Since the $U_{i_0\cdots i_p}$ are affine and the sheaves $\mathcal{F}, \mathcal{G}, \mathcal{R}$ are quasi-coherent, we also have the exact sequences of global sections, $\forall\, p \geq 0$:

$$0 \to \mathcal{F}(U_{i_0\cdots i_p}) \longrightarrow \mathcal{G}(U_{i_0\cdots i_p}) \longrightarrow \mathcal{R}(U_{i_0\cdots i_p}) \longrightarrow 0$$

and, taking products, we obtain the short exact sequence of complexes:

$$0 \longrightarrow C^{\cdot}(\mathcal{F}) \longrightarrow C^{\cdot}(\mathcal{G}) \longrightarrow C^{\cdot}(\mathcal{R}) \longrightarrow 0$$

which gives the long exact sequence of cohomology groups $0 \longrightarrow \check{H}^0(\mathcal{U}, \mathcal{F}) \longrightarrow \check{H}^0(\mathcal{U}, \mathcal{G}) \longrightarrow \cdots$

The canonical morphisms $\kappa : \check{H}^p(\mathcal{U}, *) \longrightarrow H^p(X, *)$ commute with the morphisms of long exact sequences of cohomology.

So we have commutative diagrams with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \check{H}^0(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d, "\kappa", "\cong"'] & \check{H}^0(\mathcal{U},\mathcal{G}) \arrow[r] \arrow[d, "\kappa", "\cong"'] & \check{H}^0(\mathcal{U},\mathcal{R}) \arrow[r] \arrow[d, "\kappa", "\cong"'] & \check{H}^1(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d, "\kappa"] & 0 \\
0 \arrow[r] & H^0(X,\mathcal{F}) \arrow[r] & H^0(X,\mathcal{G}) \arrow[r] & H^0(X,\mathcal{R}) \arrow[r] & H^1(X,\mathcal{F}) \arrow[r] & 0
\end{tikzcd}$$

(note $\check{H}^i(\mathcal{U},\mathcal{G}) = H^i(X,\mathcal{G}) = 0 \quad \forall\, i > 0$ because $\mathcal{G}$ is flasque)

$$\Rightarrow \quad \kappa : \check{H}^1(\mathcal{U},\mathcal{F}) \xrightarrow{\cong} H^1(X,\mathcal{F}).$$

For $i \geq 1$, we have commutative diagrams:

$$\begin{tikzcd}
0 \arrow[r] & \check{H}^i(\mathcal{U},\mathcal{R}) \arrow[r, "\delta"] \arrow[d, "\kappa"] & \check{H}^{i+1}(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d, "\kappa"] & 0 \\
0 \arrow[r] & H^i(X,\mathcal{R}) \arrow[r] & H^{i+1}(X,\mathcal{F}) \arrow[r] & 0
\end{tikzcd}$$

We know $\kappa: \check{H}^1(\mathcal{U}, \mathcal{F}) \xrightarrow{\cong} H^1(X, \mathcal{F})$ $\forall$ quasi-coherent sheaves on X. By induction, we obtain $\kappa: \check{H}^i(\mathcal{U}, \mathcal{F}) \xrightarrow{\cong} H^i(X, \mathcal{F})$ $\forall i$ and all quasi-coherent sheaves. $\square$.

Remark: When X is a separated Noetherian scheme, it satisfies the hypotheses of the theorem for any open affine covering by Ex. II.4.3 from last quarter's homework.

Corollary: If X has a covering by $r+1$ open affine sets s.t. all the intersections are affine, then $\forall$ quasi-coherent sheaves $\mathcal{F}$ and all $i > r$, $H^i(X, \mathcal{F}) = 0$.

We use the previous to compute cohomology on projective space:

Let A be a noetherian ring.

$$S := A[X_0, \dots, X_n]$$

$$X := \mathbb{P}^n_A = \operatorname{Proj} S$$

$\forall\ \mathcal{F}$ sheaf of $\mathcal{O}_X$-modules: $\Gamma_*(\mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} H^0(X, \mathcal{F}(n))$

Recall $\Gamma_*(\mathcal{O}_X) = S$ as graded S-modules.

and $\exists$ natural map

$$\widetilde{\Gamma_*(\mathcal{F})} \longrightarrow \mathcal{F}$$

which is an isomorphism if $\mathcal{F}$ is quasi-coherent.

<u>Theorem</u>: (1) $\forall\ r \in \mathbb{Z}$, $\forall\ i \neq 0, n$, $H^i(X, \mathcal{O}_X(r)) = 0$

(2) $H^n(X, \mathcal{O}_X(-n-1)) \cong A$

(3) $\forall\ r \in \mathbb{Z}$, the natural map (cup-product)

$$H^0(X, \mathcal{O}_X(r)) \times H^n(X, \mathcal{O}_X(-n-r-1)) \longrightarrow H^n(X, \mathcal{O}_X(-n-1)) = A$$

is a perfect pairing of finitely generated free A-modules.

<u>Proof</u>: Put $\mathcal{F} := \bigoplus_r \mathcal{O}_X(r)$. Then $\mathcal{F}$ is quasi-coherent, in fact, locally free of infinite rank. $\mathcal{F}$ has a $\mathbb{Z}$-grading. We compute $H^i(\mathcal{F})$, keeping track of the $\mathbb{Z}$-grading. (cohomology commutes with arbitrary direct sums: they are direct limits of finite direct sums).

Recall that all cohomology groups are

$H^0(X, \mathcal{O}_X) = A$ – modules.

We compute $\check{H}^i(\mathcal{U}, \mathcal{F})$ for $\mathcal{U} = \{U_0, \dots, U_n\}$ the usual open cover of X.

Recall $\quad U_i = D_+(X_i) \subset X$

$$U_{i_0, \dots, i_r} = D_+(X_{i_0} X_{i_1} \cdots X_{i_r}) \subset X$$

$\forall \ \{i_0, \dots, i_p\}$, we have

$$\mathcal{F}(U_{i_0 \cdots i_p}) = \bigoplus_{r \in \mathbb{Z}} \mathcal{O}_X(r)(U_{i_0 \cdots i_p})$$

$$= S[(X_{i_0} \cdots X_{i_p})^{-1}]$$

because $\mathcal{O}_X(r)(U_{i_0 \cdots i_p}) = S[(X_{i_0} \cdots X_{i_p})^{-1}]_{\text{degree } r}$

So we have the Čech complex of $\mathcal{F}$:

$$0 \longrightarrow \prod_{i=0}^{n} \mathcal{F}(U_i) \longrightarrow \prod_{0 \le i < j \le n} \mathcal{F}(U_{ij}) \longrightarrow \cdots \longrightarrow \mathcal{F}(U_{0\ldots n}) \longrightarrow 0$$

$$\qquad\qquad \| \qquad\qquad\qquad\qquad \| \qquad\qquad\qquad\qquad\qquad \|$$

$$0 \longrightarrow \prod_{i=0}^{n} S[X_i^{-1}] \longrightarrow \prod_{0 \le i < j \le n} S[X_i^{-1} X_j^{-1}] \longrightarrow \cdots \longrightarrow S[X_0^{-1} \cdots X_n^{-1}] \longrightarrow 0$$

The kernel of the first map is $H^0(X, \mathcal{F}) = S$ (already known). $H^n(X, \mathcal{F})$ is the cokernel of the last map:

$$\prod_{i=0}^{n} S[X_0^{-1} \cdots \widehat{X_i^{-1}} \cdots X_n^{-1}] \longrightarrow S[X_0^{-1} \cdots X_n^{-1}]$$

The ring $S[X_0^{-1} \cdots X_n^{-1}]$ is the free A-module with basis all the Laurent monomials, i.e., monomials with both positive and negative powers.

The cokernel of the map is the free A-module with basis the monomials where all the X_i have negative power: $\{X_0^{l_0}\cdots X_n^{l_n},\ l_i<0\ \forall i\}$

The grading on $\mathcal{F}$ or $S[X_0^{-1}\cdots X_n^{-1}]$ is given by $\sum_{i=0}^n l_i$. So $H^n(X,\mathcal{O}_X(-r))=0$ if $r\le n$

and $H^n(X,\mathcal{O}_X(-n-1-r))$ is isomorphic to the free A-module with basis $X_0^{l_0}\cdots X_n^{l_n}$ $l_i<0$ $\sum_{i=0}^n l_i=-n-1-r.$

$= \frac{1}{X_0\cdots X_n}X_0^{m_0}\cdots X_n^{m_n}$ $m_i\le 0$ $\sum m_i=-r$

The pairing between $H^0(X,\mathcal{O}_X(r))$ and $H^n(X,\mathcal{O}_X(-n-1-r))$ is induced by multiplication:

$$\Big(\bigoplus_{\substack{l_i \geq 0 \\ \sum l_i = r}} A\, X_0^{l_0}\cdots X_n^{l_n}\Big) \times \Big(\bigoplus_{\substack{m_i<0 \\ \sum m_i = -n-1-r}} A\, X_0^{m_0}\cdots X_0^{m_n}\Big) \longrightarrow A$$

the dual of $X_0^{l_0}\cdots X_0^{l_n}$ is $X_0^{-l_0-1}\cdots X_n^{-l_n-1}$.

This defines a perfect pairing.

It remains to prove that $H^i(X, \mathcal{F}) = 0$ for $0<i<n$.
Localize the Čech complex at X_n: this produces an exact sequence except on the left, because the cohomology of the localization at X_n is the cohomology of $\mathcal{F}|_{U_n}$ which is 0 because U_n is affine and $\mathcal{F}$ is quasi-coherent.

in positive degrees.

$\Rightarrow \quad \forall\, i>0 \quad H^i(X,\mathcal{F})[X_n^{-1}]=0$

(localization is exact).

We prove, by induction on n, that multiplication by X_n is injective on $H^i(X,\mathcal{F})$. This will imply that $H^i(X,\mathcal{F})=0$ for $0<i<n$.

$Z(X_n) \cong \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$

recall $\quad \mathcal{I}_{Z(X_n)} \cong \mathcal{O}_X(-Z(X_n)) \cong \mathcal{O}_X(-1)$

$\Rightarrow$ exact sequence

$$0 \longrightarrow \mathcal{I}_{Z(X_n)} \xrightarrow{X_n} \mathcal{O}_X \longrightarrow \mathcal{O}_{Z(X_n)} \longrightarrow 0$$

$$\begin{array}{c} \| \end{array}$$

$$0 \longrightarrow \mathcal{O}_X(-1) \xrightarrow{X_n} \mathcal{O}_X \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

$Z = Z(X_n)$

$\Longrightarrow \quad \forall\, r$

$$0 \longrightarrow \mathcal{O}_X(-r-1) \xrightarrow{X_n} \mathcal{O}_X(r) \longrightarrow \mathcal{O}_Z(r) \longrightarrow 0$$

induction hypothesis: $\quad H^i(Z, \mathcal{O}_Z(r)) = 0 \quad \forall\, r$
$\forall\ 0 < i < n-1$

Furthermore, the map

$$\underset{\substack{\text{hom. pol.} \\ \text{in } X_0, \dots, X_n \text{ of degree } r}}{\underset{\parallel}{H^0(X, \mathcal{O}_X(r))}} \longrightarrow \underset{\substack{\text{hom. pol. in} \\ X_0, \dots, X_{n-1} \text{ of degree } r.}}{\underset{\parallel}{H^0(Z, \mathcal{O}_X(r))}}$$

is surjective

$$\Longrightarrow \quad H^1(X, \mathcal{O}_X(r-1)) \xrightarrow{X_n} H^1(X, \mathcal{O}_X(r)) \quad \text{is injective}$$

and $\quad H^i(X, \mathcal{O}_X(r-1)) \xrightarrow{X_n} H^i(X, \mathcal{O}_X(r))$ is also

injective $\forall\ 1 < i < n$ because $H^{i-1}(Z, \mathcal{O}_Z(r)) = 0$
by induction.

$\square$