$$\Rightarrow \quad K_X|_C \sim K_C - C|_C$$

or $$K_C \sim K_X|_C + C|_C$$

take degrees: $2g-2 = C \cdot (K_X + C)$ $\square$

<u>Theorem</u> (Riemann-Roch): For any divisor D

$$\chi(D) = \chi(\mathcal{O}_X) + \frac{1}{2} D \cdot (D - K_X)$$

<u>Proof</u>: As in the proof of the existence and uniqueness of the intersection pairing, we can find two nonsingular curves C, E s.t. $D \sim C - E$: $\chi(D) = \chi(C-E)$.

For E, we have the usual exact sequence:

$$0 \to \mathcal{O}_X(-E) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_E \longrightarrow 0$$

twist by C to get

$$0 \to \mathcal{O}_X(C-E) \longrightarrow \mathcal{O}_X(C) \to \mathcal{O}_E(C) \longrightarrow 0$$

and $$\chi(C-E) = \chi(C) - \chi(\mathcal{O}_E(C))$$

similarly $$0 \to \mathcal{O}_X(-C) \longrightarrow \mathcal{O}_X \to \mathcal{O}_C \longrightarrow 0$$

twist by C : $$0 \to \mathcal{O}_X \to \mathcal{O}_X(C) \longrightarrow \mathcal{O}_C(C) \longrightarrow 0$$

and $$\chi(C) = \chi(\mathcal{O}_X) + \chi(\mathcal{O}_C(C))$$

and $$\chi(D) = \chi(C-E) = \chi(\mathcal{O}_X) + \chi(\mathcal{O}_C(C)) - \chi(\mathcal{O}_E(C))$$

R.R. for curves and the fact that $\deg \mathcal{O}_C(C) = C^2$, $\deg \mathcal{O}_E(C) = C\cdot E$

$$\Rightarrow \quad \chi(D) = \chi(\mathcal{O}_X) + 1 - g_C + C^2 - 1 + g_E - C\cdot E$$

adjunction : $2g_C - 2 = C\cdot(C+K_X) \Rightarrow g_C = \frac{1}{2} C\cdot(C+K_X) + 1$

$$\Rightarrow \quad \chi(D) = \chi(\mathcal{O}_X) + C^2 - C\cdot E - \frac{1}{2} C\cdot(C+K_X) + \frac{1}{2} E\cdot(E+K_X)$$

$$\chi(D) = \chi(\mathcal{O}_X) + \frac{1}{2}(C-E)\cdot(C-E+K_X)$$

$$= \chi(\mathcal{O}_X) + \frac{1}{2} D\cdot(D+K_X)$$

$\square$

<u>Definition:</u> We say a divisor D is numerically equivalent to 0, written $D \equiv 0$, if $C\cdot D = 0$ for all divisors C on X. Two divisors are numerically equivalent if their difference is numerically equivalent to 0.

We define $\mathrm{Num}(X) := \mathrm{Div}(X) / \equiv$.

__Remark:__ Linear equivalence implies numerical equivalence. $\mathrm{Num}(X)$ is a finitely generated free abelian group of finite rank (Ex. V.1.7, V.1.8)

__Def:__ The rank of $\mathrm{Num}(X)$ is called the Picard number of X, often denoted $\rho = \rho_X$.

__Remark:__ The intersection pairing defines a symmetric bilinear pairing on $\mathrm{Num}(X) \otimes \mathbb{R}$. The Hodge index theorem (below) shows that the signature of the intersection form on $\mathrm{Num}(X) \otimes \mathbb{R}$ is $(1, \rho - 1)$.

__Theorem:__ (The Hodge Index Theorem) Let H be an ample divisor on X and D a divisor on X s.t. $D \not\equiv 0$, $H \cdot D = 0$. Then $D^2 < 0$.

We need some preliminary results:

Lemma 1: Suppose H is an ample divisor on X. Then, for all effective nonzero divisors D on X, $D \cdot H > 0$. In particular, $H^2 > 0$.

Proof: By linearity, we may assume D is integral. Some multiple mH of H is very ample. If D' is the normalization of D, then $\mathcal{O}_{D'}(H)$ defines a morphism $D' \longrightarrow \mathbb{P}^n$ which is an embedding on some open subset of D'. In particular, $\exists$ points $p, q \in D'$ s.t.

$$h^0(mH|_{D'} - p - q) = h^0(mH|_{D'}) - 2$$

In particular $h^0(mH|_{D'}) \geq 2$ and $\deg H|_{D'} > 0$.

Hence $\deg \mathcal{O}_D(H) = \deg \mathcal{O}_{D'}(H) > 0$ and $D \cdot H > 0$

To see $H^2 > 0$, note that nH is effective and $\neq 0$ for $n \gg 0$.

$\square$

<u>Lemma 2:</u> Suppose H is ample on X. For any divisor D, if $D \cdot H > H \cdot K_X$, then $h^0(K_X - D) = h^2(D) = 0$.

<u>Proof:</u> We prove the contrapositive: If $h^0(K_X - D) > 0$, then $K_X - D$ is linearly equivalent to an effective divisor and, by Lemma 1, $(K_X - D) \cdot H \geq 0$, i.e., $K_X \cdot H \geq D \cdot H$.

$\square$

<u>Lemma 3:</u> H ample on X. D a divisor s.t. $D^2 > 0$ and $D \cdot H > 0$. Then $h^0(nD) \longrightarrow +\infty$ as $n \longrightarrow +\infty$.

<u>Proof:</u> By Riemann-Roch:

$$\chi(nD) = \chi(\mathcal{O}_X) + \frac{1}{2} nD\cdot(nD - K_X)$$

$$= \chi(\mathcal{O}_X) + \frac{n^2}{2} D^2 - \frac{n}{2} D\cdot K_X \longrightarrow \infty \text{ with } n$$

because $D^2 > 0$

So $h^0(nD) - h^1(nD) + h^2(nD) \longrightarrow \infty$

Now, for $n \gg 0$, $nD\cdot H > H\cdot K_X$ because $D\cdot H > 0$

and, by Lemma 2, $h^2(nD) = 0$.

Hence $h^0(nD) \geq h^0(nD) - h^1(nD) = \chi(nD) \longrightarrow \infty$.

$\square$

Proof of the Hodge Index Theorem:

Assume $D^2 > 0$ and obtain a contradiction.

For $n \gg 0$ $H' := D + nH$ is very ample.

Furthermore, $D \cdot H' = D^2 > 0$. So, by the Corollary, $h^0(mD)$ goes to ∞ as $m \to \infty$.

In particular $h^0(mD) > 0$ for large m and $|mD| \neq \phi$ is effective $\Rightarrow mD \cdot H > 0$ as we saw in the proof of Lemma.

$(D^2 > 0)$

$\Rightarrow$ $D \cdot H > 0$ contradiction.

Now assume $D^2 = 0$ and look for a contradiction.

Since $D \not\equiv 0$, $\exists$ E s.t. $D \cdot E \neq 0$

Replace E by $-E$ if necessary so that $D \cdot E > 0$.

Replace E by $(H^2)E - (E \cdot H)H$ so that $E \cdot H = 0$.

$\forall n$ $(nD + E) \cdot H = 0$

$$(nD+E)^2 = n^2 D^2 + 2n D \cdot E + E^2$$
$$= 2n D \cdot E + E^2$$

So for $n \gg 0$ $(nD+E)^2 > 0$ (and $(nD+E) \cdot H = 0$)

Now apply the argument in the case $D^2 > 0$ to $nD+E$, $n \gg 0$

to obtain a contradiction. $\square$

Theorem: *The Nakai–Moishezon criterion for ampleness.* D is ample iff $D^2 > 0$ and, for all irreducible curves C on X, $C \cdot D > 0$.

Example:

$$\begin{tikzcd} & \mathbb{P}^1 \times \mathbb{P}^1 =: X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathbb{P}^1 & & \mathbb{P}^1 \end{tikzcd}$$

$\mathrm{Pic}(\mathbb{P}^1) \cong \mathbb{Z}[\mathcal{O}_{\mathbb{P}^1}(1)]$

$\mathrm{Pic}(\mathbb{P}^n) \cong \mathbb{Z}[\mathcal{O}_{\mathbb{P}^n}(1)]$

$$\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{Z} \oplus \mathbb{Z} \quad \text{(homework from last quarter)}$$

$$= \mathbb{Z}[p_1^* \mathcal{O}_{\mathbb{P}^1}(1)] \oplus \mathbb{Z}[p_2^* \mathcal{O}_{\mathbb{P}^1}(1)]$$

$F_1 =$ a fiber of p_1 $\qquad$ $F_2 =$ fiber of p_2

$\mathcal{O}_X(F_1) \cong p_1^* \mathcal{O}_{\mathbb{P}^1}(1)$ $\qquad$ $\mathcal{O}_X(F_2) \cong p_2^* \mathcal{O}_{\mathbb{P}^1}(1)$.

$\forall$ D on X $\qquad$ $D \sim aF_1 + bF_2$ for some $a, b \in \mathbb{Z}$.

D is ample $\iff$ $a > 0$, $b > 0$.

Can compute the genus of $C \sim aF_1 + bF_2$ using adjunction $\left(= (a-1)(b-1)\right)$.

Ruled surfaces:

Definition: A ruled surface is a surface X with a surjective morphism $\pi : X \longrightarrow C$ where C is a curve (projective non singular curve) s.t. all the fibers of π are isomorphic to $\mathbb{P}^1_k$.

Facts about ruled surfaces: One can show that π has a section, i.e., a closed embedding $\sigma : C \hookrightarrow X$ s.t. $\pi \circ \sigma = Id_C$. Also, there exists a locally free sheaf $\mathcal{E}$ of rank 2 on C s.t. $X \cong \mathbb{P}_C(\mathcal{E})$.