It remains to prove that \( f_1, \ldots, f_n \) generate the unit ideal in \( H^0(X, \mathcal{O}_X) \).

This means that the map

\[
H^0(X, \mathcal{O}_X) \oplus \cdots \oplus H^0(X, \mathcal{O}_X)
\]

\[
(\alpha_1, \ldots, \alpha_n) \mapsto \sum_{i=1}^n \alpha_i f_i
\]

is surjective.

This is the map on global sections of the morphism of sheaves:

\[
\mathcal{O}_X \rightarrow \mathcal{O}_X
\]

on any \( U \)

\[
(\beta_1, \ldots, \beta_n) \mapsto \sum_{i=1}^n \beta_i f_i.
\]

This is a surjective morphism of sheaves because

\[
X = \bigcup_i X_{f_i} \quad \Rightarrow \quad \forall \ x \in X, \ \exists i \ \text{ s.t. } (f_i)_x \notin \mathcal{M}_x
\]
i.e., \( \{f_i\} \) generates \( G_{x_i} \).

Let \( J \) be the kernel of this morphism, so that we have the exact sequence:

\[
0 \longrightarrow J \longrightarrow G_{x_1} \bigoplus G_{x_2} \longrightarrow G_x \longrightarrow 0.
\]

Using the long exact sequence of cohomology, if \( H^1(X_1, J) = 0 \), then \( H^0(G_{x_1}) \rightarrow H^0(G_x) \) is surjective.

Filter:

\[
G_x \subset G_{x_1} \subset G_{x_2} \subset \cdots \subset G_{x_n} \subset \cdots
\]

First:

\[
0 \longrightarrow J \cap G \longrightarrow J \cap G_{x_1} \bigoplus J \cap G_{x_2} \longrightarrow \cdots \longrightarrow 0
\]
Čech cohomology: sometimes a practical way of computing coherent cohomology.

The general set-up:

- $X$ a topological space
- $F$ a sheaf of abelian groups on $X$
- $U = \{ U_i : i \in I \}$ an open covering of $X$.

The axiom of choice implies that any set has at least one well-ordering (a total ordering s.t. every non-empty subset has a minimal element).

Choose a well-ordering on $I$.

For any finite subset $\{ i_0 < i_1 < \ldots < i_n \} \subset I$,
let \( U_{i_0 \cdots i_n} \) be the intersection \( U_{i_0} \cap \cdots \cap U_{i_n} \).

with \( \bigcup_{i_0 \cdots i_n} : U_{i_0 \cdots i_n} \rightarrow X \)

and put \( \bigoplus_{i_0 \cdots i_n} := (U_{i_0 \cdots i_n})^* (\bigoplus_{U_{i_0 \cdots i_n}}) \).

Define coboundary maps:

\[
\partial^n : \prod_{i_0 \cdots i_n} \bigoplus_{i_0 \cdots i_n} \rightarrow \prod_{i_0 \cdots i_{n+1}} \bigoplus_{i_0 \cdots i_{n+1}}
\]

\[
\alpha = (\alpha_{i_0 \cdots i_n}) \mapsto d\alpha
\]

\[
(d\alpha)_{i_0 \cdots i_{n+1}} = \sum_{p=0}^{n+1} (-1)^p \alpha_{i_0 \cdots \hat{i}_p \cdots i_{n+1}} \bigg|_{U_{i_0 \cdots i_{n+1}}}
\]
The coboundaries form the long exact sequence of sheaves:

$$0 \rightarrow \mathcal{F}_0 \rightarrow \mathcal{T} \mathcal{T} \mathcal{F}_0 \rightarrow \mathcal{T} \mathcal{F}_0 \rightarrow \mathcal{F}_0 \rightarrow \mathcal{F}_0$$

The exactness is left as an exercise.

For instance, the fact that the kernel of $d^0$ is the image of $\mathcal{F}_0$ is equivalent to the sheaf axioms for $\mathcal{F}_0$. The above sequence is called the Čech resolution of $\mathcal{F}_0$.

Definition: The Čech cohomology $H^\bullet(\mathcal{U}_\alpha, \mathcal{F}_0)$ of $\mathcal{F}_0$ for the open covering $\mathcal{U}_\alpha$ is the cohomology of the complex of global sections of the exact sequence of sheaves above. In general, this would depend...
on the choice of the well-ordering of $I$.

We shall prove that for a noetherian separated scheme with an affine cover, the Čech cohomology groups of quasi-coherent sheaves are naturally isomorphic to the usual derived cohomology groups.

Mine notation: \[ \check{C}^p (\mathcal{F}) := \bigcap_{i_0 < \ldots < i_p \in I} \mathcal{F}_{i_0 \ldots i_p} \]

\[ C^p (\mathcal{F}) := \bigcap_{i_0 < \ldots < i_p \in I} \mathcal{F}_{(U_{i_0 \ldots i_p})} \]

The global sections of \[ \check{C}^p (\mathcal{F}) \]

Terminology: These are called the Čech $p$-cochains.
\[ H^k (\mathcal{V}, \mathcal{F}) = H^k (\mathcal{C}^\bullet (\mathcal{F})) \text{ by definition.} \]

We need some preliminary results, interesting in their own right:

**Lemma:** $X$-top. space. If $\mathcal{F}$ is flasque, then so are the sheaves $\mathcal{C}^\bullet (\mathcal{F})$.

**Proof:** The restriction of a flasque sheaf is flasque, direct images of flasque sheaves are flasque, products of flasque sheaves are flasque. \[ \square \]

**Lemma:** If $\mathcal{F}$ is flasque, then $\forall V \in \mathcal{V}$ and all $n \geq 0$, we have $H^n (\mathcal{V}, \mathcal{F}) = 0$. 

Proof: We have the Čech resolution,

\[ 0 \rightarrow F^i \rightarrow C^0(F) \xrightarrow{d^0} C^1(F) \rightarrow \cdots \]

rewind into short exact sequences.

\[ 0 \rightarrow F^i \rightarrow C^0 \xrightarrow{d^0} \mathcal{B}^1 \rightarrow 0 \Rightarrow \mathcal{B}^1 \text{ flasque} \]

\[ 0 \rightarrow \mathcal{B}^1 \xrightarrow{d^1} C^1 \xrightarrow{d^2} \mathcal{B}^2 \rightarrow 0 \Rightarrow \mathcal{B}^2 \text{ flasque} \]

\[ 0 \rightarrow \mathcal{B}^l \rightarrow C^l \rightarrow \mathcal{B}^{l+1} \rightarrow 0 \Rightarrow \mathcal{B}^{l+1} \text{ flasque} . \]

\[ \Rightarrow \text{ all sequences of global sections obtained from the above are exact.} \]

\[ \Rightarrow \text{ the Čech complex of global sections is exact} \]

\[ \Rightarrow H^p(U, F) = 0 \quad \forall \ p > 0 \]

\[ \blacksquare \]
Construction: $X$ top. space.

Recall that for any resolution $\mathcal{F} \to \mathcal{F}^*$ and any injective resolution $\mathcal{E} \to \mathcal{I}^*$, $\mathcal{E}$ a map $\nu^*: \mathcal{E}^* \to \mathcal{I}^*$ extending $\text{Id}: \mathcal{F}^* \to \mathcal{F}^*$, i.e.,

\[
\begin{array}{ccc}
\mathcal{F}^* & \xrightarrow{\text{Id}} & \mathcal{F}^* \\
\downarrow & & \downarrow \\
\mathcal{E}^* & \xrightarrow{\nu^*} & \mathcal{I}^*
\end{array}
\]

$\nu^*$ unique up to homotopy.

Apply this to $\mathcal{F}^* \to C^*(\mathcal{F}^*)$.

$\nu^*: C^*(\mathcal{F}^*) \to \mathcal{I}^*$ unique up to homotopy,

planning to the complexes of global sections, defines

$\kappa: H^k(U, \mathcal{F}^*) \to H^k(X, \mathcal{F}^*)$ unique.
Theorem: Let $X$ be a Noetherian scheme, $F$ a quasi-coherent sheaf of $O_X$-modules. Let $U$ be an open covering of $X$ by open affine schemes $U_i$, $i \in I$ such that $U_{i_0} \subset \cdots \subset U_{i_n} \subset U_i$, $U_{i_0} \subset \cdots \subset U_{i_n}$ is affine. Then the canonical morphism

$$k : H^i(U, F) \to H^i(X, F)$$

is an isomorphism for all $i$.

Proof: Recall that $F$ can be embedded in a flasque quasi-coherent sheaf, say $G$. Let $G/F$ be the quotient of $G$ by $F$, so that we have the exact sequence of quasi-coherent sheaves

$$0 \to F \to G \to G/F \to 0$$