Proposition 1: X a curve$/k$. X nonsingular.
Then X is complete if and only if it is projective.

Proof. We already know that if X is projective, then it is complete.

Now assume X is complete.

Let $V_1 = \operatorname{Spec} A_1, \ldots, V_n = \operatorname{Spec} A_n$ be an open affine cover of X.

Each A_i is a finitely generated algebra over k.

So it has a presentation, say

$$A_i = k[x_1, \ldots, x_m] / I$$

$$\Rightarrow \quad \operatorname{Spec} A_i \xrightarrow[\text{closed}]{} \operatorname{Spec} k[x_1, \ldots, x_m] = \mathbb{A}^m_k$$

We identify $\mathbb{A}^m$ with $U_0 \hookrightarrow \mathbb{P}^m$.

Let Y_i be the closure of the image of V_i in $\mathbb{P}^m$.

We put the reduced induced scheme structure on Y_i.

Y_i is an integral projective curve.

$$X \hookleftarrow V_i \hookrightarrow Y_i \hookrightarrow \mathbb{P}^m$$

We have the rational map $X \dashrightarrow Y_i$ well-defined on V_i

Because X is nonsingular, the rational map extends to a morphism $X \to Y_i$.

<u>Exercise</u>: Check that the product morphism

$$X \longrightarrow \prod_{i=1}^{m} Y_i$$ is a closed embedding

(hint: X complete $\Rightarrow$ image of X in $\prod Y_i$ is complete (homework from last quarter)

$\Rightarrow$ image of X is a closed subscheme of $\prod Y_i$

then show the morphism on sheaves of rings is surjective)

Now
$$X \hookrightarrow \prod_{i=1}^{n} Y_i \hookrightarrow \prod_{i=1}^{n} \mathbb{P}^n \xhookrightarrow[\text{Segre embedding}]{} \mathbb{P}^N$$

all the maps are closed embeddings, so X is projective.

$\square$.

<u>Proposition 2:</u> X complete nonsingular curve. Y a nonsingular curve. $f: X \to Y$ a morphism. Then either $f(X) \subset Y$ is a point, or $f(X) = Y$.

In the case where $f(X) = Y$, $K(Y) \hookrightarrow K(X)$ and $K(X)$ is a finite extension of $K(Y)$.

Furthermore, f is a finite morphism and Y is complete.

Proof: Since X is complete, $f(X) \subset Y$ is closed in Y.

Since X is irreducible, $f(X)$ is also irreducible.

Since Y is irreducible of dim. 1, either $f(X)$ is a point or $f(X) = Y$.

Assume $f(X) = Y$. The image of X is complete because X is complete. So Y is complete.

Since f is dominant, it sends the generic point of X to the generic point of Y. So $K(Y) \hookrightarrow K(X)$.

The extension $K(Y) \subset K(X)$ is finite because:
both fields are finitely generated extensions of k of the same transcendence degree.

It remains to prove that f is a finite morphism.

Let $V = \operatorname{Spec} B \subset Y$ be open.
Put $U := f^{-1}(V) \subset X$, and $A := \mathcal{O}_X(U)$.
We will show $U \cong \operatorname{Spec} A$ and that the ring hom. $B \to A$ is a finite extension of rings.

$f|_U : U \to V$ dominant $\Rightarrow$ $B \hookrightarrow A$.

X is integral and nonsingular $\Rightarrow$ A is an integrally closed domain.

Since X is complete (i.e., proper/k), the valuative criterion implies that $\forall R \subset K(X)$ valuation ring, $\exists x \in X$ s.t. R dominates $\mathcal{O}_{X,x}$.

Since X is nonsingular and x is a closed point, (the local ring of the generic point is $K(X)$)

$\mathcal{O}_{X,x}$ is a regular local ring of dim 1, i.e., a DVR

$\Rightarrow \quad \mathcal{O}_{X,x} = R$. ($\Rightarrow$ every valution ring of $K(X)$ is a DVR)

Since X is proper over k and Y is separated over k, (in fact Y is complete) we deduce (II.4) that X is proper over Y.

Also recall that properness is local on the base.
So $U \longrightarrow V$ is also proper.

From commutative algebra:

Every integrally closed domain is the intersection of the valuation rings of its field of fractions that contain it:

So B is the intersection of the valuation rings of $K(Y)$ that contain it. Similarly for $A \subset K(X)$.

Consider $\quad Id : A \longrightarrow \mathcal{O}_X(U) = A$.

This induces

$$\begin{tikzcd} \varphi : U \arrow[r] \arrow[d, "f"'] & \operatorname{Spec} A \arrow[d] \\ V & = \operatorname{Spec} B \end{tikzcd}$$

<u>Injectivity</u> of φ: Given a maximal ideal m of A,

Let $n \subset B$ be the inverse image of m.

B_x is a DVR containing B. $x \leftrightarrow y \in Y$

Because $U \to V$ is proper, $\exists\, x \in U$ s.t. $f(x) = y$

and we have a lift $\operatorname{Spec} A_m \to U$ with

x = image of closed point of $\operatorname{Spec} A_m$ s.t. the diagram

$$
\begin{tikzcd}
\operatorname{Spec} K(X) \arrow[r] \arrow[d] & U \arrow[d] \\
\operatorname{Spec} A_m \arrow[r] \arrow[ur] \arrow[d] & V \\
\operatorname{Spec} B_x \arrow[ur] &
\end{tikzcd}
$$

commutes.