

Lemma 1: X non singular curve/ k , $f: X \dashrightarrow Y$
a rational map (i.e., f is a morphism from an open dense
subscheme of X to Y)
to a projective variety Y .

Then f extends to a morphism $X \rightarrow Y$.

Proof: Choose an embedding $Y \hookrightarrow \mathbb{P}_k^n$.

We can replace Y with \mathbb{P}^n because if f extends to a
morphism $X \rightarrow \mathbb{P}^n$, then the morphism f factors through Y
because Y is a closed subscheme of \mathbb{P}^n .

Step 1: Let U be a non empty open set where f is well-defined.

Lemma 2: The datum of a rational map $f: X \dashrightarrow \mathbb{P}^n$ is
equivalent to the data of an invertible sheaf \mathcal{L} on X and

sections $s_0, \dots, s_n \in H^0(X, \mathcal{L})$ s.t. $\mathcal{L}|_U \cong f^* \mathcal{O}_{\mathbb{P}^n}(1)$,

$\forall i \quad s_i|_U = f^* X_i$ and $U \subseteq \{x \in X \mid \exists i \text{ s.t. } (s_i)_x \notin \mathfrak{m}_x \mathcal{L}_x\}$

Proof: Put $\mathcal{M} := f^* \mathcal{O}_{\mathbb{P}^n}(1)$ which is an invertible sheaf on U ,

and put $t_i := f^* X_i$ for $i = 0, \dots, n$.

Also put $\{P_1, \dots, P_n\} := X \setminus U$.

Via the exact sequence

$$\bigoplus_{i=1}^n \mathbb{Z}[P_i] \longrightarrow \mathcal{O}(X) \xrightarrow{\text{restriction}} \mathcal{O}(U) \longrightarrow 0$$

choose \mathcal{L}'' on X s.t. $\mathcal{L}''|_U = \mathcal{M}$.

Recall that $\mathcal{L}'' \otimes \mathcal{K}_X \cong \mathcal{K}_X$ because both sheaves are the constant sheaf with group $K(X)$.

Definition: A rational section of an invertible sheaf \mathcal{L} on an integral scheme X is a global section s of \mathcal{K}_X s.t.

$\exists \forall \neq \emptyset, \forall \text{ open } U \subset X \text{ with } s|_U \in H^0(U, \mathcal{L})$ via the embedding $\mathcal{L} \hookrightarrow \mathcal{L} \otimes \mathcal{K}_X = \mathcal{K}_X$. We say s is regular on U .

Back to the proof of Lemma 2:

We have $\mathcal{L}''|_U \hookrightarrow (\mathcal{K}_X = \mathcal{K}_X \otimes \mathcal{L}'')|_U$
 $\mathcal{M} \hookrightarrow \mathcal{K}_U = \mathcal{K}_U \otimes \mathcal{M}$

So $s_0, \dots, s_n \in H^0(U, \mathcal{M}) \hookrightarrow H^0(\mathcal{K}_U) = K(X) = H^0(\mathcal{K}_X)$ are rational sections of \mathcal{L}'' , regular on U .

For $i=1, \dots, r$, put $m_i := \text{Max} \{0, -v_{P_i}(t_j), j=0, \dots, n\}$

Then $\mathcal{O}_X(-m_1 P_1 - \dots - m_r P_r) \hookrightarrow X$ (ideal sheaf)

and $\mathcal{L}'' \hookrightarrow \mathcal{L} := \mathcal{L}'' \otimes \mathcal{O}_X(m_1 P_1 + \dots + m_r P_r)$

and $H^0(X, \mathcal{L}'') \hookrightarrow H^0(X, \mathcal{L}) \hookrightarrow H^0(X, \mathcal{K}_X) = K(X)$

at P_i : $\mathcal{L}_{P_i} = (\mathcal{L}'' \otimes \mathcal{O}_X(m_i P_i))_{P_i} = \prod_{P_i}^{m_i} \mathcal{L}''_{P_i} \subset \mathcal{K}_{X, P_i} = K(X)$ The t_j is a uniformizer at P_i .

Ex: Show that $\forall s \in K(X), s \in H^0(\mathcal{L}) \Leftrightarrow s_x \in \mathcal{L}_x \subset K(X) \forall x \in X$.

From this deduce that:

The sections of \mathcal{L} are the rational sections of \mathcal{L}'' with poles of order at most $n_i \forall i$. (You can also deduce this by analyzing the transition functions for a suitable cover)

Hence $s_1, \dots, s_n \in H^0(X, \mathcal{L})$.

By construction $U \subseteq \{x \in X \mid \exists i, (s_i)_x \notin n_x \mathcal{L}_x\}$. \square

Proof of Lemma 1 continued:

Now put $n_j := \text{Min} \{ \nu_{P_j}((s_i)_{P_j}) : i = 0, \dots, n \}$

and $\mathcal{L}' := \mathcal{L} \otimes \mathcal{O}_X(-n_1 P_1 - \dots - n_n P_n) \hookrightarrow \mathcal{L}$

At P_i : $\mathcal{L}'_{P_i} = \pi_i^{n_i} \mathcal{L}_{P_i}$

And, as in the proof of Lemma 2, we can identify the sections of \mathcal{L}' with the sections of \mathcal{L} which vanish to order at least n_i at P_i .

locally, all s_j are divisible by $\pi_i^{m_i} \forall i$, by

the definition of the u_i

\Rightarrow via the embedding of $H^0(X, \mathcal{L}^1) \hookrightarrow H^0(X, \mathcal{L})$
we can consider s_0, \dots, s_n to be global sections

of \mathcal{L}^1 (they belong to the image of $H^0(X, \mathcal{L}^1)$)

Now map X to \mathbb{P}^n via \mathcal{L}^1 and $s_0, \dots, s_n \in H^0(X, \mathcal{L}^1)$

$$\rightsquigarrow \varphi: X \rightarrow \mathbb{P}^n$$

$$\forall i \exists j \text{ s.t. } (s_j)_{P_i} \notin \mathcal{M}_{P_i} \cdot \underbrace{\mathcal{L}^1_{P_i}}_{\substack{= \\ \pi_i^{m_i} \mathcal{L}_{P_i}}}$$

So φ is well-defined everywhere

and $\varphi|_U = f$ because $\mathcal{L}^1|_U = \mathcal{L}|_U \quad \square$

Remark (Exercise). Similar arguments will prove:

If X is an integral, separated, locally factorial scheme of finite type / k , the datum of a rational map $f: X \dashrightarrow \mathbb{P}_k^n$ is equivalent to the data of an invertible sheaf \mathcal{L} and rational sections s_0, \dots, s_n of \mathcal{L} which generate \mathcal{L} on a nonempty open set U s.t.

$$f^* \mathcal{O}_{\mathbb{P}^n}(1) \cong \mathcal{L}|_U \quad \text{and} \quad f^* X_i = s_i|_U$$

Furthermore, the locus of indeterminacy of f , i.e., the complement of the largest open set to which f can be extended, has codimension ≥ 2 .

Proposition 1: X a curve/ k (integral, separated, finite type, dim. 1)

Then X is complete if and only if it is projective.

Proof: Clearly, if X is projective, then it is complete.

Assume X is complete.

Cover X with open affine subsets V_1, \dots, V_n .

$\forall i$ we can write $V_i \cong \text{Spec } A_i$ with

$$A_i = k[x_1, \dots, x_n] / I \Rightarrow V_i \hookrightarrow \mathbb{A}^n = \text{Spec } k[x_1, \dots, x_n]$$

(we can assume one n works for all i)

embed $\mathbb{A}^n \hookrightarrow \mathbb{P}_k^n$ as one of the usual open affine subsets.

Let Y_i be the closure of the image of V_i in \mathbb{P}_k^n .

Note that Y_i (with its reduced induced structure) is an integral projective curve.

$$X \hookrightarrow \bigcup_i U_i \xrightarrow{\phi} Y_i$$

defines a rational map $X \dashrightarrow Y_i$ which, using Lemma 1 extends to a morphism $X \rightarrow Y_i$.

Now show that the resulting product morphism

$$X \longrightarrow \prod_{i=1}^n Y_i \text{ is a closed embedding (exercise)}$$

This shows X is projective via the composition

$$X \hookrightarrow \prod_{i=1}^n Y_i \hookrightarrow \prod_{i=1}^n \mathbb{P}^n \xrightarrow{\text{Segre embedding}} \mathbb{P}^N$$

