

$$\Rightarrow \Omega^1_{K(X)/K(Y)} \cong \frac{K(Y)[t] dt}{(P(t)dt, P'(t)dt)}$$

Since the extension is separable, P & P' are coprime

$$\Rightarrow \Omega^1_{K(X)/K(Y)} = 0$$

Therefore the morphism $f^* \Omega^1_Y \rightarrow \Omega^1_X$ is surjective at the generic point, hence not the zero morphism. \square

Note: Since $\Omega^1_{X/Y}$ is coherent, the fact that its stalk at the generic point is zero means that $\Omega^1_{X/Y}$ is zero on some open subset of X , i.e., $(\Omega^1_{X/Y})_x$

$\bar{\omega}$ nonzero only at finitely many points of X .

$$\begin{aligned}\text{So } \chi(\Omega'_{X/k}) &= \chi(f^* \Omega'_{Y/k}) + \chi(\Omega'_{X/Y}) \\ &= \chi(f^* \Omega'_{Y/k}) + h^0(\Omega'_{X/Y})\end{aligned}$$

By Riemann-Roch:

$$\deg(\Omega'_{X/k}) + 1 - g_X = \deg(f^* \Omega'_{Y/k}) + 1 - g_Y + h^0(\Omega'_{X/Y})$$

$$\Rightarrow \boxed{2g_X - 2 = \deg(f)(2g_Y - 2) + h^0(\Omega'_{X/Y})}$$

↑ First version of the gravity formula. ↑

Now we make $h^0(\Omega'_{X/Y})$ more explicit:

Let x be a closed point of X and put $y = f(x)$.

The stalks of the exact sequence from the proposition

are:

$$0 \rightarrow \Omega'_{Y,y} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} \xrightarrow{f^*} \Omega'_{X,x} \rightarrow \Omega'_{X/Y,x} \rightarrow 0$$

\uparrow \uparrow
 $\mathcal{O}_{Y,y}$ $m \otimes 1$
 $\Omega'_{Y,y}$ m

$f^*: \Omega'_{Y,y} \rightarrow \Omega'_{X,x}$ is the differential of the morphism of local rings $f^*: \mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x}$.

Let t_x be a uniformizer at x (a local parameter), and t_y a " " " y . Thinking of $\mathcal{O}_{Y,y}$ as a subring of $\mathcal{O}_{X,x}$, we can write $t_y = u t_x^{e_x}$ where u is a unit in $\mathcal{O}_{X,x}$.

$$\text{Then } dt_y = t_x^{e_x} du + e_x u t_x^{e_x-1} dt_x.$$

Therefore, if the ramification is tame at x , then the dimension of $\Omega'_{X/Y, x}$ is $e_x - 1$. If the ramification is wild at x , then $\dim_k(\Omega'_{X/Y, x})$ is at least e_x .

Definition: The ramification divisor of f is

$$R := \sum_{x \in X} \dim_k(\Omega'_{X/Y, x}) [x]$$

The Hurwitz formula is

$$2g_x - 2 = (\deg f)(2g_y - 2) + \deg R$$

Ampleness and very ampleness on curves (non singular, complete)

Def: If D is a divisor on X , we say D is ample, respectively very ample, if $\mathcal{O}_X(D)$ is.

We say a point $x \in X$ is a base point of $|D|$

$$\begin{aligned} \left(\text{recall } |D| = \mathbb{P}H^0(\mathcal{O}_X(D)) = \{ \text{lines in } H^0(\mathcal{O}_X(D)) \} \right. \\ \left. = \{ \mathcal{L}(s) \mid s \in H^0(\mathcal{O}_X(D)) \} \right. \end{aligned}$$

$$\left. = \{ \text{effective Weil divisors lin. eq. to } D \} \right)$$

if for all $D' \in |D|$, x belongs to the support of D' ,

$$\text{i.e., } \forall s \in H^0(\mathcal{O}_X(D)), s(x) = 0 \quad (s_x \in \mathfrak{m}_x \mathcal{O}_{X(D),x})$$

$$s(x) \in \mathcal{O}_{X(D),x} / \mathfrak{m}_x \mathcal{O}_{X(D),x} \cong \mathbb{K}.$$

We say $|D|$ or $\mathcal{O}_X(D)$ is base point free if it has no base points.

If we choose a basis $s_0, \dots, s_n \in H^0(\mathcal{O}_X(D))$, the associated rational map $X \dashrightarrow \mathbb{P}_k^n \cong |D|^* = \mathbb{P}H^0(\mathcal{O}_X(D))^*$ is a morphism if and only if $|D|$ is base point free $\Leftrightarrow \mathcal{O}_X(D)$ is globally generated.

Proposition: $|D|$ is base point free iff $\forall x \in X$,

$$h^0(D-x) = h^0(D) - 1,$$

$$\text{or } \dim |D-x| = \dim |D| - 1.$$

In fact:

Proposition: $x \in X$ is a base point of $|D|$ iff

$$h^0(D-x) = h^0(D) \quad \text{or}$$

$$\dim |D-x| = \dim |D|.$$

Proof: Consider the exact sequence:

$$0 \rightarrow \mathcal{O}_X(-x) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{\{x\}} \rightarrow 0$$

twist by $\mathcal{O}_X(D)$: $\mathcal{O}_X(-x)_x = \mathfrak{m}_x$

$$0 \rightarrow \mathcal{O}_X(D-x) \rightarrow \mathcal{O}_X(D) \rightarrow \mathcal{O}_x \rightarrow 0$$

and write the cohomology sequence:

$$0 \rightarrow H^0(D-x) \rightarrow H^0(D) \xrightarrow{e|_x} k \rightarrow H^1(D-x) \rightarrow H^1(D) \rightarrow 0.$$

$$\begin{array}{c} \cong \\ \downarrow \\ \mathfrak{S} \longmapsto \mathfrak{S}(x) \in \mathcal{O}_X(D)_x / \mathfrak{m}_x \mathcal{O}_X(D)_x \cong k \end{array}$$

So there exists $s \in H^0(D)$ not vanishing at x
 iff ev_x is surjective. So x is a base point of (D)
 iff $ev_x = 0 \iff H^0(D-x) = H^0(D)$
 and $ev_x \neq 0 \iff h^0(D-x) = h^0(D) - 1$
 $\iff |D-x| = |D| - 1 \quad \square$

Note: It also follows from the proof that
 x is not a base point $\iff h^0(D-x) = h^0(D) - 1$.

Very ample: Last quarter we had some criteria
 for a morphism $X \rightarrow \mathbb{P}_k^n$ to be a closed
 embedding:

Theorem: X projective / k alg. closed, $\varphi: X \rightarrow \mathbb{P}_k^n$
 a morphism associated to an invertible sheaf \mathcal{L} and
 global sections s_0, \dots, s_n . Let $V \subset H^0(\mathcal{L})$ be the
 k -span of s_0, \dots, s_n . Then φ is a closed embedding iff

(1) elements of V separate points, i.e., $\forall x \neq y \in X$
 closed points, $\exists s \in V$ s.t. $s(x) = 0$ and $s(y) \neq 0$

$$eV_x''(s)$$

(2) elements of V separate tangent vectors, i.e., $\forall x \in X$
 closed point, the map

$\{s \in V \mid s(x) = 0\} \longrightarrow m_x \mathcal{L}_x / m_x^2 \mathcal{L}_x \cong \text{Zariski cotangent space at } x$
 is surjective.

In the case of curves, the theorem becomes:

Theorem: D divisor on the smooth (non-singular) projective curve X over k . Then D is very ample iff $\forall x, y \in X$ (closed, distinct or not),

$$h^0(D - x - y) = h^0(D) - 2, \text{ or}$$

$$\dim |D - x - y| = \dim |D| - 2.$$

Proof: We are looking at the rational map

$$\varphi: X \dashrightarrow \mathbb{P}_k^n = |D|^* = \mathbb{P}H^0(D)^*$$

associated to sections $s_0, \dots, s_n \in H^0(D)$ which form a basis.

φ is a morphism iff $\forall x \in X \quad h^0(D - x) = h^0(D) - 1$.

The morphism is a closed embedding if it separates points and tangent vectors. Choose $x \in X$

Recall the cohomology sequence:

$$0 \rightarrow H^0(D-x) \rightarrow H^0(D) \xrightarrow{ev_x} k \rightarrow H^1(D-x) \rightarrow H^1(D) \rightarrow 0$$

We see that the sections of $\mathcal{O}_X(D)$ vanishing at x can be identified with the sections of $\mathcal{O}_X(D-x)$.

For $y \in X$, $y \neq x$, $s \in H^0(D)$ vanishes at y iff $s \in H^0(D-y)$.

So $\exists s \in H^0(D)$ s.t. $s(x) = 0$ and $s(y) \neq 0$

$$\begin{aligned} (\Rightarrow) \quad H^0(D-x) &\subset H^0_{\cup}(D) \\ &\not\subset H^0(D-y) \end{aligned}$$

$H^0(D-x)$ and $H^0(D-y)$ are both hyperplanes
in $H^0(D)$ (meaning they have dim. 1 less).

So $H^0(D-x) \not\subset H^0(D-y) \Leftrightarrow H^0(D-x) \neq H^0(D-y)$
 $\Leftrightarrow H^0(D-x) \cap H^0(D-y)$
 has codim. 2 in $H^0(D)$

We can identify $H^0(D-x) \cap H^0(D-y)$ with $H^0(D-x-y)$
 (exercise: use a sequence similar to $0 \rightarrow \mathcal{O}_X(-x) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_x \rightarrow 0$)

So $\exists s$ with $s(x)=0, s(y) \neq 0 \Leftrightarrow h^0(D-x-y) = h^0(D) - 2$

Separating tangent vectors means

$$H^0(D-x) = \{s \in H^0(D) \mid s(x) = 0\} \xrightarrow{\text{evaluation}} \mathcal{M}_x \mathcal{L}_x / \mathcal{M}_x^2 \mathcal{L}_x \cong k$$

$\mathcal{L} = \mathcal{O}_X(D)$

Because the target is 1-dimensional, surjectivity means the map is not 0. This means $\exists s \in H^0(D)$

s.t. $s_x \in M_x \mathcal{O}_X(D)_x$ and $s_x \notin M_x^2 \mathcal{O}_X(D)_x$.

\Updownarrow

$x \in \text{Supp } \mathcal{L}(s)$

(if $D(s) = \text{divisor of } s$)

$(\Rightarrow) D(s) - x \geq 0$

\Updownarrow

$D(s) - 2x$ not effective.

\Updownarrow

$s \in H^0(D - x)$

$s \notin H^0(D - 2x)$

(use sequence $0 \rightarrow \mathcal{O}_X(-2x) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{\{2x\}} \rightarrow 0$)

This means that $H^0(D - 2x) \subsetneq H^0(D - x) \subset H^0(D)$

$\Leftrightarrow h^0(D - 2x) = h^0(D - x) - 1 = h^0(D) - 2 \quad \square$