Given a scheme $X$ and a point $x \in X$, let $V = \text{Spec } A$ be an affine neighborhood of $x$ and let $p$ be the prime ideal of $A$ corresponding to $x$. If $\mathcal{O}_{X,x} = A_p$ is Cohen-Macaulay, then its maximal ideal contains a regular sequence $a_1, \ldots, a_d$ with $d = \dim A_p = \text{height } p$. After possibly localizing $A$ (at the denominators of $a_1, \ldots, a_d$) we can assume $a_1, \ldots, a_d \in A$.

By Hauptidealsatz, $\dim \frac{A}{\langle a_1, \ldots, a_d \rangle} = \dim A - d$

$= \dim \left\{\overline{p}\right\}$

$= \dim \frac{A}{p}$

Hence $\text{Spec } A/\langle a_1, \ldots, a_d \rangle = \overline{\{p\}}$ is an irreducible component of $\text{Spec } A/\langle a_1, \ldots, a_d \rangle = \overline{\{(a_1)\} \cap \cdots \cap \overline{(a_d)}}$

After further localizing $A$, we can assume $\text{Spec } A/\langle a_1, \ldots, a_d \rangle$
is irreducible (each irreducible component is closed and after removing it we have a smaller neighborhood of \( x \)).

Hence, the closure \( \overline{\{P\}} \) is "set-theoretically" cut out by the \( d \) equations \( a_1, \ldots, a_d \).

Examples of Cohen-Macaulay rings:

1. Noetherian local rings of dimension 0 are Cohen-Macaulay. Note that Noetherian local rings of dimension 0 are Artin rings. A ring is an Artin ring if it satisfies the descending chain condition for ideals.
(2) One-dimensional reduced noetherian rings are Cohen-Macaulay.

(3) Two-dimensional normal noetherian rings are Cohen-Macaulay.
A ring is called normal if it is reduced and integrally closed in its total quotient ring.

(4) If $A$ is a finitely generated Cohen-Macaulay algebra over a field $k$ with an action of a finite group $G$, then the subring of invariant $A^G$ is Cohen-Macaulay.

(5) Determinantal rings are Cohen-Macaulay.
A ring is called determinantal if it is a quotient $B = A/I$, where $A$ is a regular local ring and $I$ is the ideal generated by the $rxn$ minors of a $p \times q$
matrix with coefficients in $\mathbb{A}$ s.t. the height of any minimal prime of $I$ is the expected codimension $(p-2+1)(q-2+1)$. The main example of such an ideal is the ideal of the locus of matrices of rank $< r$ in the space of all $p \times q$ matrices with entries in a field.

Curves: From now on, all schemes are over an algebraically closed field $k$.

Def: A curve $X/k$ is an integral separated scheme of finite type over $k$, of dimension 1. We say $X$ is complete if it is proper over $k$. We will prove some nice results about curves.
Lemma 1: \(X\) is non-singular curve \(\mathbb{A}^1\), \(f: X \to Y\) a rational map (i.e., \(f\) is a morphism from an open dense subscheme of \(X\) to \(Y\)) to a projective variety \(Y\).

Then \(f\) extends to a morphism \(X \to Y\).

Proof: Choose an embedding \(Y \to \mathbb{P}^n\).

We can replace \(Y\) with \(\mathbb{P}^n\) because if \(f\) extends to a morphism \(X \to \mathbb{P}^n\), then the morphism factors through \(Y\) because \(Y\) is a closed subscheme of \(\mathbb{P}^n\).

Step 1: Let \(V\) be a non-empty open set where \(f\) is well-defined.

Lemma 2: The datum of a rational map \(f: X \to \mathbb{P}^n\) is equivalent to the data of an invertible sheaf \(L\) on \(X\) and
sections $s_0, \ldots, s_n \in H^0(X, \mathcal{L})$ s.t. \( \mathcal{L}|_U \cong f^*G_{\mu_n}(1) \)

\( \forall i, s_i|_U = f^*X_i \) and \( U \subseteq \{x \in X \mid \exists i \text{ s.t. } (v)_x \not\equiv (s_i)_x \} \)

**Proof:** Put \( \mathcal{M} := f^*G_{\mu_n}(1) \) which is an invertible sheaf on \( U \),

and put \( t_i := f^*X_i \) for \( i = 0, \ldots, n \).

Also put \( \{P_1, \ldots, P_n\} := X \setminus U \).

Via the exact sequence

\[
\bigoplus_{i=1}^n \mathbb{Z}[P_i] \rightarrow \mathcal{O}(X) \rightarrow \mathcal{O}(U) \rightarrow 0
\]

choose \( \mathcal{L}'' \) on \( X \) s.t. \( \mathcal{L}''|_U = \mathcal{M} \).

Recall that \( \mathcal{L}'' \otimes K_X \cong K_X \) because both sheaves are the constant sheaf with group \( K(X) \).

**Definition:** A rational section of an invertible sheaf \( \mathcal{L} \) on an integral scheme \( X \) is a global section \( s \) of \( K_X \) s.t.
\[ \exists \, V \neq \emptyset, \forall \, \text{open } U \text{ with } \cap_{i=1}^n V_i \in \mathcal{H}^0(X, \mathcal{O}_X) \text{ via the embedding } L_c \hookrightarrow L \otimes \mathcal{O}_X = K_X. \text{ We say } \gamma \text{ is regular on } V. \]

Back to the proof of Lemma 2:

We have

\[ L'' \mid_U \hookrightarrow (K_c = K \otimes \mathcal{O}_X) \mid_U \]

\[ M \to K \mid_U = K \mid_U \otimes M \]

So \( s_0, \ldots, s_n \in H^0(U, M) \hookrightarrow H^0(K_c U) = K(X) = H^0(K_c X) \)

are rational sections of \( L'' \), regular on \( U \).

For \( i = 1, \ldots, n \), put \( m_i := \text{Max} \{ 0, -\nu_{P_i}(t_0), j=0, \ldots, n \} \)

Then

\[ \mathcal{O}(\mathcal{O}_{(-m_1 P_1, \ldots, -m_n P_n}) \to X \text{ (ideal sheaf)} \]

\[ L'' \hookrightarrow L := L'' \otimes \mathcal{O}_X(\mathcal{O}(m_1 P_1 + \cdots + m_n P_n)) \]

and

\[ H^0(X, L'') \to H^0(X, L) \to H^0(X, K_c X) = K(X) \]

at \( P_i \):

\[ L_{P_i} = (L'' \otimes \mathcal{O}_X(m_i P_i))_{P_i} = H_{P_i}^{\nu_{P_i}(t_0)} L'' \]

\[ L_{P_i} \otimes \mathcal{O}_{X_{P_i}} = K(X) \]

\[ \text{\( H_{P_i} \) is a uniformizer at } P_i \]