Serre vanishing:

Theorem: X a projective scheme over a noetherian ring A, $\mathcal{O}_X(1)$ a very ample invertible sheaf on X.
Then for any coherent sheaf on X:

(1) $H^i(X, \mathcal{F})$ is a finitely generated A-module $\forall i$.

(2) $\exists\, n_0 \in \mathbb{Z}$ (depending on $\mathcal{F}$) s.t. $\forall\, n \geq n_0$, $\forall\, i > 0$,

$$H^i(X, \mathcal{F}(n)) = 0 \quad \text{where } \mathcal{F}(n) := \mathcal{F} \otimes \mathcal{O}_X(n)$$
$$:= \mathcal{F} \otimes \mathcal{O}_X(1)^{\otimes n}$$

Proof: Let $i : X \hookrightarrow \mathbb{P}^n_A$ be an embedding such that $\mathcal{O}_X(1) \cong i^* \mathcal{O}_{\mathbb{P}^n}(1)$. Since X is projective$/A$, it is proper over A. $\Rightarrow$ i is proper $\Rightarrow$ image of i is closed

$\Rightarrow$ i is a closed embedding.

$\Rightarrow$ $i_*\mathcal{F}$ and $\mathcal{F}$ have the same cohomology.

we also know $i_*\mathcal{F}$ is coherent.

So we can assume $X = \mathbb{P}^n_A$.

If $\mathcal{F} = \mathcal{O}_X(r)$ for some r, then the theorem follows from the previous theorem.

$\mathcal{O}_X(1)$ is very ample $\Rightarrow$ ample $\Rightarrow \exists\, m \in \mathbb{Z}$ s.t. $\mathcal{F}(m)$ is generated by global sections.

this means $H^0(X, \mathcal{F}(m)) \otimes_A \mathcal{O}_X \twoheadrightarrow \mathcal{F}(m)$

twist by $\mathcal{O}_X(-m)$ to obtain

$$H^0(X, \mathcal{F}(m)) \otimes \mathcal{O}_X(-m) \twoheadrightarrow \mathcal{F}$$

take $\mathcal{E} := A^{\oplus l} \otimes \mathcal{O}_X(-m)$ where $A^{\oplus l} \twoheadrightarrow H^0(X, \mathcal{F}(m))$

We have the exact sequence

$$0 \to \mathcal{R} \to \mathcal{E} \to \mathcal{F} \to 0$$

where $\mathcal{R}$ is, by def, the kernel of $\mathcal{E} \twoheadrightarrow \mathcal{F}$.

We have the long exact sequence:

$$\begin{array}{ccccccc} \cdots \to & H^i(X, \mathcal{E}) & \to & H^i(X, \mathcal{F}) & \to & H^{i+1}(X, \mathcal{R}) & \to \cdots \\ & \wr\!\!\mid\cong & & & & & \\ & H^i(X, \mathcal{O}_X(-m))^{\oplus d} & & & & & \end{array}$$

Now we use descending induction on i.

$$H^i(X, \mathcal{F}) = H^i(X, \mathcal{R}) = 0 \quad \forall\, i > n$$

($\dim X = n$ or X has a covering by $n+1$ affine open sets and X is separated)

$\Rightarrow$ (1)

For (2) twist by n:

$$0 \to \mathcal{R}(n) \to \mathcal{E}(n) \to \mathcal{F}(n) \to 0$$

The long exact sequence:

$$\cdots \to H^i(\mathcal{E}(n)) \to H^i(\mathcal{F}(n)) \to H^{i+1}(\mathcal{R}(n)) \to \cdots$$

again use descending induction on i. $\square$

Cohomological criterion for ampleness:

<u>Theorem</u>: X a proper scheme over A noetherian, $\mathcal{L}$ an invertible sheaf on X. The following are equivalent:

(1) $\mathcal{L}$ is ample.

(2) $\forall$ $\mathcal{F}$ coherent on X, $\exists$ $n_0 \in \mathbb{Z}$, s.t. $\forall$ $i>0$ and $\forall$ $n \geq n_0$, $H^i(X, \mathcal{F} \otimes \mathcal{L}^n) = 0$

<u>Proof</u>: (1) $\Rightarrow$ (2) $\quad \mathcal{L}$ is ample.

$\Rightarrow \exists\ m > 0$ s.t. $\mathcal{L}^m$ is very ample (/A)

Then, by Serre vanishing $\exists\ l_0$ s.t. $\forall\ n \geq l_0, \forall i > 0$

$$H^i(\mathcal{F} \otimes (\mathcal{L}^m)^n) = 0.$$

$\mathcal{F}_r \quad 0 \leq r \leq m-1$ put $\mathcal{F}_r := \mathcal{F} \otimes \mathcal{L}^r$.

$\forall\ r \quad 0 \leq r \leq m-1 \quad \exists\ l_r$ s.t. $\forall\ n \geq l_r, \forall i > 0$

$$H^i(X, \mathcal{F}_r \otimes (\mathcal{L}^m)^n) = 0$$

Put $p = \max\{l_i \mid i = 0, m-1\}$, $n_0 := pm$

$\forall\ n \geq n_0$, write $n = qm + r \qquad 0 \leq r \leq m-1$, $q \geq p$

$$\Rightarrow H^i(X, \mathcal{F} \otimes \mathcal{L}^n) = H^i(X, \mathcal{F}_r \otimes \mathcal{L}^{qm}) = 0 \quad \forall i > 0$$

$(2) \Rightarrow (1)$

We will prove that for all coherent $\mathcal{F}$, $\exists\, n_0 \in \mathbb{Z}$,

s.t. $\forall\, n \geq n_0$, $\mathcal{F} \otimes \mathcal{L}^n$ is generated by global

sections.

Let $p \in X$ be a closed point. We have the exact

sequence: $\quad \{p\} \hookrightarrow X$ one point closed reduced subscheme.

$$0 \longrightarrow \mathcal{I}_p \longrightarrow \mathcal{O}_X \longrightarrow \underbrace{\mathcal{O}_p}_{\cong\, k(p) \text{ at } p} \longrightarrow 0$$

tensor with $\mathcal{F}$

$$\mathcal{F} \otimes \mathcal{I}_p \longrightarrow \mathcal{F} \longrightarrow \mathcal{F} \otimes \mathcal{O}_p \longrightarrow 0$$

put $\mathcal{I}_p \mathcal{F}$ = image of $\mathcal{F} \otimes \mathcal{I}_p$ in $\mathcal{F}$ to obtain the

exact sequence $0 \to \mathcal{I}_p \mathcal{F} \to \mathcal{F} \longrightarrow \mathcal{F} \otimes \mathcal{O}_p \to 0$.

twist with $\mathcal{L}^n$:

$$0 \longrightarrow \mathcal{I}_p \mathcal{F} \otimes \mathcal{L}^n \longrightarrow \mathcal{F} \otimes \mathcal{L}^n \longrightarrow \mathcal{F} \otimes \mathcal{L}^n \otimes \mathcal{O}_p \longrightarrow 0$$

$\exists\, n_{0,p} \in \mathbb{Z}$ s.t. $\forall\, n \geq n_{0,p}$ $H^1(\mathcal{I}_p \mathcal{F} \otimes \mathcal{L}^n) = 0$

($\mathcal{I}_p$ is coherent $\Rightarrow$ $\mathcal{I}_p \otimes \mathcal{F}$ is coherent $\Rightarrow$ $\mathcal{I}_p \mathcal{F}$ is coherent)

$\Rightarrow$ the long exact sequence of cohomology gives a surjection

$$H^0(X, \mathcal{F} \otimes \mathcal{L}^n) \twoheadrightarrow H^0(\mathcal{F} \otimes \mathcal{L}^n \otimes \mathcal{O}_p)$$

for $n \geq n_{0,p}$

$\mathcal{F} \otimes \mathcal{L}^n \otimes \mathcal{O}_p$ is a skyscraper sheaf supported at p.

$\Rightarrow$ we can identify $H^0(\mathcal{F} \otimes \mathcal{L}^n \otimes \mathcal{O}_p) \cong \underbrace{(\mathcal{F} \otimes \mathcal{L}^n \otimes \mathcal{O}_p)_p}_{\text{stalk at } p}$

We have $(\mathcal{F} \otimes \mathcal{L}^n \otimes \mathcal{O}_p)_p = \dfrac{(\mathcal{F} \otimes \mathcal{L}^n)_p}{m_p (\mathcal{F} \otimes \mathcal{L}^n)_p}$

So we can write the surjection from the previous page as:

$$H^0(X, \mathcal{F} \otimes \mathcal{L}^n) \twoheadrightarrow \frac{(\mathcal{F} \otimes \mathcal{L}^n)_p}{m_p (\mathcal{F} \otimes \mathcal{L}^n)_p}$$

Nakayama's lemma $\Longrightarrow$

$$H^0(X, \mathcal{F} \otimes \mathcal{L}^n) \twoheadrightarrow (\mathcal{F} \otimes \mathcal{L}^n)_p$$

Since $\mathcal{F}$ is coherent, $\exists$ open neighborhood $U_{p,n}$ of p s.t. $(\mathcal{F} \otimes \mathcal{L}^n)\big|_{U_{p,n}}$ is generated by global sections. We need to find a neighborhood which does not depend on n.

Take $\mathcal{F} = \mathcal{O}_X$, then $\exists\ n_{1,p}$ and a neighborhood V_p of p s.t. $\mathcal{L}^{n_{1,p}}$ is generated by its global sections on V_p.

Also, $\forall\ r = 0, 1, \dots, n_{1,p} - 1$, $\exists$ neighborhood $U_{r,p}$ of p s.t. $\mathcal{F} \otimes \mathcal{L}^{n_{0,p}+r}$ is generated by global sections on $U_{r,p}$

Put $U_p := V_p \cap U_{0,p} \cap \cdots \cap U_{n_{1,p}-1}$.

Then on U_p, all the sheaves $\mathcal{L}^{n_{1,p}}$, $\mathcal{F} \otimes \mathcal{L}^{n_{0,p}+r}$ $r = 0, \dots, n_{1,p-1}$ are globally generated. Tensor products of globally generated sheaves are also globally generated. $\forall\ n \geq n_{0,p}$, $\exists\ m \geq 0$ and $r \in \{0, \dots, n_{1,p}-1\}$ s.t. $n = n_{0,p} + r + m n_{1,p}$

Then $\mathcal{F}\otimes\mathcal{L}^n = \mathcal{F}\otimes\mathcal{L}^{n_{0,p}+r}\otimes(\mathcal{L}^{n,p})^m$

is globally generated on U_p.

Now cover X by a finite number of open neighborhoods $U_{p_1}, \dots U_{p_k}$ and put $n_0 = \max\{n_{0,p_i}\}$

$\square$

Exterior groups and sheaves :

$(X, \mathcal{O}_X)$ a ringed space. $\mathcal{F}$ an $\mathcal{O}_X$-module

The functors $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \cdot)$ and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \cdot)$ are left exact covariant functors from the category $\mathrm{Mod}(\mathcal{O}_X)$ to the categories $\mathcal{A}b$ and $\mathrm{Mod}(\mathcal{O}_X)$.