

Sheaf vanishing:

Theorem: X a projective scheme over a noetherian ring A ,

$\mathcal{O}_X(1)$ a very ample invertible sheaf on X .

Then for any coherent sheaf on X :

(1) $H^i(X, \mathcal{F})$ is a finitely generated A -module $\forall i$.

(2) $\exists n_0 \in \mathbb{Z}$ (depending on \mathcal{F}) s.t. $\forall n \geq n_0, \forall i > 0$,

$$H^i(X, \mathcal{F}(n)) = 0 \quad \text{where } \mathcal{F}(n) := \mathcal{F} \otimes \mathcal{O}_X(n) \\ := \mathcal{F} \otimes \mathcal{O}_X(1)^{\otimes n}$$

Proof: Let $i: X \hookrightarrow \mathbb{P}_A^n$ be an embedding such that

$\mathcal{O}_X(1) \cong i^* \mathcal{O}_{\mathbb{P}^n}(1)$. Since X is projective/ A , it is proper

over A . $\Rightarrow i$ is proper \Rightarrow image of i is closed

$\Rightarrow i$ is a closed embedding.

$\Rightarrow i_* \mathcal{F}$ and \mathcal{F} have the same cohomology.

we also know $i_* \mathcal{F}$ is coherent.

So we can assume $X = \mathbb{P}^n_A$.

If $\mathcal{F} = \mathcal{O}_X(n)$ for some n , then the theorem follows from the previous theorem.

$\mathcal{O}_X(1)$ is very ample \Rightarrow ample $\Rightarrow \exists m \in \mathbb{Z}$ s.t.

$\mathcal{F}(m)$ is generated by global sections.

this means $H^0(X, \mathcal{F}(m)) \otimes_A \mathcal{O}_X \twoheadrightarrow \mathcal{F}(m)$

twist by $\mathcal{O}_X(-m)$ to obtain

$H^0(X, \mathcal{F}(m)) \otimes_A \mathcal{O}_X(-m) \twoheadrightarrow \mathcal{F}$

take $\mathcal{E} := A^{\oplus l} \otimes \mathcal{O}_X(-m)$ where $A^{\oplus l} \twoheadrightarrow H^0(X, \mathcal{F}(m))$

We have the exact sequence

$$0 \rightarrow \mathcal{R} \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow 0$$

where \mathcal{R} is, by def, the kernel of $\mathcal{E} \rightarrow \mathcal{F}$.

We have the long exact sequence:

$$\dots \rightarrow H^i(X, \mathcal{E}) \rightarrow H^i(X, \mathcal{F}) \rightarrow H^{i+1}(X, \mathcal{R}) \rightarrow \dots$$

$$\cong H^i(X, \mathcal{O}_X(-n))^{\oplus d}$$

Now we use descending induction on i .

$$H^i(X, \mathcal{F}) = H^i(X, \mathcal{R}) = 0 \quad \forall i > n$$

($\dim X = n$ or X has a covering by $n+1$ affine open sets and X is separated)

\Rightarrow (1)

For (2) twist by n :

$$0 \rightarrow \mathcal{R}(n) \rightarrow \mathcal{E}(n) \rightarrow \mathcal{F}(n) \rightarrow 0$$

The long exact sequence:

$$\dots \rightarrow H^i(\mathcal{E}(n)) \rightarrow H^i(\mathcal{F}(n)) \rightarrow H^{i+1}(\mathcal{R}(n)) \rightarrow \dots$$

again use descending induction on i . □

Cohomological criterion for ampleness:

Theorem: X a proper scheme over A noetherian, \mathcal{L} an invertible sheaf on X . The following are equivalent:

- (1) \mathcal{L} is ample.
- (2) $\forall \mathcal{F}$ coherent on X , $\exists n_0 \in \mathbb{Z}$, s.t. $\forall i > 0$ and $\forall n \geq n_0$, $H^i(X, \mathcal{F} \otimes \mathcal{L}^n) = 0$

Proof: (1) \Rightarrow (2) \mathcal{L} is ample.

$\Rightarrow \exists m > 0$ s.t. \mathcal{L}^m is very ample (1A)

Then, by Serre vanishing, $\exists l_0$ s.t. $\forall n \geq l_0, \forall i > 0$

$$H^i(\mathcal{F} \otimes (\mathcal{L}^m)^n) = 0.$$

$\mathcal{F}_n \quad 0 \leq n \leq m-1$ put $\mathcal{F}_n := \mathcal{F} \otimes \mathcal{L}^n$.

$\forall n \quad 0 \leq n \leq m-1 \quad \exists l_n$ s.t. $\forall n \geq l_n, \forall i > 0$

$$H^i(X, \mathcal{F}_n \otimes (\mathcal{L}^m)^n) = 0$$

Put $\mu = \max\{l_i \mid i=0, m-1\}$, $n_0 := \mu m$

$\forall n \geq n_0$, write $n = qm + r \quad \begin{matrix} 0 \leq r \leq m-1 \\ q \geq \mu \end{matrix}$

$$\Rightarrow H^i(X, \mathcal{F} \otimes \mathcal{L}^n) = H^i(X, \mathcal{F}_r \otimes \mathcal{L}^{qm}) = 0 \quad \forall i > 0$$

(2) \Rightarrow (1)

We will prove that for all coherent \mathcal{F} , $\exists n_0 \in \mathbb{Z}$,
s.t. $\forall n \geq n_0$, $\mathcal{F} \otimes \mathcal{L}^n$ is generated by global
sections.

Let $p \in X$ be a closed point. We have the exact
sequence: $\{p\} \hookrightarrow X$ one point closed ^{reduced} subscheme.

$$0 \longrightarrow \mathcal{I}_p \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_p \longrightarrow 0$$

tensor with \mathcal{F} $\mathcal{O}_p \cong k(p)$ at p

$$\mathcal{F} \otimes \mathcal{I}_p \longrightarrow \mathcal{F} \longrightarrow \mathcal{F} \otimes \mathcal{O}_p \longrightarrow 0$$

put $\mathcal{Y}_p \mathcal{F} = \text{image of } \mathcal{F} \otimes \mathcal{I}_p \text{ in } \mathcal{F}$ to obtain the

exact sequence $0 \longrightarrow \mathcal{Y}_p \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F} \otimes \mathcal{O}_p \longrightarrow 0$.

twist with \mathcal{L}^n :

$$0 \rightarrow \mathcal{I}_p \otimes \mathcal{F} \otimes \mathcal{L}^n \rightarrow \mathcal{F} \otimes \mathcal{L}^n \rightarrow \mathcal{F} \otimes \mathcal{L}^n \otimes \mathcal{O}_p \rightarrow 0$$

$$\exists n_{0,p} \in \mathbb{Z} \text{ s.t. } \forall n \geq n_{0,p} \quad H^1(\mathcal{I}_p \otimes \mathcal{F} \otimes \mathcal{L}^n) = 0$$

(\mathcal{I}_p is coherent $\Rightarrow \mathcal{I}_p \otimes \mathcal{F}$ is coherent $\Rightarrow \mathcal{I}_p \otimes \mathcal{F}$ is coherent)

\Rightarrow the long exact sequence of cohomology gives a surjection

$$H^0(X, \mathcal{F} \otimes \mathcal{L}^n) \twoheadrightarrow H^0(\mathcal{F} \otimes \mathcal{L}^n \otimes \mathcal{O}_p)$$

for $n \geq n_{0,p}$

$\mathcal{F} \otimes \mathcal{L}^n \otimes \mathcal{O}_p$ is a skyscraper sheaf supported at p .

\Rightarrow we can identify $H^0(\mathcal{F} \otimes \mathcal{L}^n \otimes \mathcal{O}_p) \cong (\mathcal{F} \otimes \mathcal{L}^n \otimes \mathcal{O}_p)_p$
stalk at p

We have $(\mathcal{F} \otimes \mathcal{L}^n \otimes \mathcal{O}_p)_p = \frac{(\mathcal{F} \otimes \mathcal{L}^n)_p}{\mathcal{M}_p(\mathcal{F} \otimes \mathcal{L}^n)_p}$

So we can write the surjection from the previous page as:

$$H^0(X, \mathcal{F} \otimes \mathcal{L}^n) \twoheadrightarrow \frac{(\mathcal{F} \otimes \mathcal{L}^n)_p}{\mathcal{M}_p(\mathcal{F} \otimes \mathcal{L}^n)_p}$$

Nakayama's lemma \Rightarrow

$$H^0(X, \mathcal{F} \otimes \mathcal{L}^n) \twoheadrightarrow (\mathcal{F} \otimes \mathcal{L}^n)_p$$

Since \mathcal{F} is coherent, \exists open neighborhood $\mathcal{U}_{p,n}$ of p s.t. $(\mathcal{F} \otimes \mathcal{L}^n)|_{\mathcal{U}_{p,n}}$ is generated by global sections. We need to find a neighborhood which does not depend on n .

Take $\mathcal{F} = \mathcal{O}_X$, then $\exists n_{1,r}$ and a neighborhood V_r of r s.t. $\mathcal{L}^{n_{1,r}}$ is generated by its global sections on V_r .

Also, $\forall r = 0, 1, \dots, n_{1,r} - 1$, \exists neighborhood $U_{r,r}$ of r s.t. $\mathcal{F} \otimes \mathcal{L}^{n_{0,r} + r}$ is generated by global sections on $U_{r,r}$.

Put $V_r := V_r \cap U_{0,r} \cap \dots \cap U_{n_{1,r}-1,r}$.

Then on V_r , all the sheaves $\mathcal{L}^{n_{1,r}}$, $\mathcal{F} \otimes \mathcal{L}^{n_{0,r} + r}$

$r = 0, \dots, n_{1,r} - 1$ are globally generated. Tensor products of globally generated sheaves are also globally

generated. $\forall n \geq n_{0,r}$, $\exists m \geq 0$ and $r \in \{0, \dots, n_{1,r} - 1\}$

s.t. $n = n_{0,r} + r + m n_{1,r}$

$$\text{Then } \mathcal{F} \otimes \mathcal{L}^m = \mathcal{F} \otimes \mathcal{L}^{n_0, t+n} \otimes (\mathcal{L}^{n, t})^m$$

is globally generated on U_μ .

Now cover X by a finite number of open neighborhoods $U_{\mu_1}, \dots, U_{\mu_k}$ and put $n_0 = \max\{n_{0, t_i}\}$

□

Exterior groups and sheaves :

(X, \mathcal{O}_X) a ringed space. \mathcal{F} an \mathcal{O}_X -module
 The functors $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \cdot)$ and $\text{Hom}_{\mathcal{O}_X}(\cdot, \mathcal{F})$ are
 left exact covariant functors from the category
 $\text{Mod}(\mathcal{O}_X)$ to the categories $\mathcal{A}b$ and $\text{Mod}(\mathcal{O}_X)$.