\[ \Rightarrow \quad \deg \text{Div}(s) \geq 0 \]

\[ c_x (D - \text{Div}(s)) \cong \mathcal{O}_X \Rightarrow D - \text{Div}(s) \text{ is principal} \]

\[ \Rightarrow \quad \deg D = \deg \text{Div}(s) \geq 0. \]

If \( \deg D = 0 \), then \( D \sim \text{Div}(s) \) effective of degree 0.

\[ \Rightarrow \quad \text{Div}(s) = 0 \Rightarrow D \sim 0 \Rightarrow \mathcal{O}_X(D) \cong \mathcal{O}_X. \]

**Definition:** For a projective curve, its arithmetic genus is defined to be (\( X \) not necessarily nonsingular)

\[ \gamma_a := h^1(X, \mathcal{O}_X) := \dim H^1(X, \mathcal{O}_X). \]

The geometric genus is defined to be

\[ g := h^0(X, \mathcal{O}_X^*). \]
When $X$ is non-singular, then $\mathcal{O}_X \cong \mathcal{O}'_X$, and, by Serre duality, $\mathcal{F}$ quasi-coherent sheaves $\mathcal{F}$:

$$H^0(X, \mathcal{F}) \cong \text{Ext}^1_X(\mathcal{F}, \mathcal{O}_X)$$

$$H^1(X, \mathcal{F}) \cong \text{Ext}^0_X(\mathcal{F}, \mathcal{O}_X) \cong \text{Hom}_X(\mathcal{F}, \mathcal{O}_X).$$

If $\mathcal{F}$ is locally free of finite rank:

$$H^0(X, \mathcal{F}) \cong H^1(X, \mathcal{O}_X \otimes \mathcal{F}^*)$$

$$H^1(X, \mathcal{F}) \cong H^0(X, \mathcal{O}_X \otimes \mathcal{F}^*).$$

**Theorem:** Riemann-Roch for curves:

If complete non-singular curve of genus $g$. $D$ a divisor on $X$. Then
\[ X(D) := h^0(D) - h^1(D) = \deg D + 1 - g. \]

**Proof:** Let \( P \) be any (closed) point of \( X \). We show that the theorem is true for \( D \) iff it is true for \( D - P \). Starting from 0 and adding and subtracting points, this will prove the theorem for all divisors \( (\text{for } D = 0, X(0) = h^0(O_X) - h^1(O_X) = 1 - g = \deg O + 1 - g). \)

Recall that the ideal sheaf of \( P \in X \) is isomorphic to \( O_X(-P) \). This gives us the exact sequence:

\[ 0 \to O_X(-P) \to O_X \to \mathcal{O}_P \to 0 \]

(alternative notation \( i_\ast O_P \), \( i : P \to X \)) always done
Twist by $O \times (D)$ to obtain:

$$0 \rightarrow O \times (D-P) \rightarrow O \times (D) \rightarrow O_P(D) \rightarrow 0$$

Note that $O_P(D) \cong O_\mathcal{P}$ because $P$ is a smooth point.

From the long exact sequence of cohomology, we obtain:

$$\chi(D-P) = \chi(D) - 1.$$ 

Since we also have $\deg(D-P) = \deg D - 1$, we obtain

$$\chi(D-P) - \deg(D-P) = \chi(D) - \deg D,$$

and we are done. \[\square\]

**Corollary:** The degree of the canonical sheaf $\omega_X = \Omega_X^1$ is $2g - 2$. 
Proof: Let $K_X$ be a divisor s.t. $\mathcal{O}_X(K_X) \cong \omega_X$. Then

$$\chi(K_X) = \deg K_X + 1 - g$$

Then

$$h^0(K_X) - h^1(K_X) = h^0(\omega_X) - h^1(\omega_X)$$

(Genus Formula) = $h^1(\mathcal{O}_X) - h^0(\mathcal{O}_X) = g - 1$

\[\begin{align*}
\Rightarrow \deg K_X &= g - 1 + g - 1 = 2g - 2
\end{align*}\]

Def.: A canonical divisor is a divisor $K_X$ s.t.

$$\mathcal{O}_X(K_X) \cong \omega_X \cong \Omega^1_X$$

Definition: We say that $D$ is special if $h^1(D) \neq 0$, and $D$ is non-special if $h^1(D) = 0$. \[\square\]
Remark: (1) If $\deg D > 2g - 1$, then $D$ is nonspecial.

\[\deg D > 2g - 1 \implies \deg (K_X - D) = \deg K_X - \deg D\]

\[= \deg - \text{Deg}D \leq -1 < 0\]

\[\implies h^0(K_X - D) = 0 \text{ by Lemma 3.}\]

\[= h^1(D) \text{ by Base Quality.}\]

(2) If $D$ is special of degree $2g - 2$, then $\mathcal{O}(D) \cong \mathcal{O}_X$.

(apply Lemma 3 to $K_X - D$)

Theorem: Kodaira vanishing (see Griffiths-Harris). X projective nonsingular of dim. n over $\mathbb{C} = k$.

$\mathcal{L}$ ample invertible sheaf on $\mathcal{X}$. 
The following equivalent statements hold:

1. \( H^i(X, L \otimes \omega_X) = 0 \quad \forall \ i \geq 0 \)

2. \( H^i(X, L^{-1}) = 0 \quad \forall \ i < n \).

Proof for curves: Write \( L = O_x(D) \). If \( D \) is ample, then a multiple is very ample. In particular, \( \exists \ m > 0 \) s.t. \( h^0(mD) \geq 2 \).

Lemma 3 \( \Rightarrow \deg D > 0 \Rightarrow \deg (K_D + D) \geq 2g - 1 \)

\( \Rightarrow H^1(K_D + D) = 0 \); i.e., \( K_D + D \) is nonspecial. \( \square \).

We shall soon see that on a curve, a divisor is ample if and only if it has positive degree.
Definition: A variety is a separated integral scheme of finite type over \( k \).

Two varieties \( X, Y \) are birational if there exist open sets \( U \subset X, V \subset Y \) such that \( U \cong V \).

A variety \( X \) is called rational if it is birational to \( \mathbb{P}^n \) where \( n = \dim X \). (This is equivalent to \( K(X) \cong K(\mathbb{P}^n) \cong k(x_1, \ldots, x_n) \).)

Proposition: A complete non-singular curve \( X \) is rational iff it is isomorphic to \( \mathbb{P}^1 \), iff it has genus 0.
Proof: If \( X \) has genus 0, then Riemann-Roch becomes:
\[
\chi(D) = \deg D + 1.
\]
Choose two distinct points \( P, Q \in X \), then
\[
\chi(D_{P - Q}) = 1 = h^0(D_{P - Q}) - h^1(D_{P - Q})
\]
\[
\Rightarrow h^0(D_{P - Q}) > 0
\]
Lemma 3 \( \Rightarrow \) \( P - Q \approx 0 \)

Choose a rational function \( f \in K(X) \) s.t.
\[
\text{Div}(f) = P - Q.
\]
As in the proof of the Corollary to Prop. 3, \( f \) gives a morphism \( \varphi \) to \( \mathbb{P}^1 \) s.t.
\[
\text{Div}(\varphi) = \varphi^*(\{0\} - \{\infty\})
\]
and \( \varphi^*[0] = [P] \) \( \Rightarrow \) \( \deg \varphi = 1 \) \( \Rightarrow \) \( \varphi \) is an isomorphism. \( \square \).