$\Rightarrow$ Enough to show the theorem for finitely generated sheaves.

Now assume $F_i$ is a finitely generated sheaf. By induction on the number of generators, we can assume it is generated by one element, say $s \in H^0(U, F_i)$.

For some open set $U \subseteq X$, $1 \leq i \leq n$, then $F_i$ is generated by the images of $i_! Z_U$. We can complete this to an exact sequence:

$$0 \to R \to i_! Z_U \to F_i \to 0.$$
So we need to show the vanishing of
\[ H^k_{\text{subsheaves}}(i^! Z_\nu) \text{ for } k > \dim X. \]
For any nonzero subsheaf \( R \) of \( i^! Z_\nu \), its stalks are abelian subgroups of \( \mathbb{Z} \), hence generated by positive integers. Let \( d \) be the least positive integer occurring in the stalks of \( R \). Then \( d \) lifts to a section \( \eta \) of \( R \) on some nonempty open set \( V \), i.e.,
\[
j^! \overline{\mathcal{E}} \xrightarrow{\eta} d Z_\nu \quad \text{put} \quad Z = X \setminus V
\]
\[
\Rightarrow \quad 0 \to j^! i^! \mathcal{E} \to \mathcal{R} \xrightarrow{\eta} d Z \xrightarrow{i^! \eta} \mathcal{E} \to 0
\]
\[
\Rightarrow \quad \text{since} \quad \text{dim } Z < \text{dim } X
\]
\[
H^k_{\text{subsheaves}}(d Z, i_\nu^! \mathcal{E}) = H^k_{\text{subsheaves}}(j_\nu^* \mathcal{E}) = 0
\]
by induction: \( \text{dim } Z < \text{dim } X \).
So we only need to prove the vanishing of

\[ H^k (\mathbb{Z}_U) \neq 0 \quad \text{for} \quad k > \dim X \]

and all \( U \subseteq X \).

We again have

\[ 0 \rightarrow i! \mathbb{Z}_U \rightarrow \mathbb{Z}_X \rightarrow j_* \mathbb{Z}_Y \rightarrow 0 \]

where \( Y = X \setminus U \), \( i : U \hookrightarrow X \), \( j : Y \rightarrow X \).

\[ H^k (\mathbb{Z}_X) = 0 \quad \text{for} \quad k > 0 \quad \text{because constant sheaves are flasque} \]

\[ H^k (j_* \mathbb{Z}_Y) = 0 \quad \text{for} \quad k > \dim X - 1 \]

by induction because \( \dim Y \leq \dim X - 1 \).

The long exact sequence of cohomology now gives:

\[ H^k (i! \mathbb{Z}_U) = 0 \quad \text{for} \quad k > \dim X. \]
The cohomology of quasi-coherent sheaves on Noetherian affine schemes:

\[ X := \text{Spec } A \quad A \text{ a noetherian ring.} \]

**Theorem:** For all quasi-coherent sheaves \( \mathcal{F} \) on \( X \),

\[ H^i(X, \mathcal{F}) = 0 \quad \forall \ i > 0. \]

**Lemma:** (Prop 3.4 in Hartshorne) For any injective \( A \)-module \( I \), \( \tilde{I} \) is flasque. (reason: the localization maps \( I \rightarrow I[\mathfrak{p}] \) are injective)

**Proof of the theorem:** If \( M := H^0(X, \mathcal{F}) \), then \( \mathcal{F} = \tilde{M} \).

Choose a resolution \( I^* \) of \( M \) by injective \( A \)-modules.
Then we have a resolution
\[ 0 \to M \to \tilde{I}^0 \to \tilde{I}^1 \to \ldots \]
by flasque sheaves of $\mathcal{O}_X$-modules because the $\mathcal{I}$ functor is exact with global sections complex.
\[ 0 \to M \to I^0 \to I^1 \to \ldots \]
which is exact because this is the original resolution. \[ \Rightarrow H^k(\mathcal{M}) = 0 \text{ for } k > 0. \]

Some tools for computing cohomology on noetherian schemes: 
- $X$ a noetherian scheme

Lemma: If $\mathcal{F}$ is a quasi-coherent sheaf on $X$, then $\mathcal{F}$ can be embedded in a flasque, quasi-coherent sheaf.
Proof: Choose a finite affine cover of $X$:

$$X = \bigcup_{i=1}^{n} U_i, \quad U_i = \text{Spec} A_i.$$

Put $M_i := H^0(U_i, F)$, then $\mathcal{F}|_{U_i} \cong \tilde{M}_i$.

Embed $\tilde{M}_i$ in an injective $A_i$-module $I_i$.

$\tilde{M}_i \hookrightarrow I_i$ on $U_i = \text{Spec} A_i$.

If $\alpha_i : U_i \hookrightarrow X$, then

$$F \hookrightarrow \bigoplus_{i=1}^{n} M_i \otimes F|_{U_i} \hookrightarrow \bigoplus_{i=1}^{n} M_i \otimes I_i$$

$(F(U)) \hookrightarrow \bigoplus_{i=1}^{n} F(U \cap U_i)$

flasque and quasi-coherent.

\[ \square \]
Corollary: Quasi-coherent sheaves on noetherian schemes have quasi-coherent flasque resolutions.

Lene's theorem: $X$ a noetherian scheme. The following are equivalent.

(a) $X$ is affine
(b) $H^i(X, \mathcal{F}) = 0 \ \forall \ i > 0$ and $\mathcal{F}$ quasi-coherent.
(c) $H^i(X, \mathcal{J}) = 0 \ \forall \ i > 0$ and $\mathcal{J}$ coherent sheaf of ideals. (replace with quasi-coherent if $X$ is not noetherian)

Proof: We only need to prove $(c) \Rightarrow (a)$

We use Ex. II. 2. 17 from last quarter:
a scheme $X$ is affine iff $\exists f_1, \ldots, f_n \in H^0(X, \mathcal{O}_X)$ generating the unit ideal i.e. $\forall i$

$X_{f_i} := \{ x \in X \mid f_i x \not\in \mathfrak{m}_x \}$ is affine.

Let $P \in X$ be a closed point, $U$ an affine neighborhood of $P$ and put $Y := X \setminus U$. Let $k_P$ be the skyscraper sheaf with support $\{ P \}$ and stalk $k = k(P)$. We have the exact sequence

$$
0 \rightarrow J_{Y \setminus \{ P \}} \rightarrow J_Y \rightarrow k_P \rightarrow 0
$$

$Y_U[\{ P \}]$ is the sheaf of ideals of $Y$ for some scheme structure on $Y$. 

The map: \( J_y \rightarrow k_p \):

If \( V \not\in P \), then \( k_p(V) = 0 \) by def.
and \( J_y(V) \rightarrow k_p(V) \) is zero.

If \( V \in P \), then \( k_p(V) = k \) and \( \forall y \in P \)
The map is \( J_y(V) \rightarrow J_y(U \cap V) = C_x(U \cap V) \)
\[ \rightarrow C \times, P \rightarrow k_p = C X_P / M_p = k_p(V) \]

The kernel is \( J_y U \{ P \} \).

The long exact sequence of cohomology:

\[ 0 \rightarrow H^0(J_y U \{ P \}) \rightarrow H^0(J_y) \rightarrow H^0(k_p) \rightarrow H^1(J_y U \{ P \}) \]
\[ \begin{array}{c}
\text{II} \\
\text{II} \\
\text{II} \\
\end{array} \]
\[ \begin{array}{c}
k_p \\
k_p \\
0 \end{array} \]

\( \Rightarrow \) \( H^0(J_y) \rightarrow k_p \)
In particular, \( \exists f \in H^0(D_Y) \) mapping to \( 1 \in \mathfrak{m}_p \). This implies \( f_p \not\in \mathfrak{m}_p \).

\[ \Rightarrow \quad p \in X_f := \{ x \in X \mid f_x \not\in \mathfrak{m}_x \} \]

We also have \( X_f = U_f := \{ x \in U \mid f_x \not\in \mathfrak{m}_x \} \) because \( Y = X \setminus U \) and \( f \in D_Y \Rightarrow f_x \in \mathfrak{m}_x \forall x \in Y \)

\[ \Rightarrow \quad Y \cap X_f = \emptyset \]

\[ \Rightarrow \quad X_f \subset U \]

\[ \Rightarrow \quad X_f = U_f \]

Since \( U \) is affine, so is \( U_f = X_f \).

Since \( X \) is noetherian, we can cover \( X \) with a finite #, say \( X_f_1, \ldots, X_f_n \) of such affine open sets.