Since the $U_i$ are affine and the sheaves $F_i$, $E_i$ are quasi-coherent, we also have the exact sequence of global sections, for $\forall i \geq 0$:

$$0 \to \mathcal{F}(U_i \cap \cdots \cap i) \to \mathcal{E}(U_i \cap \cdots \cap i) \to \mathcal{R}(U_i \cap \cdots \cap i) \to 0$$

and, taking products, we obtain the short exact sequence of complexes:

$$0 \to C^\cdot(\mathcal{F}) \to C^\cdot(\mathcal{E}) \to C^\cdot(\mathcal{R}) \to 0$$

which gives the long exact sequence of cohomology groups:

$$0 \to H^0(U, F_i) \to H^0(U, E_i) \to \cdots$$

The canonical morphisms $\kappa : H^k(U, *) \to H^k(X, *)$ commute with the morphisms of long exact sequences of cohomology.
So we have commutative diagrams with exact rows:

\[ 0 \to H^0(U, \mathcal{F}) \to H^0(U, \mathcal{G}) \to H^1(U, \mathcal{F}) \to 0 \]

\[ \Rightarrow K \: \xrightarrow{\cong} H^1(U, \mathcal{F}) \xrightarrow{\cong} H^1(U, \mathcal{G}) \]

\[ 0 \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{G}) \to H^1(X, \mathcal{F}) \to 0 \]

(Note \( H^i(U, \mathcal{G}) = H^i(X, \mathcal{G}) = 0 \) \( \neq i > 0 \) because \( \mathcal{G} \) is flasque)

\[ \Rightarrow K : H^1(U, \mathcal{F}) \xrightarrow{\cong} H^1(X, \mathcal{F}) \]

For \( i > 1 \), we have commutative diagrams:

\[ 0 \to H^i(U, \mathcal{F}) \xrightarrow{\delta} H^{i+1}(U, \mathcal{F}) \to 0 \]

\[ \Rightarrow K \]

\[ 0 \to H^i(X, \mathcal{F}) \to H^{i+1}(X, \mathcal{F}) \to 0 \]
We know \( k : H^i(U, \mathcal{F}) \cong H^i(X, \mathcal{F}) \) for quasi-coherent sheaves on \( X \). By induction, we obtain \( k : H^i(U, \mathcal{F}) \cong H^i(X, \mathcal{F}) \) for all quasi-coherent sheaves.

**Remark:** When \( X \) is a separated Noetherian scheme, it satisfies the hypotheses of the theorem for any open affine covering by Ex. II.4.3 for last quarter's homework.

**Corollary:** If \( X \) has a covering by \( n+1 \) open affine sets \( U_1, \ldots, U_n \) such that all the intersections are affine, then for any quasi-coherent sheaves \( \mathcal{F} \) and all \( i > n \), \( H^i(X, \mathcal{F}) = 0 \).
We use the previous to compute cohomology on projective space:

Let \( A \) be a noetherian ring.

\[
S := A[X_0, \ldots, X_n]
\]

\( X := \mathbb{P}^n_A = \text{Proj} S \)

A \( \mathcal{F} \) sheaf of \( G_X \)-modules:

\[
\Gamma_*(\mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} H^0(X, \mathcal{F}(n))
\]

Recall \( \Gamma_*(\mathcal{O}_X) = S \) as graded \( S \)-module.

And \( \exists \) natural map

\[
\Gamma_*(\mathcal{F}) \rightarrow \mathcal{F}
\]

which is an isomorphism if \( \mathcal{F} \) is quasi-coherent.
Theorem: (1) \( \forall n \in \mathbb{Z}, \forall i \neq 0, n, H^i(X, \mathcal{O}_X(n)) = 0 \)

(2) \( H^n(X, \mathcal{O}_X(-n-1)) = A \)

(3) \( \forall n \in \mathbb{Z}, \) the natural map (cup-product)

\[
H^0(X, \mathcal{O}_X(n)) \times H^1(X, \mathcal{O}_X(-n-1)) \to H^n(X, \mathcal{O}_X(-n-1)) = A
\]

is a perfect pairing of finitely generated free \( A \)-modules.

Proof: Put \( \mathcal{T} = \bigoplus \mathcal{O}_X(n) \). Then \( \mathcal{T} \) is quasi-coherent, in fact, locally free of infinite rank. \( \mathcal{T} \) has a \( \mathbb{Z} \)-grading. We compute \( H^i(\mathcal{T}) \), keeping track of the \( \mathbb{Z} \)-grading.

(colomnology commutes with arbitrary direct sums: they are direct limits of finite direct sums).


Recall that all cohomology groups are
\[ H^0(X, G_X) = A - \text{modules}. \]

We compute \( H^i(U, G_X) \) for \( U = \{ U_0, \ldots, U_n \} \) the usual open cover of \( X \).

Recall
\[ U_i = D_+(X_i) \subset X \]
\[ U_{i_0 \ldots i_n} = D_+(X_{i_0} X_{i_1} \ldots X_{i_n}) \subset X \]

For \( \{ i_0, \ldots, i_p \} \), we have
\[ \mathcal{F} \left( U_{i_0 \ldots i_p} \right) = \bigoplus_{n \in \mathbb{Z}} G_X(n) \left( U_{i_0 \ldots i_p} \right) \]
\[ = S \left[ (X_{i_0} \ldots X_{i_p})^{-1} \right] \]

because \( G_X(n) \left( U_{i_0 \ldots i_p} \right) = S \left[ (X_{i_0} \ldots X_{i_p})^{-1} \right] \) degree \( n \)
So we have the Čech complex of $\mathcal{F}$:

\[ 0 \rightarrow T \mathcal{F}(U_i) \rightarrow T \mathcal{F}(U_{ij}) \rightarrow \cdots \rightarrow \mathcal{F}(U_0 \ldots U_n) \rightarrow 0 \]

The kernel of the first map is $H^0(X, \mathcal{F}) = S$ (already known). $H^n(X, \mathcal{F})$ is the cokernel of the last map:

\[ 0 \rightarrow \bigoplus_{i=0}^n S[X_0^{-1} \ldots \hat{X}_i \ldots X_n^{-1}] \rightarrow S[X_0^{-1} \ldots X_n^{-1}] \]

The ring $S[X_0^{-1} \ldots X_n^{-1}]$ is the free $A$-module with basis all the Laurent monomials, i.e., monomials with both
positive and negative powers.
The cokernel of the map is the free $A$-module
with basis the monomials where all the $X_i$ have
negative power: \( \{X_0^{l_0} \cdots X_n^{l_n} \mid l_i < 0 \ \forall i \} \)

The grading on $F = \mathbb{R} [X_0, \ldots, X_n]$ is given by
\[
\sum_{i=0}^{n} l_i.
\]
So \( H^i (X, \mathcal{O}_X (-n)) = 0 \) if \( n \leq i \)
and \( H^i (X, \mathcal{O}_X (-n+1-n)) \) is isomorphic to the
free $A$-module with basis \( X_0^{l_0} \cdots X_n^{l_n} \mid l_i < 0 \)
\[
\sum_{i=0}^{n} l_i = -n+1-n.
\]

The pairing between $H^0 (X, \mathcal{O}_X (n))$ and
\( H^m (X, \mathcal{O}_X (-n+1-n)) \) is induced by multiplication.
\[(\bigoplus A X_{0}^{l_{0}} \cdots X_{n}^{l_{n}}) \times (\bigoplus A X_{0}^{w_{0}} \cdots X_{0}^{w_{n}})\]

\[l_{i} \geq 0\]

\[\sum l_{i} = n\]

\[m_{i} < 0\]

\[\sum m_{i} = -n-1-n\]

\[
\xrightarrow{A}
\]

the dual of \(X_{0}^{l_{0}} \cdots X_{0}^{l_{n}}\) is \(X_{0}^{l_{0}-1} \cdots X_{n}^{l_{n}-1}\).

This defines a perfect pairing.

It remains to prove that \(H^{i}(X, \mathcal{O}) = 0\) for \(0 < i < n\).

Localize the Čech complex at \(X_{n}\): this produces an exact sequence, because the cohomology of the localization at \(X_{n}\) is the cohomology of \(\mathcal{O}/\mathcal{I}_{n}\), which is 0 because \(\mathcal{I}_{n}\) is affine and \(\mathcal{F}\) is quasi-coherent. In positive degrees.
\( \Rightarrow \forall i \geq 0 \quad H^i(X, \mathcal{O})[X^{-1}] = 0 \)

(localization is exact).

We prove, by induction on \( n \), that multiplication by \( X_n \) is surjective on \( H^i(X, \mathcal{O}) \). This will imply that \( H^i(X, \mathcal{O}) = 0 \) for \( 0 < i < n \).

\[
Z(X_n) = \mathbb{P}^{n-1} \xrightarrow{\text{recall}} \mathbb{P}^n
\]

\[
\Rightarrow \text{exact sequence}
\]

\[
0 \rightarrow JZ(X_n) \xrightarrow{X_n} \mathcal{O}_X \xrightarrow{} \mathcal{O}_Z(X_n) \rightarrow 0
\]

\[
0 \rightarrow \mathcal{O}_X(-1) \xrightarrow{X} \mathcal{O}_X \xrightarrow{} \mathcal{O}_Z \rightarrow 0
\]

\( Z = Z(X_n) \)
\[ 0 \to G_\times(n-1) \xrightarrow{X_n} G_\times(n) \to G_\times(n) \to 0 \]

**Induction Hypothesis:** \( H^i(Z, G_\times(n)) = 0 \quad \forall \ n \quad \forall \ 0 < i < n-1 \)

Furthermore, the map

\[ H^0(X, G_\times(n)) \to H^0(Z, G_\times(n)) \]

is surjective from pol. in \( X_0, \ldots, X_n \) of degree \( \ell \). From pol. in \( X_0, \ldots, X_{n-1} \) of degree \( \ell \).

\[ \Rightarrow \quad H^i(X, G_\times(n-1)) \xrightarrow{X_n} H^i(X, G_\times(n)) \text{ is injective} \]

and \( H^i(X, G_\times(n-1)) \xrightarrow{X_n} H^i(X, G_\times(n)) \) is also injective \( \forall \ 1 < i < n \) because \( H^{i-1}(Z, G_\times(n)) = 0 \) by induction. \( \square \)