$$f^*: \mathrm{Cl}(Y) \longrightarrow \mathrm{Cl}(X).$$

This coincides with pull-backs of invertible sheaves and the pull-back $f^*: \mathrm{CaCl}(Y) \longrightarrow \mathrm{CaCl}(X)$ under the identifications $\mathrm{Cl}(X) = \mathrm{CaCl}(X) = \mathrm{Pic}(X)$.

(2) <u>Proposition 3</u>: Suppose $f: X \longrightarrow Y$ is a finite morphism of nonsingular curves. Then, for any divisor D on Y, we have $\deg(f^*D) = (\deg f)(\deg D)$,

where $\deg f := \deg K(X)/K(Y)$

and $\deg\left(\sum n_P[P]\right) = \sum n_P$

<u>Proof</u>: By linearity, we need to show that, $\forall$ closed $Q \in Y$,

$$\deg f^*Q = \deg f.$$

Let $V = \operatorname{Spec} B$ be an open neighborhood of Q, and put $U = \operatorname{Spec} A := f^{-1}(V)$.

Repeating the argument in the proof of Prop. 2, we know that A is the integral closure of B in $K(X)$.

Let $\mathfrak{m}$ be the maximal ideal of B corresponding to Q. Localize A and B at $\mathfrak{m}$:

$$\mathcal{O}_{Y,Q} = B_{\mathfrak{m}} \qquad A' := A \otimes_B B_{\mathfrak{m}}$$

Since B_m is a DVR (in particular it is a PID) and A' is torsion-free, we obtain that A' is a free B_m-module of finite rank. The rank of A' over B_m is $r := \deg f$ because if we further localize B_m and A' at all nonzero elements of B_m we obtain the field extension $K(Y) \subset K(X)$. (the localization of A' is a finite ring extension of $K(Y)$ which is an integral domain $\underset{\text{comm. alg.}}{\Rightarrow}$ it is a field $= K(X) = \mathrm{Frac}(A')$).

Furthermore, if t is a uniformizer at Q, then

$$A'/tA' \text{ is a vector space over } \mathcal{O}_{Y,Q}/t\mathcal{O}_{Y,Q} = k(Q) = k$$

of dim. r.

The points P_i of X mapping to Q are in one-to-one correspondence with the maximal ideals of A pulling back to m, i.e., containing $m \subset B \subset A$, these are in bijection with the maximal ideals m_i of A'.

<u>Now:</u> $A' = \bigcap_i A'_{m_i}$ because A' is an integrally closed domain.

$$\Rightarrow \quad tA' = \bigcap_i (tA'_{m_i} \cap A')$$

By the Chinese remainder theorem:

$$A'/tA' \cong \bigoplus_i A'/tA'_{m_i} \cap A' = \bigoplus_i A'_{m_i}/tA'_{m_i}$$

(map $A' \longrightarrow A'_{m_i} \twoheadrightarrow A'_{m_i}/tA'_{m_i}$ and factor through $tA'_{m_i} \cap A'$ to get the second isom.)

$$= \bigoplus_i \mathcal{O}_{P_i} / t\,\mathcal{O}_{P_i}$$

$$\Rightarrow \deg f = r = \operatorname{rank} A' = \dim_k A'/tA' = \sum_i \dim \mathcal{O}_{P_i}/t\,\mathcal{O}_{P_i}$$

$$= \sum_i \nu_{P_i}(t)$$

$$= \deg f^* Q \text{ by def.}$$

$\square$.

<u>Corollary</u>: The degree of a principal divisor is 0 (on a complete nonsingular curve).

<u>Proof</u>: Let $f \in K(X)$. If $f \in k$, then $\operatorname{Div}(f) = 0$ and we are done. If $f \notin k$, then, since k is algebraically closed, f is transcendental over k, hence

f defines a nonconstant morphism $\varphi : X \to \mathbb{P}^1$ as follows:

Write $\mathrm{Div}(f) = \sum_i m_i P_i - \sum n_j Q_j$

where all the integers m_i and n_j are nonnegative and $\{P_i\} \cap \{Q_j\} = \emptyset$.

Put $U := X \setminus \{Q_j\}$, $U' := X \setminus \{P_i\}$.

$\forall\ P \in U$, $f \in \mathcal{O}_{X,P} \Rightarrow f \in \mathcal{O}_X(U) = \bigcap_{P \in U} \mathcal{O}_{X,P}$.

$\Rightarrow \exists$ morphism $\varphi_U : U \longrightarrow \mathbb{A}^1$

defined by $\mathcal{O}_X(U) \hookleftarrow k[x]$

$$f \longmapsfrom x$$

Similarly define $\varphi_{U'} : U' \longrightarrow \mathbb{A}^1$ using $\frac{1}{f}$

These glue to $\varphi : X \longrightarrow \mathbb{P}^1$. $\quad X = U \cup U'$

We have $\mathrm{Div}(f) = \varphi^*(\{0\} - \{\infty\})$

has degree $0 = (\deg \varphi) \deg(\{0\} - \{\infty\})$. $\square$

Exercise: In the proof above, show that

$$\varphi^* \mathcal{O}_{\mathbb{P}^1}(1) \cong \mathcal{O}_X\Big(\sum_i m_i \cdot P_i\Big) \cong \mathcal{O}_X\Big(\sum_j n_j \cdot Q_j\Big).$$

More nice results about curves: X nonsingular complete$/k$

Lemma 3: D a divisor on X. If $h^0(D) := \dim_k H^0(D) \neq 0$, then $\deg D \geq 0$. If $h^0(D) \neq 0$ and $\deg D = 0$, then $D \sim 0$, i.e., $\mathcal{O}_X(D) \cong \mathcal{O}_X$.

Proof: If $h^0(D) \neq 0$, then for $s \in H^0(D)$, $s \neq 0$

$\mathrm{Div}(s)$ is an effective divisor, $= \mathrm{Div}(\mathcal{Z}(s))$