$$\Rightarrow \quad \deg \mathrm{Div}(s) \geq 0$$

$$\mathcal{O}_X(D - \mathrm{Div}(s)) \cong \mathcal{O}_X \Rightarrow D - \mathrm{Div}(s) \text{ is principal}$$

$$\Rightarrow \deg D = \deg \mathrm{Div}(s) \geq 0.$$

If $\deg D = 0$, then $D \sim \mathrm{Div}(s)$ effective of degree 0

$$\Rightarrow \mathrm{Div}(s) = 0 \quad \Rightarrow D \sim 0 \Rightarrow \mathcal{O}_X(D) \cong \mathcal{O}_X \quad \square.$$

Definition: For a projective curve, its arithmetic genus is defined to be (X not necessarily nonsingular)

$$p_a := h^1(X, \mathcal{O}_X) := \dim H^1(X, \mathcal{O}_X).$$

The geometric genus is defined to be

$$g := h^0(X, \Omega^1_X).$$

When X is nonsingular, then $\omega_X \cong \Omega^1_X$, and, by Serre duality, $\forall$ quasi-coherent sheaves $\mathcal{F}$:

$$H^0(X, \mathcal{F}) \cong \operatorname{Ext}^1_X(\mathcal{F}, \omega_X)^*$$

$$H^1(X, \mathcal{F}) \cong \operatorname{Ext}^0_X(\mathcal{F}, \omega_X)^* \cong \operatorname{Hom}_X(\mathcal{F}, \omega_X)^*$$

If $\mathcal{F}$ is locally free of finite rank:

$$H^0(X, \mathcal{F}) \cong H^1(X, \omega_X \otimes \mathcal{F}^*)^*$$

$$H^1(X, \mathcal{F}) \cong H^0(X, \omega_X \otimes \mathcal{F}^*)^*.$$

<u>Theorem</u>: Riemann-Roch for curves:

X complete nonsingular curve of genus g.

D a divisor on X. Then

$$\chi(D) := h^0(D) - h^1(D) = \deg D + 1 - g.$$

Proof: Let P be any (closed) point of X.

We show that the theorem is true for D iff it is true for $D-P$. Starting from 0 and adding and subtracting points, this will prove the theorem for all divisor (for $D=0$, $\chi(0) = h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) = 1-g$
$= \deg 0 + 1 - g$).

Recall that the ideal sheaf of $P \hookrightarrow X$ is isom. to $\mathcal{O}_X(-P)$.

This gives us the exact sequence:

$$0 \longrightarrow \mathcal{O}_X(-P) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_P \longrightarrow 0$$

$$\mathcal{O}_P = i_* \mathcal{O}_P \quad i : P \hookrightarrow X$$

(abuse of notation always done)

twist by $\mathcal{O}_X(D)$ to obtain:

$$0 \longrightarrow \mathcal{O}_X(D-P) \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathcal{O}_P(D) \longrightarrow 0$$

Note that $\mathcal{O}_P(D) \cong \mathcal{O}_P$ because $P \cong \operatorname{Spec} k$

= skyscraper sheaf supported at P with group k.

From the long exact sequence of cohomology, we obtain: $\chi(D-P) = \chi(D) - 1$.

Since we also have $\deg(D-P) = \deg D - 1$,

we obtain $\chi(D-P) - \deg(D-P) = \chi(D) - \deg D$,

and we are done. $\square$

<u>Corollary</u>: The degree of the canonical sheaf $\omega_X = \Omega^1_X$ is $2g-2$.

Proof: Let K_X be a divisor s.t. $\mathcal{O}_X(K_X) \cong \omega_X$.

Then
$$\chi(K_X) = \deg K_X + 1 - g$$
$$\rotatebox{90}{=}$$
$$h^0(K_X) - h^1(K_X) = h^0(\omega_X) - h^1(\omega_X)$$
$$(\text{Serre Duality}) = \underbrace{h^1(\mathcal{O}_X)}_{\underset{g}{\shortparallel}} - h^0(\mathcal{O}_X) = g-1$$

$$\Rightarrow \deg K_X = g-1 + g-1 = 2g-2 \qquad \square.$$

Def: A canonical divisor is a divisor K_X s.t.
$$\mathcal{O}_X(K_X) \cong \omega_X \cong \Omega^1_X.$$

Definition: We say that D is special if $h^1(D) \neq 0$, and D is nonspecial if $h^1(D) = 0$.

Remark: (1) If $\deg D \geq 2g-1$, then D is nonspecial.

$$\deg D \geq 2g-1 \Rightarrow \deg(K_X - D) = \deg K_X - \deg D$$
$$= 2g-2-\operatorname{Deg} D \leq -1 < 0$$

$\Rightarrow h^0(K_X - D) = 0$ by Lemma 3.

$= h^1(D)$ by Serre Duality.

(2) If D is special of degree $2g-2$, then $\mathcal{O}_X(D) \cong \omega_X$.

(apply Lemma 3 to $K_X - D$)

Theorem: Kodaira vanishing (see Griffiths-Harris)

X projective nonsingular of dim. n over $\mathbb{C} = k$.

$\mathcal{L}$ ample invertible sheaf on X.

The following equivalent statements hold:

(1) $H^i(X, \mathcal{L} \otimes \omega_X) = 0 \quad \forall\, i > 0$

(2) $H^i(X, \mathcal{L}^{-1}) = 0 \quad \forall\, i < n.$

Proof for curves: Write $\mathcal{L} = \mathcal{O}_X(D)$. If D is ample, then a multiple is very ample.

In particular, $\exists\ m > 0$ s.t. $h^0(mD) \geq 2$.

Lemma 3 $\Longrightarrow \deg D > 0 \Rightarrow \deg(K_D + D) \geq 2g-1$

$\Rightarrow h^1(K_X + D) = 0$, i.e., $K_X + D$ is nonspecial. $\square$.

We shall soon see that on a curve, a divisor is ample iff it has positive degree.

__Definition:__ A variety is a separated integral scheme of finite type/k.

Two varieties X, Y are birational if $\exists$ nonempty open sets $U \subset X$, $V \subset Y$ s.t. $U \cong V$.

A variety X is called rational if it is birational to $\mathbb{P}^n$ where $n = \dim X$. (This is equivalent to

$$K(X) \cong K(\mathbb{P}^n) \cong k(x_1, \ldots, x_n))$$

__Proposition:__ A complete nonsingular curve X is rational iff it is isomorphic to $\mathbb{P}^1$, iff it has genus 0.

Proof: If X has genus 0, then Riemann-Roch becomes: $\chi(D) = \deg D + 1$.

Choose two distinct points $P, Q \in X$, then

$$\chi(P-Q) = 1 = h^0(P-Q) - h^1(P-Q)$$

$$\Rightarrow \quad h^0(P-Q) > 0$$

Lemma 3 $\Rightarrow$ $P - Q \overset{\text{lin.}}{\sim} 0$

Choose a rational function $f \in K(X)$ s.t.

$$\mathrm{Div}(f) = P - Q.$$

As in the proof of the Corollary to Prop. 3, f gives a morphism φ to $\mathbb{P}^1$ s.t. $\mathrm{Div}(f) = \varphi^*(\{0\} - \{\infty\})$ and $\varphi^*\{0\} = [P]$ $\Rightarrow$ $\deg \varphi = 1$ $\Rightarrow$ φ is an isom. $\square$