Injectivity of $\varphi : U \to \operatorname{Spec} A$

Given $x, y \in U$ s.t. $\mathfrak{m}_x \cap A = \mathfrak{m}_y \cap A$

we need to show $x = y$.

Put $\mathfrak{m} := \mathfrak{m}_x \cap A = \mathfrak{m}_y \cap A$.

Then we also have $A_{\mathfrak{m}} = \mathcal{O}_{X,x} = \mathcal{O}_{X,y} \subset K(X)$

Lemma 2: Suppose X is a quasi-projective variety $/k$.
Given $\underset{\text{closed}}{x, y} \in X$, if $\mathcal{O}_{X,x} = \mathcal{O}_{X,y} \subset K(X)$, then $x = y$.

Proof: Choose an embedding $X \hookrightarrow \mathbb{P}^n_k$.
Replace X with its closure. Now $x, y \in X \underset{\text{closed}}{\subset} \mathbb{P}^n_k$

After possibly a linear change of coordinates, we can assume $x, y \in U_0 = D_+(X_0)$.

Now replace X with $X \cap V_0$. So we can assume X is a closed subscheme of $\mathbb{A}^n_k$.

Then $X = \operatorname{Spec} k[x_1, \dots, x_n]/I_X$

$x, y \longleftrightarrow$ maximal ideals M_x, M_y of

$A(X) := k[x_1, \dots, x_n]/I_X$

Now $\mathcal{O}_{X,x} = A_{M_x} = \mathcal{O}_{X,y} = A_{M_y}$

$\Rightarrow \quad M_x = A \cap M_x A_{M_x} = A \cap M_y A_{M_y} = M_y$

$\Rightarrow \quad x = y.$ $\square$

<u>Back to the proof of Proposition 2:</u>

We already proved $\varphi : U \longrightarrow \operatorname{Spec} A$ is a bijection.

$\Rightarrow$ homeo.

$$\varphi^{\#} : \mathcal{O}_{\operatorname{Spec} A} \longrightarrow \varphi_* \mathcal{O}_U$$

is an isomorphism of sheaves on the same topological space because it is an isom. on all local rings.

It remains to prove that the extension of rings $B \subset A$ is finite (or integral because here A and B are both finitely generated k-algebras).

$\forall$ valuation ring R of $K(X)$ containing B, $\exists\, x \in X$ whose local ring is R. $\Rightarrow f(x) \in \operatorname{Spec} B = V$

$\Rightarrow$ The ring R also contains A. (because $\operatorname{Spec} A = f^{-1}(\operatorname{Spec} B)$)

$\Rightarrow$ Every valuation ring of $K(X)$ containing B, also contains A.

From commutative algebra:

$$A \subset \bigcap_{R \supset B} R \subset K(X)$$

is the integral closure of B in $K(X)$

$\Rightarrow$ $B \subset A$ is integral $\Rightarrow$ finite.

In fact, since A is integrally closed, A is the integral closure of B in $K(X)$. $\square$

Divisors on curves. A Weil divisor is a linear combination of points: $W = \sum_{i=1}^{m} n_i [P_i]$

Definition: Assume $f: X \to Y$ is a finite morphism of nonsingular curves over k.

We define a homomorphism of groups

$$f^*: \mathrm{Div}(Y) \longrightarrow \mathrm{Div}(X)$$

as follows.

For any closed point $Q \in Y$, let $t \in \mathcal{O}_{Y,Q} \subset K(Y) \subset K(X)$ be a uniformizer or local parameter, i.e., a generator of the maximal ideal $m_Q \subset \mathcal{O}_{Y,Q}$. Define

$$f^*[Q] := \sum_{f(P)=Q} v_P(t)\,[P] \quad \in \mathrm{Div}(X)$$

and extend by linearity to $\mathrm{Div}(Y)$.

The definition makes sense because

(1) The sum is finite: we saw earlier that finite morphisms are quasi-finite, i.e., every point of Y has

finitely many preimages.

(2) the definition does not depend on the choice of uniformizer $t \in \mathfrak{m}_Q$, because:

given another uniformizer s, $\exists$ unit a of $\mathcal{O}_{Y,Q}$

s.t. $t = as$ $\qquad f^{\#}: \mathcal{O}_{Y,Q} \to \mathcal{O}_{X,P} \quad \forall\ P \mapsto Q$

$$v_P(t) = v_P(a) + v_P(s)$$

$v_P(a) = 0$ because units map to units.

Some nice properties of f^*:

(1) <u>Exercise:</u> Pull-backs of principal divisors are again principal.

So we have a well-defined homomorphism of groups

$$f^*: Cl(Y) \longrightarrow Cl(X).$$

This coincides with pull-backs of invertible sheaves and the pull-back $f^*: CaCl(Y) \longrightarrow CaCl(X)$ under the identifications $Cl(X) = CaCl(X) = Pic(X)$.

(2) <u>Proposition 3</u>: Suppose $f: X \longrightarrow Y$ is a finite morphism of nonsingular curves. Then, for any divisor D on Y, we have $\deg(f^*D) = (\deg f)(\deg D)$,

where $\deg f := \deg K(X)/K(Y)$

and $\deg\left(\sum n_p[p]\right) = \sum n_p$

<u>Proof</u>: By linearity, we need to show that, $\forall$ closed $Q \in Y$,

$$\deg f^*Q = \deg f.$$

Let $V = \operatorname{Spec} B$ be an open neighborhood of Q, and put $U = \operatorname{Spec} A := f^{-1}(V)$.

Repeating the argument in the proof of Prop. 2, we know that A is the integral closure of B in $K(X)$.

Let m be the maximal ideal of B corresponding to Q. Localize A and B at m:

$$\mathcal{O}_{Y,Q} = B_m \qquad A' := A \otimes_B B_m$$