<u>Definition</u>: (1) Given an abelian category $\mathcal{A}$, the category $\mathcal{C}(\mathcal{A})$ is the category whose objects are complexes

$$A^\bullet = (\dots, A^{n-1} \longrightarrow A^n \xrightarrow{d^n} A^{n+1} \longrightarrow \dots)$$

of objects and morphisms of $\mathcal{A}$ where $\forall\, n \in \mathbb{Z}$ $\ d^{n+1} d^n = 0$, the morphisms in $\mathcal{C}(\mathcal{A})$ are morphisms of complexes, i.e., $\varphi^\bullet : A^\bullet \longrightarrow B^\bullet$ is the data of a collection of morphisms $\varphi^n : A^n \longrightarrow B^n$ $\forall n$ s.t. $\varphi^{n+1} d^n = d^n \varphi^n$:

$$\begin{tikzcd}
A^{n-1} \arrow[r, "\varphi^{n-1}"'] \arrow[d, "d^{n-1}"'] \arrow[dr, phantom, "\circlearrowleft"] & B^{n-1} \arrow[d, "d^{n-1}"] \\
A^n \arrow[r, "\varphi^n"'] \arrow[d, "d^n"'] \arrow[dr, phantom, "\circlearrowleft"] & B^n \arrow[d, "d^n"] \\
A^{n+1} \arrow[r, "\varphi^{n+1}"'] & B^{n+1}
\end{tikzcd}$$

(2) A homotopy between two morphisms $\varphi^\cdot, \psi^\cdot : A^\cdot \to B^\cdot$ is a collection of maps $h^n : A^n \to B^{n-1}$ s.t.

$$\forall n \qquad \psi^n - \varphi^n = d^{n-1} h^n + h^{n+1} d^n$$

$$\begin{tikzcd}
A^{n-1} \arrow[r] \arrow[d, "d^{n-1}"'] & B^{n-1} \arrow[d, "d^{n-1}"] \\
A^n \arrow[r, "\psi^n - \varphi^n"] \arrow[ur, "h^n"] \arrow[d, "d^n"'] & B^n \arrow[d, "d^n"] \\
A^{n+1} \arrow[r] \arrow[ur, "h^{n+1}"] & B^{n+1}
\end{tikzcd}$$

A morphism is called null-homotopic if it is homotopic to the zero morphism ($\forall$ $A, B \in \mathcal{A}$

$A \to 0 \to B$ is the zero homomorphism).

(3) The homotopy category $\mathcal{K}(\mathcal{A})$ is the category with the same objects as $\mathcal{C}(\mathcal{A})$, and whose morphisms are the homotopy classes of morphisms of complexes.

(homotopy defines an equivalence relation on morphisms in $\mathcal{C}(\mathcal{A})$, $\mathrm{Hom}_{\mathcal{K}(\mathcal{A})}(A^\cdot, B^\cdot) = \mathrm{Hom}_{\mathcal{C}(\mathcal{A})}(A^\cdot, B^\cdot) / \text{homotopy}$)

(4) Given a complex $A^\cdot \in \mathrm{Ob}(\mathcal{C}(\mathcal{A}))$, the i-th cohomology object $h^i(A^\cdot)$ is the quotient

$$h^i(A^\cdot) := \operatorname{Ker} d^i \big/ \operatorname{Im} d^{i-1}$$

Any morphism of complexes $\varphi^\cdot : A^\cdot \to B^\cdot$ induces morphisms $h^i(\varphi^\cdot) : h^i(A^\cdot) \longrightarrow h^i(B^\cdot)$ obtained from the commutativity of

$$\begin{array}{ccc} A^{n-1} & \xrightarrow{\varphi^{n-1}} & B^{n-1} \\ d^{n-1}\downarrow & & \downarrow d^{n-1} \\ A^n & \xrightarrow{\varphi^n} & B^n \\ d^n \downarrow & & \downarrow d^n \\ A^{n+1} & \xrightarrow{\varphi^{n+1}} & B^{n+1} \end{array}$$

Given an exact sequence of complexes:

$$0 \to A^\cdot \to B^\cdot \to C^\cdot \to 0$$

there is a sequence of coboundary morphisms

$\delta^i : h^i(C^\cdot) \longrightarrow h^{i+1}(A^\cdot)$ such that the sequence

$$\cdots \to h^i(A^\cdot) \longrightarrow h^i(B^\cdot) \longrightarrow h^i(C^\cdot) \xrightarrow{\delta^i} h^{i+1}(A^\cdot) \longrightarrow \cdots$$

is exact.

A morphism of complexes is a quasi-isomorphism if it induces isomorphisms in cohomology.

(5) The derived category $\mathcal{D}(\mathcal{A})$ of $\mathcal{A}$ is the "localization" of $\mathcal{H}(\mathcal{A})$ at the "multiplicative set" of quasi-isomorphisms.

This means that a morphism $\Phi : A^\cdot \longrightarrow B^\cdot$ in $\mathcal{D}(\mathcal{A})$ is a finite sequence

$$\begin{tikzcd}[column sep=small]
 & M_0 \arrow[dl, "\text{q.-isom}"'] \arrow[dr] & & M_2 \arrow[dl, "\text{q.i.}"'] \arrow[dr] & & & M_n \arrow[dl, "\text{q.i.}"'] \arrow[dr] & \\
A^\bullet & & M_1 & & M_3 \quad \cdots & M_{n-1} & & B^\bullet
\end{tikzcd}$$

Hence two complexes gives isomorphic objects of $\mathcal{D}(\mathcal{A})$ if and only if they can be connected by a chain of quasi-isomorphisms.

The category $\mathcal{A}$ embeds in $\mathcal{C}(\mathcal{A})$, $\mathcal{H}(\mathcal{A})$, $\mathcal{D}(\mathcal{A})$ by sending an object A to the complex

$$\cdots \longrightarrow 0 \longrightarrow A = A^0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

(6) An injective resolution of $A \in \mathrm{Ob}(\mathcal{A})$ is a complex $I^\bullet$ of injective objects which is 0 in negative degrees, together with a quasi-isomorphism $\varepsilon : A \to I^\bullet$

This means that the sequence

$$0 \longrightarrow A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$$

is exact.

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & I^2 & \longrightarrow & \cdots \\ & & & & \uparrow & & & & & & \\ 0 & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

A projective resolution is the dual notion (reverse all the arrows).

If $\mathcal{A}$ has enough injectives, then every object has an injective resolution.

Choose I^0 s.t. $0 \to A \to I^0$

Choose I^1 s.t. $I^0/A \hookrightarrow I^1$, $I^0 \to I^0/A \to I^1$

" I^2 s.t. $I^1 / \text{image of } I^0 \hookrightarrow I^2 \ldots$

Dually, if $\mathcal{A}$ has enough projectives, then every object has a projective resolution.

Derived functors:

Suppose $\mathcal{A}$ has enough injectives and let $F : \mathcal{A} \to \mathcal{B}$ be a covariant left exact functor to another abelian category.

[F is left exact if, F is additive, i.e.;

$$F_{A,B} : \mathrm{Hom}_{\mathcal{A}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{B}}(F(A), F(B))$$

is a hom. of abelian groups.

and $\forall$ exact sequences $0 \to A \to B \to C \to 0$

$0 \to F(A) \to F(B) \to F(C)$ is exact.]

The right derived functors R^iF of F can be defined as $R^iF(A) := H^i(F(I^\bullet))$ where $I^\bullet$ is an injective resolution of A.

<u>Lemma 1</u>: This is well-defined, i.e., independent of the choice of injective resolution.

Follows from:

<u>Lemma 2</u>: $\forall$ injective resolution $I^\cdot$ of an object B and an arbitrary resolution $C^\cdot$ of an object A, and any morphism $u: A \to B$, $\exists$ $v^\cdot : C^\cdot \to I^\cdot$ s.t.

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[d] \\
C^\cdot \arrow[r, "v^\cdot"] & I^\cdot
\end{tikzcd}$$

commutes.

Furthermore, $v^\cdot$ is unique up to homotopy.

<u>Proof</u>:

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d, "0"] & 0 \arrow[r] \arrow[d, "0"] & A \arrow[r] \arrow[d, "u"] & C^0 \arrow[r] & C^1 \arrow[r] & C^2 \arrow[r] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & B \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & I^2 \arrow[r] & \cdots
\end{tikzcd}$$

$v^{-1} := u$

Suppose we have constructed v^i for $i \leq n-1$

$$\begin{tikzcd}
C^{n-2} \arrow[r, "d"] \arrow[d, "v^{n-2}"'] \arrow[dr, phantom, "\circlearrowleft"] & C^{n-1} \arrow[r, "d"] \arrow[d, "v^{n-1}"] \arrow[dr, phantom, "\circlearrowleft"] & C^n \arrow[d, dashed, "v^n"] \\
I^{n-2} \arrow[r, "d"] & I^{n-1} \arrow[r, "d"] & I^n
\end{tikzcd}$$

$d^{n-1} v^{n-1} d^{n-2} = d^{n-1} d^{n-2} v^{n-2} = 0$ so $d^{n-1} v^{n-1}$ is zero on $\operatorname{im} d^{n-2} = \ker d^{n-1}$, i.e., factors through $C^{n-1}/\ker d^{n-1} \hookrightarrow C^n$.
Since I^n is injective, this factorization, hence also $d^{n-1} v^{n-1}$ extends to $v^n : C^n \longrightarrow I^n$.