Sure vanishing:

Theorem: \( X \) a projective scheme over a noetherian ring \( A \), \( \mathcal{O}_X(1) \) a very ample invertible sheaf on \( X \).

Then for any coherent sheaf on \( X \):

1. \( H^i(X, \mathcal{F}) \), is a finitely generated \( A \)-module \( \forall \ i \).
2. \( \exists \ n_0 \in \mathbb{Z} \) (depending on \( \mathcal{F} \)) s.t. \( \forall \ n \geq n_0, \forall \ i > 0, \)
   \[ H^i(X, \mathcal{F}(n)) = 0 \quad \text{where} \quad \mathcal{F}(n) := \mathcal{F} \otimes \mathcal{O}_X(n) \]

Proof: Let \( i : X \hookrightarrow \mathbb{P}^n_A \) be an embedding such that \( \mathcal{O}_X(1) \cong i^* \mathcal{O}_{\mathbb{P}^n}(1) \). Since \( X \) is projective over \( A \), it is proper over \( A \). \( \Rightarrow \) \( i \) is proper \( \Rightarrow \) image of \( i \) is closed

\( \Rightarrow \) \( i \) is a closed embedding.
\[ \Rightarrow i^* F \text{ and } F \text{ have the same cohomology.} \]

we also know \( i^* F \) is coherent.

So we can assume \( X = \mathbb{P}^n \).

If \( F = O_X (n) \) for some \( n \), then the theorem follows from the previous theorem.

\( O_X (1) \) is very ample \( \Rightarrow \) ample \( \Rightarrow \exists m \in \mathbb{Z} \text{ s.t. } \)

\( F (m) \) is generated by global sections.

this means \( H^0 (X, F (m)) \otimes \mathbb{C} X \rightarrow \mathbb{F} (m) \)

twist by \( \mathbb{C} (-m) \) to obtain

\[ H^0 (X, F (m)) \otimes \mathbb{C} (-m) \rightarrow F \]

take \( E := A^d \otimes \mathbb{C} (-m) \) where \( A^d \rightarrow H^0 (X, F (m)) \)
We have the exact sequence

\[ 0 \to R \to \mathcal{E} \to \mathcal{F} \to 0 \]

where \( R \) is, by def, the kernel of \( \mathcal{E} \to \mathcal{F} \).

We have the long exact sequence:

\[ \ldots \to H^i(X, \mathcal{E}) \to H^i(X, \mathcal{F}) \to H^{i+1}(X, R) \to \ldots \]

\[ \cong \]

\[ H^i(X, \mathcal{O}_X(-n)) \]

Now we use descending induction on \( i \).

\[ H^i(X, \mathcal{F}) = H^i(X, \mathcal{O}) = 0 \quad \forall \ i \geq n \]

\[ (\dim X = n \text{ or } X \text{ has a covering by } n+1 \text{ aff. open sets and } X \text{ is separated}) \]

\[ \Rightarrow (1) \]
For (2) twist by \( n \):
\[
0 \rightarrow \mathcal{R}(n) \rightarrow \mathcal{E}(n) \rightarrow \mathcal{F}(n) \rightarrow 0
\]

The long exact sequence:
\[
\ldots \rightarrow H^i(\mathcal{E}(n)) \rightarrow H^i(\mathcal{F}(n)) \rightarrow H^{i+1}(\mathcal{R}(n)) \rightarrow \ldots \]
again use descending induction on \( i \).


cohomological criterion for ampleness:

Theorem: \( X \) a proper scheme over \( A \) noetherian, \( \mathcal{L} \)

an invertible sheaf on \( X \). The following are equivalent:

1. \( \mathcal{L} \) is ample.
2. \( \forall \mathcal{F} \) coherent on \( X \), \( \exists n_0 \in \mathbb{Z}, n \) s.t. \( \forall i > 0 \) and \( \forall n \geq n_0 \), \( H^i(X, \mathcal{F} \otimes \mathcal{L}^n) = 0 \)
Proof: (1) ⇒ (2) X is ample.

⇒ \exists m > 0 \text{ s.t. } L^m \text{ is very ample} (A)

Then, by base vanishing, \exists b_0 \text{ s.t. } \forall n \geq b_0, \forall i > 0

H^i(F_0 \otimes (L^m)^n) = 0.

\therefore 0 \leq i \leq m - 1 \text{ put } F_i := F_0 \otimes L^i.

\forall n \geq b_0, \exists b_i \text{ s.t. } \forall n \geq b_i, \forall i > 0

H^i(X, F_i \otimes (L^m)^n) = 0

Put \beta = \max \{ b_i : 0 \leq i \leq m - 1 \}, n_0 := \beta \cdot m

\forall n \geq n_0, \text{ write } n = q \cdot m + r \quad 0 \leq r \leq m - 1, q \geq \beta

⇒ H^i(X, F_0 \otimes L^n) = H^i(X, F_{q \cdot m} \otimes L^r) = 0 \quad \forall i > 0
We will prove that for all coherent $F$, $\exists n_0 \in \mathbb{Z}$, s.t. $\forall n \geq n_0$, $F \otimes \mathbb{L}^n$ is generated by global sections.

Let $p \in X$ be a closed point. We have the exact reduced sequence:

$$0 \rightarrow J_p \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_p \rightarrow 0$$

form with $F$

$$F \otimes \mathcal{O}_p \rightarrow F \rightarrow F \otimes \mathcal{O}_p \rightarrow 0$$

such that $J_p F = \text{image of } F \otimes \mathcal{O}_p \text{ in } F$ to obtain the exact sequence

$$0 \rightarrow J_p F \rightarrow F \rightarrow F \otimes \mathcal{O}_p \rightarrow 0.$$
twist with $L^\mu$:

\[ 0 \to T_p \mathcal{F} \otimes L^\mu \to \mathcal{F} \otimes L^\mu \to \mathcal{F} \otimes L^\mu \otimes \mathcal{O}_p \to 0 \]

\[ \exists m_0 \in \mathbb{Z} \text{ s.t. } \forall n \geq m_0, \quad H^1(T_p \mathcal{F} \otimes L^\mu) = 0 \]

(If $\mathcal{F}$ is coherent $\Rightarrow$ $T_p \mathcal{F}$ is coherent $\Rightarrow$ $T_p \mathcal{F}$ is coherent)

$\Rightarrow$ the long exact sequence of cohomology gives a

morphism

\[ H^0(X, T_p \mathcal{F} \otimes L^\mu) \to H^0(\mathcal{F} \otimes L^\mu \otimes \mathcal{O}_p) \]

for $n \geq m_0, \mu$

$\mathcal{F} \otimes L^\mu \otimes \mathcal{O}_p$ is a skyscraper sheaf supported at $p$.

$\Rightarrow$ we can identify $H^0(\mathcal{F} \otimes L^\mu \otimes \mathcal{O}_p)$ with the stalk at $p$. 
We have \((F_0 \otimes \mathcal{L}_m)^\dagger = \frac{((F_0 \otimes \mathcal{L}_m)^\dagger)}{M_p (F_0 \otimes \mathcal{L}_m)^\dagger}\).

So we can write the injection from the previous page as:

\[
H^0 (X, F_0 \otimes \mathcal{L}_m) \rightarrow \frac{((F_0 \otimes \mathcal{L}_m)^\dagger)}{M_p (F_0 \otimes \mathcal{L}_m)^\dagger}
\]

Nakayama's lemma \Rightarrow

\[
H^0 (X, F_0 \otimes \mathcal{L}_m) \rightarrow (F_0 \otimes \mathcal{L}_m)^\dagger
\]

Since \(F_0\) is coherent, \(\exists\) open neighborhood \(U_{p,n}\) of \(\phi\) s.t. \((F_0 \otimes \mathcal{L}_m)^\dagger\) is generated by global sections. We need to find a neighborhood which does not depend on \(n\).
Take $\mathcal{F}_p = \mathcal{O}_X$, then \( \exists \ m_{n, p} \) and a neighborhood \( U_p \) of \( p \) s.t. \( L^{m_{n, p}} \) is generated by its global sections on \( U_p \).

Also, \( \forall \ n = 0, 1, \ldots, m_{n, p} - 1 \), \( \exists \) neighborhood \( U_{n, p} \) of \( p \) s.t. \( \mathcal{F}_p \otimes L^{m_{n, p} + n} \) is generated by global sections on \( U_{n, p} \).

Put \( U_p := U_{n, p} \cap U_{0, p} \cap \cdots \cap U_{n_{1}, p} \).

Then on \( U_p \), all the sheaves \( L^{m_{n, p}}, \mathcal{F}_p \otimes L^{m_{n, p} + n} \)

\( n = 0, \ldots, m_{n, p} - 1 \) are globally generated. Tensor products of globally generated sheaves are also globally generated. \( \forall \ m \geq m_{0, p}, \exists \ m > 0 \) and \( n \in \{0, \ldots, m_{n, p} - 1\} \)

s.t. \( m = m_{0, p} + n + m_{n, p} \).
Then $F \otimes \mathbb{L}^m = F \otimes \mathbb{L}^{m,1} \otimes (\mathbb{L}^{m,1})^m$ is globally generated on $U_x$.

Now cover $X$ by a finite number of open neighborhoods $U_{x_1}, \ldots, U_{x_k}$ and put $m_0 = \max \{m_{0,x_i}\}$.

**Exterior groups and sheaves:**

$(X, \mathcal{O}_X)$ a ringed space. $F$ an $\mathcal{O}_X$-module.

The functors $\text{Hom}_{\mathcal{O}_X}(F, \cdot)$ and $\text{Hom}_{\mathcal{O}_X}(F, \cdot)$ are left exact covariant functors from the category $\text{Mod}(\mathcal{O}_X)$ to the categories $\text{Ab}$ and $\text{Mod}(\mathcal{O}_X)$. 