$\Longrightarrow$ Enough to show the theorem for finitely generated sheaves.

Now assume $\mathcal{F}$ is a finitely generated sheaf.
By induction on the number of generators, we can assume it is generated by one element, say $s \in H^0(U, \mathcal{F})$ for some open set $U \overset{i}{\hookrightarrow} X$.

Note: If $\mathcal{F}$ is generated by $s_i \in \mathcal{F}(U_i)$, $1 \le i \le n$, then $\mathcal{F} / \text{image of } i_{n!} \mathbb{Z}_{U_n} s_n$ is generated by the images of $s_1, \dots, s_{n-1}$.

This means we have a surjection

$$i_! \mathbb{Z}_U \twoheadrightarrow \mathcal{F}.$$
$$1 \longmapsto s$$

We can complete this to an exact sequence:

$$0 \longrightarrow \mathcal{R} \longrightarrow i_! \mathbb{Z}_U \longrightarrow \mathcal{F} \longrightarrow 0.$$

So we need to show the vanishing of

$$H^k(\text{subsheaves of } i_! \mathbb{Z}_U) \text{ for } k > \dim X.$$

For any nonzero subsheaf $\mathcal{R}$ of $i_! \mathbb{Z}_U$, its stalks are abelian subgroups of $\mathbb{Z}$, hence generated by positive integers. Let d be the least positive integer occuring in the stalks of $\mathcal{R}$. Then d lifts to a section of $\mathcal{R}$ on some nonempty open set V, i.e.,

$$j_V^{-1} \mathcal{R} \xleftarrow{\simeq} d\mathbb{Z}_V \quad , \text{ put } Z = X \setminus V$$

$$\Rightarrow \quad 0 \to j_{V!} j_V^{-1} \mathcal{R} \longrightarrow \mathcal{R} \longrightarrow j_{Z*} j_Z^{-1} \mathcal{R} \to 0$$

$j_V : V \hookrightarrow X \qquad j_Z : Z \hookrightarrow X.$

$\simeq j_{V!} \mathbb{Z}_V$

$H^k(j_{Z*} j_Z^{-1} \mathcal{R}) = H^k(j_Z^{-1} \mathcal{R}) = 0$
by induction: $\dim Z < \dim X$

So we only need to prove the vanishing of

$$H^k(i_! \mathbb{Z}_U) \quad \forall \quad k > \dim X$$

and all $U \underset{\text{open}}{\subset} X$.

We again have

$$0 \longrightarrow i_! \mathbb{Z}_U \longrightarrow \mathbb{Z}_X \longrightarrow j_* \mathbb{Z}_Y \longrightarrow 0$$

where $Y = X \setminus U$ $\quad i : U \hookrightarrow X$, $j : Y \hookrightarrow X$.

$H^k(\mathbb{Z}_X) = 0$ for $k > 0$ because constant sheaves are flasque, $H^k(j_* \mathbb{Z}_Y) = 0$ for $k > \dim X - 1$ by induction because $\dim Y \leq \dim X - 1$.

The long exact sequence of cohomology now gives:

$$H^k(i_! \mathbb{Z}_U) = 0 \quad \text{for } k > \dim X. \qquad \square$$

The cohomology of quasi-coherent sheaves on Noetherian affine schemes:

$X := \operatorname{Spec} A$ $\quad$ A noetherian ring.

Theorem: For all quasi-coherent sheaves $\mathcal{F}$ on X,

$H^i(X, \mathcal{F}) = 0 \quad \forall\, i > 0.$

Lemma: (Prop 3.4 in Hartshorne) For any injective A-module I, $\tilde{I}$ is flasque.

(reason: the localization maps $I \longrightarrow I[f^{-1}]$ are surjective)

Proof of the theorem: If $M := H^0(X, \mathcal{F})$, then $\mathcal{F} = \tilde{M}$.

Choose a resolution $I^{\cdot}$ of M by injective A-modules.

Then we have a resolution

$$0 \longrightarrow \tilde{M} \longrightarrow \tilde{I}^0 \longrightarrow \tilde{I}^1 \longrightarrow \cdots$$

by flasque sheaves of $\mathcal{O}_X$-modules because the $\sim$ functor is exact with global sections complex:

$$0 \longrightarrow M \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots$$

which is exact because this is the original resolution. $\Rightarrow$ $H^k(\tilde{M}) = 0$ for $k > 0$ $\square$.

<u>Some tools for computing cohomology on noetherian schemes:</u> X a noetherian scheme

<u>Lemma:</u> If $\mathcal{F}$ is a quasi-coherent sheaf on X, then $\mathcal{F}$ can be embedded in a flasque, quasi-coherent sheaf.

Proof: Choose a finite affine cover of X:

$$X = \bigcup_{i=1}^{n} U_i \qquad U_i = \operatorname{Spec} A_i .$$

Put $M_i := H^0(U_i, \mathcal{F})$, then $\mathcal{F}|_{U_i} \cong \tilde{M}_i$.

Embed M_i in an injective A_i-module I_i.

$$\tilde{M}_i \hookrightarrow \tilde{I}_i \quad \text{on } U_i = \operatorname{Spec} A_i$$

If $u_i : U_i \hookrightarrow X$, then

$$\mathcal{F} \hookrightarrow \bigoplus_{i=1}^{n} u_{i*} \mathcal{F}|_{U_i} \hookrightarrow \bigoplus_{i=1}^{n} u_{i*} \tilde{I}_i$$

$\left(\mathcal{F}(U) \hookrightarrow \bigoplus_{i=1}^{n} \mathcal{F}(U \cap U_i) \quad \forall U \right)$

flasque and quasi-coherent.

$\square$.

Corollary: Quasi-coherent sheaves on noetherian schemes have quasi-coherent flasque resolutions.

Serre's theorem: X a noetherian scheme. The following are equivalent.

(a) X is affine

(b) $H^i(X, \mathcal{F}) = 0$ $\forall$ $i > 0$ and $\mathcal{F}$ quasi-coherent.

(c) $H^i(X, \mathcal{I}) = 0$ $\forall$ $i > 0$ and $\mathcal{I}$ coherent sheaf of ideals. (replace with quasi-coherent if X is not noetherian)

Proof: We only need to prove (c) $\Rightarrow$ (a)

We use Ex. II. 2.17 from last quarter:

a scheme X is affine iff $\exists\ f_1, \dots, f_n \in H^0(X, \mathcal{O}_X)$ generating the unit ideal s.t. $\forall i$

$$X_{f_i} := \{x \in X \mid f_x \notin \mathfrak{m}_x\} \text{ is affine.}$$

Let $P \in X$ be a closed point, U an affine neighborhood of P and put $Y := X \setminus U$. Let k_P be the skyscraper sheaf with support $\{P\}$ and stalk $k =: k(P)$.

We have the exact sequence:

$$0 \longrightarrow \mathcal{J}_{Y \amalg \{P\}} \longrightarrow \mathcal{J}_Y \longrightarrow k_P \longrightarrow 0$$

$Y \cup \{P\} =$

sheaf of ideals of Y for some scheme structure on Y.

The map: $\mathcal{I}_Y \longrightarrow k_P$:

If $V \not\ni P$, then $k_P(V) = 0$ by def.

and $\mathcal{I}_Y(V) \longrightarrow k_P(V)$ is zero.

If $V \ni P$, then $k_P(V) = k$ and $U \cap Y \ni P$

The map is $\mathcal{I}_Y(V) \longrightarrow \mathcal{I}_Y(U \cap V) = \mathcal{O}_X(U \cap V)$

$$\longrightarrow \mathcal{O}_{X,P} \longrightarrow k_P = \mathcal{O}_{X,P}/m_P = k_P(V)$$

The kernel is $\mathcal{I}_{Y \cup \{P\}}$.

The long exact sequence of cohomology:

$$0 \longrightarrow H^0(\mathcal{I}_{Y\cup\{P\}}) \longrightarrow H^0(\mathcal{I}_Y) \longrightarrow \underset{\underset{k_P}{\parallel}}{H^0(k_P)} \longrightarrow \underset{\underset{0 \text{ by (c)}}{\parallel}}{H^1(\mathcal{I}_{Y\cup\{P\}})}$$

$\Rightarrow \quad H^0(\mathcal{I}_Y) \longrightarrow\!\!\!\!\!\rightarrow k_P$

In particular, $\exists\ f \in H^0(\mathcal{I}_Y)$ mapping to $1 \in k_P$. This implies $f_P \notin m_P$

$$\Rightarrow \quad P \in X_f := \{ x \in X \mid f_x \notin m_x \}$$

We also have $X_f = U_f := \{ x \in U \mid f_x \notin m_x \}$

because $Y = X \setminus U$ and $f \in \mathcal{I}_Y \Rightarrow f_x \in m_x \ \forall x \in Y$

$$\Rightarrow Y \cap X_f = \emptyset$$

$$\Rightarrow X_f \subset U.$$

$$\Rightarrow X_f = U_f.$$

Since U is affine, so is $U_f = X_f$.

Since X is noetherian, we can cover X with a finite #, say $X_{f_1}, \dots, X_{f_r}$ of such affine open sets.