(5) If \( X \) is a noetherian scheme and \( \mathcal{F} \) is coherent, then \( \forall x \in X \) and \( \forall i \) and \( \forall y \in \mathcal{Y}_x \)-module:

\[
\text{Ext}_X^i(\mathcal{F}, y)_x \cong \text{Ext}_X^i(\mathcal{F}_x, y_x)
\]

Dualizing sheaves:

**Def:** \( X \) proper scheme of dimension \( n \) over a field \( k \).

A dualizing sheaf for \( X \) is a coherent sheaf \( \omega_X \), together with a “trace” morphism

\[
t : \text{Hom}(\mathcal{F}_x, \omega_x) \to k
\]

s.t. \( \forall \) coherent \( \mathcal{F} \), the natural pairing

\[
\text{Hom}(\mathcal{F}_x, \omega_x) \times H^n(X, \mathcal{F}) \to H^n(X, \omega_x)
\]
followed by it is a perfect pairing, i.e., it gives an isomorphism
\[ \text{Hom} (\mathcal{E}^\vee, \omega_X) \cong H^m (X, \mathcal{E})^\vee. \]

Fact:

1. Dualizing sheaves do not always exist, but are unique when they do.
2. If \( X \) is projective, then
   \[ \omega_X \cong \text{Ext}_{\mathcal{O}_{\mathbb{P}^n}}^r (\mathcal{O}_X, \mathcal{O}_{\mathbb{P}^n}) \]
   where \( X \hookrightarrow \mathbb{P}^n \), \( r = m - n \) is the codimension of \( X \) in \( \mathbb{P}^n \), \( \mathcal{O}_{\mathbb{P}^n} := \bigwedge^m \mathcal{O}_{\mathbb{P}^n} \).
(3) In particular, apply (2) to $X = \mathbb{P}^n$:

$$\omega_{\mathbb{P}^n} = \text{Ext}^0_{\mathbb{P}^n}(\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}^*)$$

$$= \text{Hom}_{\mathbb{P}^n}(\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}^*) = \mathcal{O}_{\mathbb{P}^n}^*$$

Recall the Euler sequence:

$$0 \to \Omega^1_{\mathbb{P}^n}(1) \to H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \otimes \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1) \to 0$$

Twist back by $-1$:

$$0 \to \Omega^1_{\mathbb{P}^n}(-1) \to H^0(\mathcal{O}_{\mathbb{P}^n}(-1)) \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \to \mathcal{O}_{\mathbb{P}^n}(-1) \to 0$$

Take top exterior power (past homework):
\[ \Lambda^{m+1}(H^0(\mathcal{O}_P^{\mu+1}(-1)) \otimes \mathcal{O}_P^{\mu+1}) \cong \Lambda^{m+1} \mathcal{O}_P^{\mu+1} \otimes \mathcal{O}_P^{\mu+1} \]

\[ \Lambda^{m+1}(\mathcal{O}_P^{\mu+1}(-1) \oplus (\mu+1)) \cong \mathcal{O}_P^{\mu+1} \]

\[ \mathcal{O}_P^{\mu+1} \cong \mathcal{O}_P^{\mu+1} \otimes (\mu+1) \]

\[ \mathcal{O}_P^{\mu+1}(-1) \cong \mathcal{O}_P^{\mu+1} \]

\[ \mathcal{O}_P^{\mu+1}(-\mu-1) \cong \mathcal{O}_P^{\mu+1} = \mathcal{O}_P^{\mu+1} \]

(4) More generally, if $X$ is a nonsingular projective (mod.) variety over a field, then the dualizing sheaf of $X$ is its canonical sheaf, i.e., the top exterior power of its sheaf of differentials $\Omega^1_X$. 
(5) Local Duality:

Theorem: If $X$ is a Cohen-Macaulay and projective
of pure dimension $n$ (i.e., all irreducible components
have dimension $n$), then, for all coherent sheaves $\mathcal{F}$
and all $i$, there are functional isomorphisms:

$$\text{Ext}^i(\mathcal{F}, \mathcal{O}_X) \cong H^{n-i}(X, \mathcal{F}^*)^*.$$ 

For all locally free sheaves of finite rank:

$$H^i(X, \mathcal{F}) \cong H^{n-i}(X, \mathcal{O}_X \otimes \mathcal{F}^*)^*.$$ 

(recall $\mathcal{F}^* := \hom_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$)
(6) If $X$ is projective and a local complete intersection of codimension $n$ in $\mathbb{P}^m$, then 

$$\mathcal{O}_X = \mathcal{O}_{\mathbb{P}^m} \otimes \mathcal{N}(\mathcal{I}_X/\mathcal{I}_X^2)$$.

**Def:** $(\mathcal{I}_X/\mathcal{I}_X^2)^*$ is the normal sheaf of $X$ in $\mathbb{P}^n$.

$\mathcal{I}_X/\mathcal{I}_X^2$ is the conormal sheaf.

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**Regular sequence and the Cohen-Macaulay condition:**

**Def:** given a ring $A$ and an $A$-module $M$, a sequence $a_1, \ldots, a_n$ of elements of $A$ is called regular for $M$ if

1. $a_1$ is not a zero divisor in $M$ ($a_1: M \rightarrow M$)
2. $\forall i \geq 2$, $a_i$ is not a zero divisor in $M/(a_1, \ldots, a_{i-1})M$. 
Def: If $A$ is a local ring with maximal ideal $m$, the depth of $M$ is the maximum length of a regular sequence $\{a_1, \ldots, a_n\} \subseteq m$.

A noetherian local ring $A$ is called Cohen–Macaulay if its depth as a module over itself is equal to its dimension.

Facts: Regular local rings are Cohen–Macaulay and quotients of Cohen–Macaulay rings by ideals generated by regular sequences are Cohen–Macaulay.

Def: (1) A scheme is Cohen–Macaulay if all its local rings are Cohen–Macaulay.
(2) A scheme is a local complete intersection if \( \forall x \in X \exists \) affine neighborhood \( U = \text{Spec} A \ni x \)

s.t. \( A = B/\mathcal{I} \) where \( \mathcal{I} \) can be generated by a sequence \( a_1, \ldots, a_n \in \mathcal{I} \) s.t.

\( \forall y \in \text{Spec} B \quad (a_1)_y, \ldots, (a_n)_y \in (\mathcal{I})_y \) is a regular sequence.

and \( \text{Spec} B \) is a regular scheme.

Intuitively, Cohen–Macaulay means that the closed points of \( X \) can be cut out by regular sequences.
Fact: If $a_1, \ldots, a_n$ is a regular sequence and $I := (a_1, \ldots, a_n) \subset A$, then $I/\mathfrak{m} = A/\mathfrak{m}$ is a free $A$-module of rank $n$ and, for all $d$, the natural map

$$\text{Sym}^d(I/\mathfrak{m}) \to I^d/\mathfrak{m}^{d+1}$$

is an isomorphism.

To say that a noetherian local ring of dimension $n$ is Cohen-Macaulay means that there exists a regular sequence $a_1, \ldots, a_n$ such that the quotient $A/(a_1, \ldots, a_n)$ has dimension zero. (When the ring is regular, a sequence $a_1, \ldots, a_n$ is a field.)