

Surfaces :

In this section, by a surface we mean a nonsingular projective surface over an algebraically closed field.

Now divisors are supported on curves.

As usual, the datum of an effective divisor is equivalent to the datum of a closed subscheme of codim 1.

We start by intersecting divisors on a surface:

We define a bilinear pairing on $\mathcal{C}l(X) \cong \text{CaCl}(X) \cong \text{Pic}(X)$ which is an extension of the naive notion of intersecting transverse curves on a surface.

Recall that for two closed subschemes Y and Z of a scheme X , the intersection scheme $Y \cap Z$ is the closed

subscheme with sheaf of ideals $\mathcal{I}_Y + \mathcal{I}_Z \hookrightarrow \mathcal{O}_X$.

Definition: We say that two curves (closed subschemes of codim. 1) C and D on X (a surface) intersect transversely at a (closed) point $x \in C \cap D$ if the local equations of C and D at x generate the maximal ideal \mathfrak{m}_x of $\mathcal{O}_{X,x}$.

This implies, in particular, that C and D are both nonsingular at x (for instance, the local equations of D at x will generate the maximal ideal of $\mathcal{O}_{C,x}$).

We say that C and D are transverse if they are transverse at all of their points of intersection.

Theorem: There exists a unique symmetric bilinear form

$$\text{Div}(X) \times \text{Div}(X) \longrightarrow \mathbb{Z}$$

$$(C, D) \longmapsto C \cdot D$$

such that, if C and D are transverse curves, then

$$C \cdot D = \# C \cap D \text{ and, if } C \sim C' \text{ (linear equivalence)}$$

then, for all $D \in \text{Div}(X)$, $C \cdot D = C' \cdot D$.

Lemma: Suppose $C, D \subset X$ are curves meeting transversely and C is nonsingular. Then

$$\# C \cap D = \deg \mathcal{O}_C(D).$$

Proof: We have the usual exact sequence

$$0 \longrightarrow \mathcal{O}_C(-D) \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_{C \cap D} \longrightarrow 0.$$

$$\quad \quad \quad \parallel$$

$$\quad \quad \quad \mathcal{O}_C(-C \cap D)$$

Since C and D are transverse, their intersection is reduced, hence $\mathcal{O}_{C \cap D}$ has length $\# C \cap D$.

The exact sequence above now gives:

$$\deg \mathcal{O}_C(-D) = -\deg \mathcal{O}_C(D) = -\deg \mathcal{O}_{C \cap D}$$

$$\Rightarrow \deg \mathcal{O}_C(D) = \# C \cap D.$$

□.

If the curves C and D are not necessarily transverse but have no common components, assuming C nonsingular, as in the lemma, we can define:

$$\begin{aligned} C \cdot D &= \deg \mathcal{O}_C(D) = \sum_{x \in C \cap D} \text{length}_x \mathcal{O}_{C \cap D, x} \\ &= \sum_{x \in C \cap D} \text{length}_x (\mathcal{O}_{X, x} / (f_x, g_x)) \end{aligned}$$

where f_x, g_x are the local equations of C and D at x .

Proof of theorem about the existence and uniqueness
of the intersection pairing:

Uniqueness: Choose a very ample divisor H on X , and let $C, D \in \text{Div}(X)$ be arbitrary.
 $\exists m > 0$ s.t. $C + mH$ and $D + mH$ are generated by global sections. By homework from last quarter (Ex. II.7.5), $C + (m+1)H, D + (m+1)H$ are very ample.

Replace H by $(m+1)H$ so that $C + H$ and $D + H$ are very ample.

Now apply Bertini's theorem: $\exists C' \in |C + H|,$

$$D' \in |D+H|, E' \neq F' \in |H|$$

all irreducible and nonsingular.

$$\text{We have } C \sim C' - E' \quad D \sim D' - F'$$

and, putting $l(C \cap D) := \text{length}(C \cap D)$

$$\begin{aligned} C \cdot D &= (C' - E') \cdot (D' - F') \\ &= C' \cdot D' - C' \cdot F' - E' \cdot D' + E' \cdot F' \end{aligned}$$

$$\textcircled{A} \quad C \cdot D = l(C \cap D') - l(C' \cap F') - l(E' \cap D') + l(E' \cap F')$$

which proves uniqueness.

To prove the existence, we take formula \textcircled{A} as our definition for $C \cdot D$ and show that it is independent

of the choices of the curves C', D', E', F' and the ample divisor H .

Independence of the choice of C', D', E', F' for H fixed:

For another choice of curves C'', D'', E'', F'' (similar to C', D', E', F') we need to show the intersection numbers are the same.

For instance, we show $l(C' \cap D') = l(C'' \cap D'')$:

this follows from the lemma via

$$l(C' \cap D') = \deg \mathcal{O}_{C'}(D') = \deg \mathcal{O}_{C'}(D'') = C' \cdot D''$$

$$= \deg \mathcal{O}_{D''}(C') = \deg \mathcal{O}_{D''}(C'') = C'' \cdot D'' = l(C'' \cap D'').$$

Now we show independence of the choice of H :

For another very ample divisor H' , choose curves

$$C'' \in |C + H'|, \quad D'' \in |D + H'|, \quad E'' \neq F'' \in |H'|.$$

(after possibly replacing H' by a multiple)

non singular, irred.

We need to show:

$$\begin{aligned} & l(C' \cap D') - l(C' \cap F') - l(E' \cap D') + l(E' \cap F') \\ = & l(C'' \cap D'') - l(C'' \cap F'') - l(E'' \cap D'') + l(E'' \cap F'') \end{aligned}$$

$$\text{We have } C \sim C' - E' \sim C'' - E''$$

$$D \sim D' - F' \sim D'' - E''$$

via the Lemma, we obtain

$$C \cdot (D' - F') = C \cdot (D'' - F'') \text{ etc.}$$

exercise: finish the proof.

□.

Def: The self-intersection of any divisor is defined as

$$D^2 := D \cdot D. \quad \text{If } D \text{ is a curve, i.e., an}$$

effective divisor, this is equal to $\deg \mathcal{O}_D(D)$.

Example: $X = \mathbb{P}^2$. $L \subset \mathbb{P}^2$ a line.

$$\text{Then } L^2 = \deg \mathcal{O}_L(L) = \deg \mathcal{O}_L(1) = \deg \mathcal{O}_{\mathbb{P}^1}(1) = 1$$

$$\text{a} \quad = \# L \cap L' \quad L' \text{ a line } \neq L.$$

If C is a curve of degree d in \mathbb{P}^2 .

C is the scheme of zeros of $P \in H^0(\mathcal{O}_{\mathbb{P}^2}(d))$

$C \sim dL$ because $\mathcal{O}(dL) = \mathcal{O}(d) = \mathcal{O}(1)^{\otimes d}$

$$\text{So } C^2 = (dL)^2 = d^2$$

If D is a curve of degree e , then

$$C \cdot D = (dL) \cdot (eL) = de \quad \text{Bézout's theorem}$$

for curves in \mathbb{P}^2 when they are transverse.

Proposition: The adjunction formula:

For any nonsingular curve C on X of genus g , we have

$$2g - 2 = C \cdot (C + K_X)$$

Proof: We have the sequence of differentials on C :

$$0 \longrightarrow \mathcal{J}_C / \mathcal{J}_C^2 \longrightarrow \Omega'_X|_C \longrightarrow \Omega'_C \longrightarrow 0$$

$$\begin{aligned} \mathcal{J}_C / \mathcal{J}_C^2 &= \mathcal{J}_C \otimes \mathcal{O}_X / \mathcal{J}_C = \mathcal{J}_C|_C = \mathcal{O}_X(-C)|_C \\ &\cong \mathcal{O}_C(-C) \end{aligned}$$

$$\Rightarrow 0 \longrightarrow \mathcal{O}_C(-C) \longrightarrow \Omega'_X|_C \longrightarrow \Omega'_C \longrightarrow 0$$

Take top exterior powers:

$$\Lambda^2 \Omega'_X|_C \cong \Omega'_C \otimes \mathcal{O}_C(-C)$$

$$\omega_X|_C \cong \omega_C \otimes \mathcal{O}_C(-C)$$

$$\Rightarrow K_X|_C \sim K_C - C|_C$$

$$a \quad K_C \sim K_X|_C + C|_C$$

$$\text{take degrees: } 2g-2 = C \cdot (K_X + C) \quad \square$$

Theorem (Riemann-Roch): For any divisor D

$$\chi(D) = \chi(\mathcal{O}_X) + \frac{1}{2} D \cdot (D - K_X)$$

Proof: As in the proof of the existence and uniqueness of the intersection pairing, we can find two nonsingular curves C, E s.t. $D \sim C - E$: $\chi(D) = \chi(C - E)$.

For E , we have the usual exact sequence: