Remark: (1) If $\deg D \geq 2g-1$, then $D$ is nonspecial.

$$\deg D \geq 2g-1 \Rightarrow \deg(K_X - D) = \deg K_X - \deg D$$
$$= 2g-2-\deg D \leq -1 < 0$$
$$\Rightarrow h^0(K_X - D) = 0 \text{ by Lemma 3.}$$
$$= h^1(D) \text{ by Serre Duality.}$$

(2) If $D$ is special of degree $2g-2$, then $\mathcal{O}_X(D) \cong \omega_X$.
(apply Lemma 3 to $K_X - D$)

---

Theorem: Kodaira vanishing (see Griffiths-Harris)

$X$ projective nonsingular of dim. $n$ over $\mathbb{C} = k$.
$\mathcal{L}$ ample invertible sheaf on $X$.

The following equivalent statements hold:

(1) $H^i(X, \mathcal{L} \otimes \omega_X) = 0 \quad \forall\, i > 0$

(2) $H^i(X, \mathcal{L}^{-1}) = 0 \quad \forall\, i < n.$

---

<u>Proof for curves:</u> Write $\mathcal{L} = \mathcal{O}_X(D)$. If $D$ is ample, then a multiple is very ample. In particular, $\exists\ m > 0$ s.t. $h^0(mD) \geq 2$.

Lemma 3 $\Longrightarrow \deg D > 0 \Rightarrow \deg(K_D + D) \geq 2g - 1$

$\Rightarrow h^1(K_X + D) = 0$, i.e., $K_X + D$ is nonspecial. $\square$.

We shall soon see that on a curve, a divisor is ample iff it has positive degree.

**Definition:** A variety is a separated integral scheme of finite type $/k$.

Two varieties $X, Y$ are birational if $\exists$ nonempty open sets $U \subset X$, $V \subset Y$ s.t. $U \cong V$.

A variety $X$ is called rational if it is birational to $\mathbb{P}^n$ where $n = \dim X$. (This is equivalent to

$$K(X) \cong K(\mathbb{P}^n) \cong k(x_1, \dots, x_n))$$

**Proposition 3:** Two complete nonsingular curves $X$ and $Y$ are birational iff they are isomorphic.

Proof: Let $U \subset X$ be open such that there is a birational morphism $f: U \longrightarrow Y$.

Then, by Proposition 2, $f$ extends to a morphism

$$\varphi: X \longrightarrow Y.$$

Furthermore, $\varphi$ is not constant, hence surjective and it has degree 1 because $f$ is birational, hence induces an isomorphism

$$U \xrightarrow{\cong} \underset{\text{open}}{V \subset} Y$$

which means

$$K(X) \cong K(U) \cong K(V) \cong K(Y).$$

Therefore $f$ is an isomorphism. $\square$

Proposition 4: A complete nonsingular curve has genus 0 iff it is isomorphic to $\mathbb{P}^1$.

Proof: If $X$ has genus $0$, then Riemann-Roch becomes: $\chi(D) = \deg D + 1$.

Choose two distinct points $P, Q \in X$, then

$$\chi(P-Q) = 1 = h^0(P-Q) - h^1(P-Q)$$

$$\Rightarrow \quad h^0(P-Q) > 0$$

Lemma 3 $\Rightarrow$ $P - Q \overset{\text{lin.}}{\sim} 0$

Choose a rational function $f \in K(X)$ s.t.

$$\mathrm{Div}(f) = P - Q.$$

As in the proof of the Corollary to Prop. 3, $f$ gives a morphism $\varphi$ to $\mathbb{P}^1$ s.t. $\mathrm{Div}(f) = \varphi^*(\{0\} - \{\infty\})$ and $\varphi^*\{0\} = [P]$ $\Rightarrow$ $\deg \varphi = 1 \Rightarrow \varphi$ is an isom. $\square$

Next case: Elliptic curves.

Definition: A smooth (= nonsingular) complete curve is called elliptic if it has genus 1.

Some facts about elliptic curves:

(1) The canonical sheaf has degree $2g-2=0$. Since the space of global sections $H^0(\omega_X)$ has dim. 1, we deduce that $\omega_X = \mathcal{O}_X(K_X)$ is trivial:

$$\omega_X \cong \mathcal{O}_X.$$

(2) Fix a closed point $x_0 \in X$. Then the map

$$X \longrightarrow \mathrm{Pic}^0(X) \subset \mathrm{Pic}(X)$$

subgroup of degree 0 divisors

$$x \longmapsto \mathcal{O}_X(x - x_0)$$

is a bijection.

<u>Proof</u>: First represent any $\mathcal{L} \in \mathrm{Pic}^0 X$ by a divisor $D$ of degree $0$: $\mathcal{L} \cong \mathcal{O}_X(D)$.

For any divisor $D$ of degree $0$, the divisor $D + x_0$ has degree $1 = 2g-1$, hence it is nonspecial, i.e., $h^1(D+x_0) = 0$. Now Riemann-Roch implies $h^0(D+x_0) = 1$, i.e., $H^0(\mathcal{O}_X(D+x_0))$ has dim. $1$.

The divisor of zeros of a nonzero section of $\mathcal{O}_X(D+x_0)$ is the unique effective divisor of degree $1$ linearly equivalent to $D+x_0$, i.e., $\exists!\, x \in X$ s.t.

$$D + x_0 \sim x.$$

$\square$.

Via the bijection $\begin{array}{rcl} X & \longrightarrow & \mathrm{Pic}^0(X) \\ x & \mapsto & \mathcal{O}_X(x - x_0) \end{array}$

we can define a group structure on the set of closed points of $X$. The group structure depends on the choice of $x_0$:

$x_0$ is its origin.

We say that $X$ is a principal homogeneous space or a torsor on $\mathrm{Pic}^0(X)$.

This is similar to the situation of an affine space and its associated vector space.

---

## Residues and the trace map of Serre Duality:

A small reminder about completions: