

\Rightarrow Enough to show the theorem for
finitely generated sheaves.

Now assume \mathcal{F} is a finitely generated sheaf.

By induction on the number of generators, we can
assume it is generated by one element, say $s \in H^0(U, \mathcal{F})$

for some open set $U \subset X$.

Note: If \mathcal{F} is generated by $s_i \in \mathcal{F}(U_i)$,
 $1 \leq i \leq n$, then $\mathcal{F}/\text{image of } i_n! \mathcal{Z}_{U_n} s_n$ is generated
by the images of s_1, \dots, s_{n-1} .

This means we have a surjection

$$i_! \mathcal{Z}_U \twoheadrightarrow \mathcal{F}.$$

$$1 \longmapsto s$$

We can complete this to an exact sequence:

$$0 \rightarrow \mathcal{R} \rightarrow i_! \mathcal{Z}_U \rightarrow \mathcal{F} \rightarrow 0.$$

So we need to show the vanishing of

$$H^k(\text{subsheaves of } i_! \mathbb{Z}_V) \text{ for } k > \dim X.$$

For any nonzero subsheaf \mathcal{R} of $i_! \mathbb{Z}_V$, its stalks are abelian subgroups of \mathbb{Z} , hence generated by positive integers. Let d be the least positive integer occurring in the stalks of \mathcal{R} . Then d lifts to a section of \mathcal{R} on some nonempty open set V , i.e.,

$$j_V^{-1} \mathcal{R} \xrightarrow{\cong} d \mathbb{Z}_V, \text{ where } Z = X \setminus V$$

$$\Rightarrow 0 \rightarrow j_V! j_V^{-1} \mathcal{R} \rightarrow \mathcal{R} \rightarrow j_Z * j_Z^{-1} \mathcal{R} \rightarrow 0$$

$$j_V : V \hookrightarrow X$$

$$j_Z : Z \hookrightarrow X.$$

$$\cong j_V! \mathbb{Z}_V$$

$$H^k(j_Z * j_Z^{-1} \mathcal{R}) = H^k(j_Z^{-1} \mathcal{R}) = 0$$

by induction: $\dim Z < \dim X$

So we only need to prove the vanishing of

$$H^k(i_! \mathbb{Z}_U) \quad \forall k > \dim X$$

and all $U \subset X$
open.

We again have

$$0 \longrightarrow i_! \mathbb{Z}_U \longrightarrow \mathbb{Z}_X \longrightarrow j_* \mathbb{Z}_Y \longrightarrow 0$$

where $Y = X \setminus U$ $i: U \hookrightarrow X$, $j: Y \hookrightarrow X$.

$H^k(\mathbb{Z}_X) = 0$ for $k > 0$ because constant sheaves are flasque, $H^k(j_* \mathbb{Z}_Y) = 0$ for $k > \dim X - 1$

by induction because $\dim Y \leq \dim X - 1$.

The long exact sequence of cohomology now gives:

$$H^k(i_! \mathbb{Z}_U) = 0 \quad \text{for } k > \dim X. \quad \square$$

The cohomology of quasi-coherent sheaves on

Noetherian affine schemes:

$X := \text{Spec } A$ A noetherian ring.

Theorem: For all quasi-coherent sheaves \mathcal{F} on X ,

$$H^i(X, \mathcal{F}) = 0 \quad \forall i > 0.$$

Lemma: (Prop 3.4 in Hartshorne) For any injective A -module I , \tilde{I} is flasque.
(reason: the localization maps $I \rightarrow I[f^{-1}]$ are surjective)

Proof of the theorem: If $M := H^0(X, \mathcal{F})$, then $\mathcal{F} = \tilde{M}$.

Choose a resolution I^\bullet of M by injective A -modules.

Then we have a resolution

$$0 \rightarrow \tilde{M} \rightarrow \tilde{I}^0 \rightarrow \tilde{I}^1 \rightarrow \dots$$

by flasque sheaves of \mathcal{O}_X -modules because the \sim functor is exact with global sections complex:

$$0 \rightarrow M \rightarrow I^0 \rightarrow I^1 \rightarrow \dots$$

which is exact because this is the original resolution. $\Rightarrow H^k(\tilde{M}) = 0$ for $k > 0$ \square .

Some tools for computing cohomology on noetherian schemes: X a noetherian scheme

Lemma: If \mathcal{F} is a quasi-coherent sheaf on X , then \mathcal{F} can be embedded in a flasque, quasi-coherent sheaf.

Proof: Choose a finite affine cover of X :

$$X = \bigcup_{i=1}^n U_i \quad U_i = \text{Spec } A_i.$$

Put $M_i := H^0(U_i, \mathcal{F})$, then $\mathcal{F}|_{U_i} \cong \tilde{M}_i$.

Embed M_i in an injective A_i -module I_i .

$$\tilde{M}_i \hookrightarrow \tilde{I}_i \quad \text{on } U_i = \text{Spec } A_i$$

If $\mu_i: U_i \hookrightarrow X$, then

$$\mathcal{F} \hookrightarrow \bigoplus_{i=1}^n \mu_{i*} \mathcal{F}|_{U_i} \hookrightarrow \bigoplus_{i=1}^n \mu_{i*} \tilde{I}_i$$

$$\left(\begin{array}{l} \mathcal{F}(U) \hookrightarrow \bigoplus_{i=1}^n \mathcal{F}(U \cap U_i) \\ \mathcal{F}_U \end{array} \right)$$

flasque and quasi-coherent.

□.

Corollary: Quasi-coherent sheaves on noetherian schemes have quasi-coherent flasque resolutions.

Levine's theorem: X a noetherian scheme. The

following are equivalent.

(a) X is affine

(b) $H^i(X, \mathcal{F}) = 0 \quad \forall i > 0$ and \mathcal{F} quasi-coherent.

(c) $H^i(X, \mathcal{I}) = 0 \quad \forall i > 0$ and \mathcal{I} coherent sheaf of ideals. (replace with quasi-coherent if X is not noetherian)

Proof: We only need to prove (c) \Rightarrow (a)

We use Ex. II. 2.17 from last quarter:

a scheme X is affine iff $\exists f_1, \dots, f_n \in H^0(X, \mathcal{O}_X)$

generating the unit ideal s.t. $\forall i$

$X_{f_i} := \{x \in X \mid f_i \notin \mathfrak{m}_x\}$ is affine.

Let $P \in X$ be a closed point, U an affine neighborhood of P and put $Y := X \setminus U$. Let \mathcal{I}_P be the skyscraper sheaf with support $\{P\}$ and stalk $k = k(P)$.

We have the exact sequence.

$$0 \longrightarrow \mathcal{I}_{Y \amalg \{P\}} \longrightarrow \mathcal{I}_Y \longrightarrow k_P \longrightarrow 0$$

$Y \amalg \{P\} = Y \cup \{P\}$
 \mathcal{I}_Y sheaf of ideals of Y for the reduced induced scheme structure on Y .

The map: $\mathcal{I}_Y \longrightarrow k_P$:

If $V \not\supseteq P$, then $k_P(V) = 0$ by def.

and $\mathcal{I}_Y(V) \longrightarrow k_P(V)$ is zero.

If $V \supseteq P$, then $k_P(V) = k$ and $U \cap V \ni P$

The map is $\mathcal{I}_Y(V) \longrightarrow \mathcal{I}_Y(U \cap V) = \mathcal{O}_X(U \cap V)$

$\longrightarrow \mathcal{O}_{X,P} \longrightarrow k_P = \mathcal{O}_{X,P}/\mathfrak{m}_P = k_P(V)$

The kernel is $\mathcal{I}_{Y \cup \{P\}}$.

The long exact sequence of cohomology:

$$0 \longrightarrow H^0(\mathcal{I}_{Y \cup \{P\}}) \longrightarrow H^0(\mathcal{I}_Y) \longrightarrow H^0(k_P) \longrightarrow H^1(\mathcal{I}_{Y \cup \{P\}})$$

\parallel \parallel

k_P 0 by (c)

$$\Rightarrow H^0(\mathcal{I}_Y) \twoheadrightarrow k_P$$

In particular, $\exists f \in H^0(\mathcal{O}_Y)$ mapping to $1 \in k_P$. This implies $f_P \notin \mathfrak{m}_P$

$$\Rightarrow P \in X_f := \{x \in X \mid f_x \notin \mathfrak{m}_x\}$$

We also have $X_f = U_f := \{x \in U \mid f_x \notin \mathfrak{m}_x\}$
because $Y = X \setminus U$ and $f \in \mathcal{O}_Y \Rightarrow f_x \in \mathfrak{m}_x \forall x \in Y$
 $\Rightarrow Y \cap X_f = \emptyset$

$$\Rightarrow X_f \subset U.$$

$$\Rightarrow X_f = U_f.$$

Since U is affine, so is $U_f = X_f$.

Since X is noetherian, we can cover X with a finite number, say X_{f_1}, \dots, X_{f_r} of such affine open sets.