(5) If X is a noetherian scheme and $\mathcal{F}$ is coherent, then $\forall\, x \in X$ and $\forall\, i$ and $\forall\, \mathcal{G}$ $\mathcal{O}_X$-module:

$$\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \cong \mathrm{Ext}^i_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$$

Dualizing sheaves:

Def: X proper scheme of dimension n over a field k.
A dualizing sheaf for X is a coherent sheaf ω_X, together with a "trace" morphism

$$t : H^n(X, \omega_X) \longrightarrow k$$

s.t. $\forall$ coherent $\mathcal{F}$, the natural pairing

$$\mathrm{Hom}(\mathcal{F}, \omega_X) \times H^n(X, \mathcal{F}) \longrightarrow H^n(X, \omega_X)$$

followed by t is a perfect pairing, i.e., it gives an isomorphism

$$\mathrm{Hom}(\mathcal{F}, \omega_X) \cong H^n(X, \mathcal{F})^*.$$

Facts:

(1) Dualizing sheaves do not always exist, but are unique when they do.

(2) If X is projective, then

$$\omega_X \cong \mathcal{E}xt^r_{\mathbb{P}^m}(\mathcal{O}_X, \Omega^m_{\mathbb{P}^m})$$

where $X \hookrightarrow \mathbb{P}^m$, $r = m-n$ is the codimension of X in $\mathbb{P}^m$, $\Omega^m_{\mathbb{P}^m} := \Lambda^m \Omega^1_{\mathbb{P}^m}$

(3) In particular, apply (2) to $X = \mathbb{P}^n$:

$$\omega_{\mathbb{P}^n} \cong \mathcal{E}xt^0_{\mathbb{P}^n}(\mathcal{O}_{\mathbb{P}^n}, \Omega^n_{\mathbb{P}^n})$$
$$\cong \mathcal{H}om_{\mathbb{P}^n}(\mathcal{O}_{\mathbb{P}^n}, \Omega^n_{\mathbb{P}^n}) \cong \Omega^n_{\mathbb{P}^n}$$

Recall the Euler sequence:

$$0 \to \Omega^1_{\mathbb{P}^n}(1) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \otimes \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1) \longrightarrow 0$$

$$H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \otimes \mathcal{O}_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)}$$

twist back by -1:

$$0 \to \Omega^1_{\mathbb{P}^n} \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n} \to 0$$

Take top exterior powers (past homework):

$$\Lambda^{n+1}\left(H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \otimes \mathcal{O}_{\mathbb{P}^n}(-1)\right) \cong \Lambda^n \Omega^1_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}$$

$$\Lambda^{n+1}\left(\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus(n+1)}\right) \cong \Omega^n_{\mathbb{P}^n}$$

or $$\mathcal{O}_{\mathbb{P}^n}(-1)^{\otimes(n+1)} \cong \Omega^n_{\mathbb{P}^n}$$

or $$\mathcal{O}_{\mathbb{P}^n}(-n-1) \cong \Omega^n_{\mathbb{P}^n} = \omega_{\mathbb{P}^n}$$

(4) More generally, if X is a nonsingular projective (irred.) variety over a field, then the dualizing sheaf of X is its canonical sheaf, i.e., the top exterior power of its sheaf of differentials Ω^1_X.

(5) <u>Serre Duality:</u>

<u>Theorem:</u> If X is Cohen-Macaulay and projective of pure dimension n (i.e., all irreducible components have dimension n), then, for all coherent sheaves $\mathcal{F}$ and all i, there are functorial isomorphisms:

$$\mathrm{Ext}^i(\mathcal{F}, \omega_X) \cong H^{n-i}(X, \mathcal{F})^*.$$

For all locally free sheaves of finite rank:

$$H^i(X, \mathcal{F}) \cong H^{n-i}(X, \omega_X \otimes \mathcal{F}^*)^*$$

(recall $\mathcal{F}^* := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$)

(6) If X is projective and a local complete intersection of codimension r in $\mathbb{P}^n$, then

$$\omega_X \cong \omega_{\mathbb{P}^n} \otimes \bigwedge^r \left(\mathcal{I}_X/\mathcal{I}_X^2\right)^*.$$

Def: $\left(\mathcal{I}_X/\mathcal{I}_X^2\right)^*$ is the normal sheaf of X in $\mathbb{P}^n$

$\mathcal{I}_X/\mathcal{I}_X^2$ is the conormal sheaf.

Regular sequences and the Cohen-Macaulay condition:

Def: given a ring A and an A-module M, a sequence $a_1, \dots, a_r$ of elements of A is called regular for M if

(1) a_1 is not a zero divisor in M ($a_1 : M \hookrightarrow M$)

(2) $\forall i \geq 2$, a_i is not a zero divisor in $M/(a_1, \dots, a_{i-1})M$.

Def: If A is a local ring with maximal ideal m, the depth of M is the maximum length of a regular sequence $\{a_1, \dots, a_n\} \subset m$.

A noetherian local ring is called Cohen-Macaulay if its depth as a module over itself is equal to its dimension.

Facts: Regular local rings are Cohen-Macaulay and quotients of Cohen-Macaulay rings by ideals generated by regular sequences are Cohen-Macaulay.

Def: (1) A scheme is Cohen-Macaulay if all its local rings are Cohen-Macaulay.

(2) A scheme is a local complete intersection if $\forall\ x \in X$ $\exists$ affine neighborhood $U = \operatorname{Spec} A \ni x$.

s.t. $A = B/I$ where I can be generated

by a sequence $a_1, \dots, a_n \in I$ s.t.

$\forall\ y \in \operatorname{Spec} B$ $(a_1)_y, \dots, (a_n)_y \in (\tilde{I})_y$

is a regular sequence.

and $\operatorname{Spec} B$ is a regular scheme.

Intuitively, Cohen-Macaulay means that the closed points of X can be cut out by regular sequences.

Fact: If $a_1, \dots, a_n$ is a regular sequence and $I := (a_1, \dots, a_n) \subset A$, then I/I^2 is a free A-module of rank n and, for all d, the natural map

$$\mathrm{Sym}^d(I/I^2) \longrightarrow I^d/I^{d+1}$$

is an isomorphism.

To say that a noetherian local ring of dimension n is Cohen-Macaulay means that there exists a regular sequence $a_1, \dots, a_n$ s.t. the quotient $A/(a_1, \dots, a_n)$ has dimension zero.

(When the ring is regular, $\exists$ sequence s.t. the quotient is a field)