R local ring with maximal ideal m.

The completion of R is $\hat{R} := \varprojlim_n R/m^n$.

When R is a k-algebra, noetherian, regular of dim. n,

then $\hat{R} \cong k[[x_1, \dots, x_n]]$

the ring of formal power series in n variables over k.

The field of fractions of $\hat{R}$ is isomorphic to

$$k((x_1, \dots, x_n))$$

the field of Laurent power series over k. (see comm. alg. books)

<u>Residues</u>: X smooth complete curve $/k$ as before.

Recall $\omega_X \cong \Omega^1_{X/k}$ the sheaf of Kähler differentials.

The sheaf $\omega_X \otimes K(X) := \omega_X \otimes_{\mathcal{O}_X} \mathcal{K}_X$ $(\cong \mathcal{K}_X)$ is the sheaf of rational (or meromorphic) differential forms on X.

For a closed point $x \in X$, let $t \in \mathcal{O}_{X,x} \subset K(X)$ be a uniformizer. Then $\widehat{\mathcal{O}}_{X,x} = k[[t]]$, and

$$\omega_{X,x} = \Omega^1_{\mathcal{O}_{X,x}/k} = \mathcal{O}_{X,x}\, dt, \text{ hence}$$

$$\widehat{\omega}_{X,x} = \widehat{\mathcal{O}}_{X,x}\, dt = k[[t]]\, dt$$

and $\widehat{\omega_{X,x} \otimes K(X)} = k((t))\, dt$, and the germ at x of any section of $\omega_X \otimes K(X)$ has a Laurent expansion

$$\left(\frac{a_{-n}}{t^n} + \frac{a_{-n+1}}{t^{n-1}} + \cdots + \frac{a_{-1}}{t} + \sum_{i \geq 0} a_i t^i \right) dt.$$

Definition: The residue of $w \in H^0(\omega_X \otimes K(X))$ at x is the coefficient a_{-1} in the Laurent expansion of ω_x at x. It is denoted $\mathrm{Res}_x(w)$.

One can show that $\mathrm{Res}_x(w)$ is independent of the choice of the uniformizer t. (references in III.7)

As in the case of rational functions, the number of poles (or zeros) of a meromorphic differential form $w \in H^0(\omega_X \otimes K) \cong H^0(\mathcal{K}_X)$ is finite. So $\mathrm{Res}_x(w)$ is nonzero at most for finitely many $x \in X$.

__Theorem__ (Residue theorem):

For all $w \in H^0(\omega_X \otimes K)$:

$$\sum_{x \in X} \mathrm{Res}_x(w) = 0$$

__Proof__: See references in Hartshorne.

__The trace map of Serre Duality for curves:__

We have the exact sequence:

$$0 \to \mathcal{O}_X \to \mathcal{K}_X \to \mathcal{K}_X/\mathcal{O}_X \to 0$$

One can show (exercise):

$$\mathcal{K}_X/\mathcal{O}_X \cong \bigoplus_{x \in X} K/\mathcal{O}_{X,x}$$

where $K/\mathcal{O}_{X,x}$ is the skyscraper sheaf at x with group $K/\mathcal{O}_{X,x}$.

Tensor with ω_X to get the exact sequence

$$0 \to \omega_X \to \omega_X \otimes \mathcal{K}_X \to (\omega_X \otimes \mathcal{K}_X)/\omega_X \to 0$$

with long exact sequence of cohomology:

$$0 \to H^0(\omega_X) \to H^0(\omega_X \otimes \mathcal{K}_X) \to H^0((\omega_X \otimes \mathcal{K}_X)/\omega_X) \to H^1(\omega_X) \to 0$$

Note that $H^1(\omega_X \otimes \mathcal{K}_X) = H^1(\mathcal{K}_X) = 0$.

and $H^1((\omega_X \otimes \mathcal{K}_X)/\omega_X) = 0$ because these are flasque sheaves (exercise).

We have the residue map

$$H^0(\omega_X \otimes \mathcal{K}_X/\omega_X) = \bigoplus_{x \in X} (\omega_{X,x} \otimes K)/\omega_{X,x} \xrightarrow{\sum_{x \in X} \mathrm{Res}_x} k$$

vanishes on the image of $H^0(\omega_X \otimes K)$ by the Residue theorem. So, from the long exact sequence of cohomology, the residue factors through a map

$$H^1(\omega_X) \xrightarrow[\cong]{Tr} k$$

which is the trace map of Serre Duality in this case.

The Hurwitz theorem:

Some preparation:

Definition: Let $f: X \longrightarrow Y$ be a finite morphism of nonsingular curves. Recall $\deg f := [K(X) : K(Y)]$.

For a point $x \in X$, let y be its image in Y.

The ramification index of f at x is

$$e_x := v_x(t_y)$$

where t_y is a uniformizer at y. Note that $e_x \geq 1 \ \forall x$.

We say that f is ramified at x if $e_x \geq 2$, otherwise, we say f is unramified at x.

If char $k = 0$ or if char $k \nmid e_x$, we say the ramification is tame at x, otherwise, we say it is wild.

With the above definition, the map $f^*: \mathrm{Div}(Y) \to \mathrm{Div}(X)$ can be written as $f^*([y]) = \sum_{x \mapsto y} e_x [x]$.

Recall that $f^* \mathcal{O}_Y(D) \cong \mathcal{O}_X(f^* D)$ (exercise) so that f^* on Div induces f^* on Pic.

Def: We say f is separable if $K(Y) \subset K(X)$ is separable.

Proposition: $f: X \to Y$ a finite separable morphism of nonsingular curves. Then we have the exact sequence

$$0 \longrightarrow f^* \Omega^1_{Y/k} \longrightarrow \Omega^1_{X/k} \longrightarrow \Omega^1_{X/Y} \longrightarrow 0$$

Proof: The sequence exists in general (203B) and is exact except possibly on the left.
To see exactness on the left, since both sheaves are invertible, it suffices to see that the map $f^*\Omega^1_Y \to \Omega^1_X$ is nonzero (203B homework). To see this, we localize to

the generic point of X where the stalk of $\Omega^1_{X/Y}$ is $\Omega^1_{K(X)/K(Y)}$.

We have $\Omega^1_{K(X)/K(Y)} = 0$ because $K(X)/K(Y)$ is separable. To check this, write $K(X) = K(Y)[\alpha]$.

α algebraic $/K(Y)$, i.e.,

$$K(X) \cong K(Y)[T] \big/ P(T)$$

T variable
P polynomial with coeff in $K(Y)$

$$\Rightarrow \quad \Omega^1_{K(X)/K(Y)} \cong \left(K(Y)[T] \big/ P(T) \right) dT \Big/ P'(T)\, dT$$

$$\Rightarrow \quad \Omega^1_{K(X)/K(Y)} \cong \frac{K(Y)[T]\, dT}{(P(T)dT,\ P'(T)\, dT)}$$

Since the extension is separable, P & P' are coprime

$$\Rightarrow \quad \Omega^1_{K(X)/K(Y)} = 0$$

Therefore the morphism $f^*\Omega^1_Y \to \Omega^1_X$ is surjective at the generic point, hence not the zero morphism. $\square$

<u>Note</u>: Since $\Omega^1_{X/Y}$ is coherent, the fact that its stalk at the generic point is zero means that $\Omega^1_{X/Y}$ is zero on some open subset of X, i.e., $(\Omega^1_{X/Y})_x$

is nonzero only at finitely many points of X.

So
$$\chi(\Omega^1_{X/k}) = \chi(f^*\Omega^1_{Y/k}) + \chi(\Omega^1_{X/Y})$$
$$= \chi(f^*\Omega^1_{Y/k}) + h^0(\Omega^1_{X/Y})$$

By Riemann-Roch:

$$\deg(\Omega^1_{X/k}) + 1 - g_X = \deg(f^*\Omega^1_{Y/k}) + 1 - g_X + h^0(\Omega^1_{X/Y})$$

$$\Rightarrow \boxed{2g_X - 2 = \deg(f)(2g_Y - 2) + h^0(\Omega^1_{X/Y})}$$

First version of the Hurwitz formula.

Now we make $h^0(\Omega^1_{X/Y})$ more explicit: