We have $\mathrm{Div}(f) = \varphi^*(\{0\} - \{\infty\})$

has degree $0 = (\deg \varphi) \deg(\{0\} - \{\infty\})$. $\square$

<u>Exercise</u>: In the proof above, show that

$$\varphi^* \mathcal{O}_{\mathbb{P}^1}(1) \cong \mathcal{O}_X\left(\sum_i m_i \cdot P_i\right) \cong \mathcal{O}_X\left(\sum_j n_j \cdot Q_j\right).$$

<u>More nice results about curves</u>: X nonsingular complete$/k$

<u>Lemma 3</u>: D a divisor on X. If $h^0(D) := \dim_k H^0(D) \neq 0$, then $\deg D \geq 0$. If $h^0(D) \neq 0$ and $\deg D = 0$, then $D \sim 0$, i.e., $\mathcal{O}_X(D) \cong \mathcal{O}_X$.

<u>Proof</u>: If $h^0(D) \neq 0$, then for $s \in H^0(D)$, $s \neq 0$

$\mathrm{Div}(s)$ is an effective divisor, $= \mathrm{Div}(\mathcal{Z}(s))$

$\Rightarrow \quad \deg \mathrm{Div}(s) \geq 0$

$\mathcal{O}_X(D - \mathrm{Div}(s)) \cong \mathcal{O}_X \Rightarrow D - \mathrm{Div}(s)$ is principal

$\Rightarrow \deg D = \deg \mathrm{Div}(s) \geq 0$.

If $\deg D = 0$, then $D \sim \mathrm{Div}(s)$ effective of degree 0

$\Rightarrow \mathrm{Div}(s) = 0 \Rightarrow D \sim 0 \Rightarrow \mathcal{O}_X(D) \cong \mathcal{O}_X$ $\square$.

<u>Definition:</u> For a projective curve, its arithmetic genus is defined to be (X not necessarily nonsingular)

$$p_a := h^1(X, \mathcal{O}_X) := \dim H^1(X, \mathcal{O}_X).$$

The geometric genus is defined to be

$$g := h^0(X, \Omega^1_X).$$

When X is nonsingular, then $\omega_X \cong \Omega^1_X$, and, by Serre duality, $\forall$ quasi-coherent sheaves $\mathcal{F}$:

$$H^0(X, \mathcal{F}) \cong \mathrm{Ext}^1_X(\mathcal{F}, \omega_X)^*$$

$$H^1(X, \mathcal{F}) \cong \mathrm{Ext}^0_X(\mathcal{F}, \omega_X)^* \cong \mathrm{Hom}_X(\mathcal{F}, \omega_X)^*$$

If $\mathcal{F}$ is locally free of finite rank:

$$H^0(X, \mathcal{F}) \cong H^1(X, \omega_X \otimes \mathcal{F}^*)^*$$

$$H^1(X, \mathcal{F}) \cong H^0(X, \omega_X \otimes \mathcal{F}^*)^* .$$

<u>Theorem</u>: Riemann-Roch for curves:

X complete nonsingular curve of genus g.

D a divisor on X. Then

$$\chi(D) := h^0(D) - h^1(D) = \deg D + 1 - g.$$

Proof: Let P be any (closed) point of X.

We show that the theorem is true for D iff it is true for $D-P$. Starting from 0 and adding and subtracting points, this will prove the theorem for all divisor (for $D=0$, $\chi(0) = h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) = 1-g$
$= \deg 0 + 1 - g$).

Recall that the ideal sheaf of $P \hookrightarrow X$ is isom. to $\mathcal{O}_X(-P)$.

This gives us the exact sequence:

$$0 \longrightarrow \mathcal{O}_X(-P) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_P \longrightarrow 0$$

$$\begin{array}{c} \mathcal{O}_P \\ \| \\ i_* \mathcal{O}_P \end{array}$$

(abuse of notation always done $\quad i : P \hookrightarrow X$)

twist by $\mathcal{O}_X(D)$ to obtain:

$$0 \longrightarrow \mathcal{O}_X(D-P) \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathcal{O}_P(D) \longrightarrow 0$$

Note that $\mathcal{O}_P(D) \cong \mathcal{O}_P$ because $P \cong \operatorname{Spec} k$

= skyscraper sheaf supported at P with group k.

From the long exact sequence of cohomology, we obtain: $\chi(D-P) = \chi(D) - 1$.

Since we also have $\deg(D-P) = \deg D - 1$,

we obtain $\chi(D-P) - \deg(D-P) = \chi(D) - \deg D$,

and we are done. $\square$

<u>Corollary</u>: The degree of the canonical sheaf $\omega_X = \Omega^1_X$ is $2g-2$.

Proof: Let K_X be a divisor s.t. $\mathcal{O}_X(K_X) \cong \omega_X$.

Then $\chi(K_X) = \deg K_X + 1 - g$

$$\begin{aligned} \chi(K_X) &= h^0(K_X) - h^1(K_X) = h^0(\omega_X) - h^1(\omega_X) \\ &\text{(Serre Duality)} = \underbrace{h^1(\mathcal{O}_X)}_{= g} - h^0(\mathcal{O}_X) = g-1 \end{aligned}$$

$$\Rightarrow \deg K_X = g-1 + g-1 = 2g-2 \qquad \square$$

Def: A canonical divisor is a divisor K_X s.t.

$$\mathcal{O}_X(K_X) \cong \omega_X \cong \Omega^1_X.$$

Definition: We say that D is special if $h^1(D) \neq 0$, and D is nonspecial if $h^1(D) = 0$.

Remark: (1) If $\deg D \geq 2g-1$, then D is nonspecial.

$$\deg D \geq 2g-1 \implies \deg(K_X - D) = \deg K_X - \deg D$$
$$= 2g-2-\operatorname{Deg} D \leq -1 < 0$$

$\Rightarrow h^0(K_X - D) = 0$ by Lemma 3.

$= h^1(D)$ by Serre Duality.

(2) If D is special of degree $2g-2$, then $\mathcal{O}_X(D) \cong \omega_X$.

(apply Lemma 3 to $K_X - D$)

Theorem: Kodaira vanishing (see Griffiths-Harris)

X projective nonsingular of dim. n over $\mathbb{C} = k$.

$\mathcal{L}$ ample invertible sheaf on X.