0 \to H^0(D-x) \to H^0(D) \xrightarrow{ev_x} k \to H^1(D-x) \to H^1(D) \to 0$$

We see that the sections of $\mathcal{O}_X(D)$ vanishing at x can be identified with the sections of $\mathcal{O}_X(D-x)$.

For $y \in X$, $y \neq x$, $s \in H^0(D)$ vanishes at y iff $s \in H^0(D-y)$.

So $\exists\ s \in H^0(D)$ s.t. $s(x)=0$ and $s(y) \neq 0$

$$\Longleftrightarrow \quad H^0(D-x) \subset H^0(D), \quad H^0(D-x) \not\subset H^0(D-y) \subset H^0(D)$$

$H^0(D-x)$ and $H^0(D-y)$ are both hyperplanes in $H^0(D)$ (meaning they have dim. 1 less).

So $H^0(D-x) \not\subset H^0(D-y) \iff H^0(D-x) \neq H^0(D-y)$
$\iff H^0(D-x) \cap H^0(D-y)$ has codim. 2 in $H^0(D)$

We can identify $H^0(D-x) \cap H^0(D-y)$ with $H^0(D-x-y)$
(exercise: use a sequence similar to $0 \to \mathcal{O}_X(-x) \to \mathcal{O}_X \to \mathcal{O}_x \to 0$)

So $\exists\ s$ with $s(x)=0,\ s(y)\neq 0 \iff h^0(D-x-y) = h^0(D)-2$

Separating tangent vectors means $\qquad \mathcal{L} = \mathcal{O}_X(D)$

$$H^0(D-x) = \{ s \in H^0(D) \mid s(x) = 0 \} \xrightarrow[\text{evaluation}]{} \mathfrak{m}_x \mathcal{L}_x / \mathfrak{m}_x^2 \mathcal{L}_x \cong k$$