Some nice properties of f^*:

Exercise: (1) Pull-backs of principal divisors are again principal, so that we obtain an induced morphism

$$f^*: Cl(Y) \longrightarrow Cl(X).$$

This coincides with pull-backs of invertible sheaves and the pull-back $f^*: CaCl(Y) \longrightarrow CaCl(X)$ under the identifications $Cl(X) = CaCl(X) = Pic(X)$

(2) Proposition 3: $f: X \to Y$ finite morphism of integral nonsingular curves. Then for any divisor D on Y, we have $\deg(f^*D) = \deg(f) \deg D$.

Recall that if $D = \sum n_P [P]$, then $\deg D = \sum n_P$.

and $\deg f = \deg K(X)/K(Y)$.

<u>Proof:</u> By linearity, we need to show that for $Q \in Y$,

$$\deg f^* Q = \deg f.$$

As in the proof of Proposition 2, let $V = \operatorname{Spec} B$ be an affine open set of Y, and put $U = f^{-1}(V) = \operatorname{Spec} A$.

We know that A is the integral closure of B in $K(X)$.

Let m be the maximal ideal of B corresponding to Q and localize B and A at m to obtain

$$\mathcal{O}_{Y,Q} = B_m \qquad A' := A \otimes_B B_m$$

Since B_m is a DVR (in particular it is a PID) and A' is torsion-free, we obtain that A' is a free B_m-module of finite rank. The rank of A' over B_m is $r := \deg f$ because if we further localize B_m and A' at all nonzero elements of B_m we obtain the field extension $K(Y) \subset K(X)$. (the localization of A' is a finite ring extension of $K(Y)$ which is an integral domain $\underset{\text{comm. alg.}}{\Rightarrow}$ it is a field $= K(X) = \mathrm{Frac}(A')$).

Furthermore, if t is a uniformizer at Q, then

$$A'/tA' \text{ is a vector space over } \mathcal{O}_{Y,Q}/t\mathcal{O}_{Y,Q} = k(Q) = k \text{ of dim. } r.$$

The points P_i of X mapping to Q are in one-to-one correspondence with the maximal ideals of A pulling back to m, i.e., containing $m \subset B \subset A$, these are in bijection with the maximal ideals m_i of A'.

<u>Now</u>: $A' = \bigcap_i A'_{m_i}$ because A' is an integrally closed domain.

$$\Rightarrow \quad tA' = \bigcap_i \left(tA'_{m_i} \cap A' \right)$$

By the Chinese remainder theorem:

$$A'/tA' \cong \bigoplus_i A'/tA'_{m_i} \cap A' = \bigoplus_i A'_{m_i}/tA'_{m_i}$$

(map $A' \longrightarrow A'_{m_i} \twoheadrightarrow A'_{m_i}/tA'_{m_i}$ and factor through $tA'_{m_i} \cap A'$ to get the second isom.)

$$= \bigoplus_i \mathcal{O}_{P_i} / t\,\mathcal{O}_{P_i}$$

$$\Rightarrow \deg f = r = \operatorname{rank} A' = \dim_k A'/tA' = \sum_i \dim \mathcal{O}_{P_i}/t\,\mathcal{O}_{P_i}$$

$$= \sum_i \nu_{P_i}(t)$$

$$= \deg f^* Q \text{ by def.}$$

$\square$.

<u>Corollary</u>: The degree of a principal divisor is 0 (on a complete nonsingular curve).

<u>Proof</u>: Let $f \in K(X)$. If $f \in k$, then $\operatorname{Div}(f) = 0$ and we are done. If $f \notin k$, then, since k is algebraically closed, f is transcendental over k, hence

f defines a nonconstant morphism $\varphi : X \to \mathbb{P}^1$ as follows:

Write $\mathrm{Div}(f) = \sum_i m_i P_i - \sum n_j Q_j$

where all the integers m_i and n_j are nonnegative and $\{P_i\} \cap \{Q_j\} = \emptyset$.

Put $U := X \setminus \{Q_j\}$, $U' := X \setminus \{P_i\}$.

$\forall\ P \in U$, $f \in \mathcal{O}_{X,P} \Rightarrow f \in \mathcal{O}_X(U) = \bigcap_{P \in U} \mathcal{O}_{X,P}$.

$\Rightarrow\ \exists$ morphism $\varphi_U : U \to \mathbb{A}^1$

defined by $\mathcal{O}_X(U) \hookleftarrow k[x]$

$$f \longleftarrow\!\shortmid\ x$$

Similarly define $\varphi_{U'} : U' \to \mathbb{A}^1$ using $\frac{1}{f}$

These glue to $\varphi : X \to \mathbb{P}^1$. $\quad X = U \cup U'$

We have $\mathrm{Div}(f) = \varphi^*(\{0\} - \{\infty\})$

has degree $0 = (\deg \varphi) \deg(\{0\} - \{\infty\})$. $\square$

<u>Exercise</u>: In the proof above, show that

$$\varphi^* \mathcal{O}_{\mathbb{P}^1}(1) \cong \mathcal{O}_X\left(\sum_i m_i \cdot P_i\right) \cong \mathcal{O}_X\left(\sum_j n_j \cdot Q_j\right).$$

<u>More nice results about curves</u>: X nonsingular complete$/k$

<u>Lemma 3</u>: D a divisor on X. If $h^0(D) := \dim_k H^0(D) \neq 0$, then $\deg D \geq 0$. If $h^0(D) \neq 0$ and $\deg D = 0$, then $D \sim 0$, i.e., $\mathcal{O}_X(D) \cong \mathcal{O}_X$.

<u>Proof</u>: If $h^0(D) \neq 0$, then for $s \in H^0(D)$, $s \neq 0$

$\mathrm{Div}(s)$ is an effective divisor, $= \mathrm{Div}(\mathcal{Z}(s))$

$\Rightarrow \quad \deg \mathrm{Div}(s) \geq 0$

$\mathcal{O}_X(D - \mathrm{Div}(s)) \cong \mathcal{O}_X \Rightarrow D - \mathrm{Div}(s)$ is principal

$\Rightarrow \deg D = \deg \mathrm{Div}(s) \geq 0.$

If $\deg D = 0$, then $D \sim \mathrm{Div}(s)$ effective of degree 0

$\Rightarrow \mathrm{Div}(s) = 0 \quad \Rightarrow D \sim 0 \Rightarrow \mathcal{O}_X(D) \cong \mathcal{O}_X$ $\square$.

<u>Definition:</u> For a projective curve, its arithmetic genus is defined to be (X not necessarily nonsingular)

$$p_a := h^1(X, \mathcal{O}_X) := \dim H^1(X, \mathcal{O}_X).$$

The geometric genus is defined to be

$$g := h^0(X, \Omega^1_X).$$