Given a scheme X and a point $x \in X$, let $U = \operatorname{Spec} A$ be an affine neighborhood of x and let $\mathfrak{p}$ be the prime ideal of A corresponding to x. If $\mathcal{O}_{X,x} = A_{\mathfrak{p}}$ is Cohen-Macaulay, then its maximal ideal contains a regular sequence $a_1, \dots, a_d$ with $d = \dim A_{\mathfrak{p}} = \operatorname{height} \mathfrak{p}$. After possibly localizing A (at the denominators of $a_1, \dots, a_d$) we can assume $a_1, \dots, a_d \in A$.

By Hauptidealsatz, $\dim A/\langle a_1, \dots, a_d \rangle = \dim A - d$

$$= \dim \overline{\{\mathfrak{p}\}}$$

$$= \dim A/\mathfrak{p}$$

Hence $\operatorname{Spec} A/\mathfrak{p} = \overline{\{\mathfrak{p}\}}$ is an irreducible component of $\operatorname{Spec} A/\langle a_1, \dots, a_d \rangle = Z(a_1, \dots, a_d) = Z(a_1) \cap \cdots \cap Z(a_d)$

After further localizing A, we can assume $\operatorname{Spec} A/\langle a_1, \dots, a_d \rangle$

is irreducible (each irreducible component is closed and after removing it we have a smaller neighborhood of x).

Hence the closure $\overline{\{p\}}$ is "set-theoretically" cut out by the d equations $a_1, \dots, a_d$.

Examples of Cohen-Macaulay rings:

(1) Noetherian local rings of dimension 0 are Cohen-Macaulay. Note that Noetherian local rings of dimension 0 are Artin rings. A ring is an Artin ring if it satisfies the descending chain condition for ideals.

(2) One-dimensional reduced noetherian rings are Cohen Macaulay.

(3) Two dimensional normal noetherian rings are Cohen-Macaulay

A ring is called normal if it is reduced and integrally closed in its total quotient ring.

(4) If A is a finitely generated Cohen-Macaulay algebra over a field k with an action of a finite group G, then the subring of invariants A^G is Cohen-Macaulay.

(5) Determinantal rings are Cohen-Macaulay.

A ring is called determinantal if it is a quotient

$$B = A/I$$ where A is a regular local ring and

I is the ideal generated by the $r \times r$ minors of a $p \times q$

matrix with coefficients in A s.t. the height of any minimal prime of I is the expected codimension $(p-r+1)(q-r+1)$.

The main example of such an ideal is the ideal of the locus of matrices of rank $< r$ in the space of all $p \times q$ matrices with entries in a field.

<u>Curves:</u> From now on, all schemes are over an algebraically closed field k.

<u>Def</u>: A curve $X/_k$ is an integral separated scheme of finite type over k, of dimension 1. We say X is complete if it is proper over k. We will prove some nice results about curves.

Lemma 1: X nonsingular curve$/k$, $f: X \dashrightarrow Y$ a rational map (i.e., f is a morphism from an open dense subscheme of X to Y) to a projective variety Y.
Then f extends to a morphism $X \longrightarrow Y$.

Proof: Choose an embedding $Y \hookrightarrow \mathbb{P}^n_k$.
We can replace Y with $\mathbb{P}^n$ because if f extends to a morphism $X \longrightarrow \mathbb{P}^n$, then the morphism factors through Y because Y is a closed subscheme of $\mathbb{P}^n$.
Step 1: Let U be a nonempty open set where f is well-defined.
Lemma 2: The datum of a rational map $f: X \dashrightarrow \mathbb{P}^n$ is equivalent to the data of an invertible sheaf $\mathcal{L}$ on X and

sections $s_0, \dots, s_n \in H^0(X, \mathcal{L})$ s.t. $\mathcal{L}|_U \cong f^*\mathcal{O}_{\mathbb{P}^n}(1)$,

$\forall i \quad s_i|_U = f^* X_i$ and $U \subseteq \{x \in X \mid \exists i \text{ s.t. } (s_i)_x \notin m_x \mathcal{L}_x\}$

Proof: Put $\mathcal{M} := f^*\mathcal{O}_{\mathbb{P}^n}(1)$ which is an invertible sheaf on U,

and put $t_i := f^* X_i$ for $i = 0, \dots, n$.

Also put $\{P_1, \dots, P_r\} := X \setminus U$.

Via the exact sequence

$$\bigoplus_{i=1}^{r} \mathbb{Z}[P_i] \longrightarrow \mathrm{Cl}(X) \xrightarrow[\text{restriction}]{} \mathrm{Cl}(U) \longrightarrow 0$$

choose $\mathcal{L}''$ on X s.t. $\mathcal{L}''|_U = \mathcal{M}$.

Recall that $\mathcal{L}'' \otimes \mathcal{K}_X \cong \mathcal{K}_X$ because both sheaves are the constant sheaf with group $K(X)$.

Definition: A rational section of an invertible sheaf $\mathcal{L}$ on an integral scheme X is a global section s of $\mathcal{K}_X$ s.t.

$\exists\ V \neq \phi$, V open $\subset X$ with $s|_V \in H^0(X, \mathcal{L})$ via the embedding $\mathcal{L} \hookrightarrow \mathcal{L} \otimes \mathcal{K}_X = \mathcal{K}_X$. We say s is regular on V.

Back to the proof of Lemma 2:

We have
$$\begin{array}{ccc} \mathcal{L}''|_U & \hookrightarrow & (\mathcal{K}_X = \mathcal{K}_X \otimes \mathcal{L}'')|_U \\ & & \| \\ \mathcal{M} & \hookrightarrow & \mathcal{K}_U = \mathcal{K}_U \otimes \mathcal{M} \end{array}$$

So $s_0, \ldots, s_n \in H^0(U, \mathcal{M}) \hookrightarrow H^0(\mathcal{K}_U) = K(X) = H^0(\mathcal{K}_X)$

are rational sections of $\mathcal{L}''$, regular on U.

For $i = 1, \ldots, r$, put $m_i := \mathrm{Max}\{0, -v_{P_i}(t_j), j = 0, \ldots, n\}$

Then $\mathcal{O}_X(-m_1 P_1 - \cdots - m_r P_r) \hookrightarrow X$ (ideal sheaf)

and $\mathcal{L}'' \hookrightarrow \mathcal{L} := \mathcal{L}'' \otimes \mathcal{O}_X(m_1 P_1 + \cdots + m_r P_r)$

and $H^0(X, \mathcal{L}'') \hookrightarrow H^0(X, \mathcal{L}) \hookrightarrow H^0(X, \mathcal{K}_X) = K(X)$

at P_i: $\mathcal{L}_{P_i} = (\mathcal{L}'' \otimes \mathcal{O}_X(m_i P_i))_{P_i} = \pi_{P_i}^{m_i} \mathcal{L}''_{P_i}$ $\quad \pi_{P_i}$ is a uniformizer at P_i

$\subset \mathcal{K}_{X, P_i} = K(X)$