Cohomology:

Cohomology measures the failure of exactness of sequences of global sections of sheaves.

Examples: (1) $U \subset \mathbb{C}$ an open subset in the analytic topology. $\mathcal{O}_U$ denotes the sheaf of holomorphic functions on U. $\mathcal{O}_U^* \subset \mathcal{O}_U$ denotes the subsheaf of nowhere zero holomorphic functions. We have the exact sequence of sheaves:

$$0 \longrightarrow 2\pi i\, \mathbb{Z}_U \longrightarrow \mathcal{O}_U \xrightarrow{\exp.} \mathcal{O}_U^* \longrightarrow 1$$

where exp is the exponential map, 0, 1, $2\pi i\,\mathbb{Z}_U$ are constant sheaves.

There are many examples of open sets U where the associated sequence of global sections is not exact on the right. For instance, we can take U to be an open disc minus its center.

We will see that we can complete the sequence of global sections to the exact sequence of cohomology

$$0 \longrightarrow \underset{\Gamma(2\pi i \mathbb{Z}_U)}{\overset{\shortparallel}{H^0(2\pi i \mathbb{Z}_U)}} \longrightarrow \underset{\Gamma(\mathcal{O}_U)}{\overset{\shortparallel}{H^0(\mathcal{O}_U)}} \xrightarrow{\exp} H^0(\mathcal{O}_U^*) \longrightarrow H^1(2\pi i \mathbb{Z}_U)$$

$$\longrightarrow H^1(\mathcal{O}_U) \longrightarrow H^1(\mathcal{O}_U^*) \longrightarrow H^2(2\pi i \mathbb{Z}_U) \longrightarrow \cdots$$

(2) Let X be a scheme, recall that we defined the

sheaf of total quotient rings of X : $\mathcal{K}_X$.

$U = \operatorname{Spec} A \underset{\text{open}}{\subset} X$ $\qquad \mathcal{K}_X(U)$ = total ring of fractions of A

= localization of A at the set of nonzero divisors.

We have the exact sequence of sheaves:

$$1 \longrightarrow \mathcal{O}_X^* \longrightarrow \mathcal{K}_X^* \longrightarrow \mathcal{K}_X^* / \mathcal{O}_X^* \longrightarrow 1.$$

This gives an exact sequence of global sections

$$1 \longrightarrow H^0(\mathcal{O}_X^*) \longrightarrow H^0(\mathcal{K}_X^*) \longrightarrow \underbrace{H^0(\mathcal{K}_X^*/\mathcal{O}_X^*)}_{\text{group of Cartier divisors}}$$

the image of $H^0(\mathcal{K}_X^*)$ in $H^0(\mathcal{K}_X^*/\mathcal{O}_X^*)$ is the subgroup of principal Cartier divisors.

We saw examples last quarter (e.g. projective spaces) where the map on the right is not surjective.

We can complete the sequence to an exact sequence of cohomology:

$$1 \to H^0(\mathcal{O}_X^*) \to H^0(\mathcal{K}_X^*) \to H^0(\mathcal{K}_X^*/\mathcal{O}_X^*) \to H^1(\mathcal{O}_X^*) \to$$
$$\to H^1(\mathcal{K}_X^*) \to \cdots$$

<u>We need a few preliminaries from homological algebra:</u>

<u>Def:</u> A category $\mathcal{A}$ is a collection of objects $\mathrm{Ob}(\mathcal{A})$ with the datum, for each $A, B \in \mathrm{Ob}(\mathcal{A})$, a "class" (i.e., collection) of objects $\mathrm{Hom}(A,B)$ s.t. we can

compose $\varphi \in \mathrm{Hom}(A,B)$ with $\psi \in \mathrm{Hom}(B,C)$

i.e., $\exists$ map $\mathrm{Hom}(A,B) \times \mathrm{Hom}(B,C) \longrightarrow \mathrm{Hom}(A,C)$

$$(\varphi, \psi) \longmapsto \psi \circ \varphi.$$

We say $\mathcal{A}$ is locally small, if $\forall\, A, B \in \mathrm{Ob}(\mathcal{A})$ $\mathrm{Hom}(A,B)$ is a set. We say $\mathcal{A}$ is small, if, in addition $\mathrm{Ob}(\mathcal{A})$ is a set.

<u>Definition:</u> (1) A $\mathbb{Z}$-category or pre-additive category is a locally small category $\mathcal{A}$ such that $\forall\, A, B \in \mathrm{Ob}(\mathcal{A})$ the set $\mathrm{Hom}(A,B)$ has the structure of an abelian group s.t. the composition laws are bilinear.

(2) An additive category is a $\mathbb{Z}$-category which has

a 0 object, i.e., an object 0 s.t. $\mathrm{Hom}(A,0)=\mathrm{Hom}(0,A)=0$ $\forall\ A \in \mathrm{Ob}(\mathcal{A})$; and such that $\forall\ A, B \in \mathrm{Ob}(\mathcal{A})$ $\exists$ product $A\times B$.

Equivalently, an additive category is a locally small category with a 0 object and s.t. $\forall\ A, B \in \mathrm{Ob}(\mathcal{A})$ $\exists$ a product $A\times B$ (or $A \prod B$) an a coproduct $A+B$ (or $A \amalg B$) such that the canonical map

$$A+B \longrightarrow A\times B \quad \text{is an isomorphism.}$$
$$\in \mathrm{Hom}(A+B, A\times B)$$

(in addition, $\forall\ A \in \mathrm{Ob}(\mathcal{A})$, $\mathrm{Hom}(A,A)$ has an identity element Id_A, i.e., $\forall\ B \in \mathrm{Ob}(\mathcal{A})$ and $\varphi \in \mathrm{Hom}(A,B)$

$$\varphi \circ \mathrm{Id}_A = \varphi \quad \text{and} \quad \mathrm{Id}_B \circ \varphi = \varphi \ )$$

The product or coproduct is then called a biproduct or direct sum and denoted $A \oplus B$.

Given an additive category $\mathcal{A}$, $A \in Ob(\mathcal{A})$

we have the functor $\quad Hom(\cdot, A) : \mathcal{A} \longrightarrow \underline{\mathcal{Ab}}$

$$B \longmapsto Hom(B, A)$$

$\underline{\mathcal{Ab}}$:= category of abelian groups.

$$Hom(A, \cdot) : \mathcal{A} \longrightarrow \underline{\mathcal{Ab}}$$

$$B \longmapsto Hom(A, B)$$

Given a functor $F : \mathcal{A} \longrightarrow \underline{\mathcal{Ab}}$

we say F is representable if $\exists\ A \in Ob(\mathcal{A})$

and $\varphi : Hom(\cdot, A) \longrightarrow F$ a natural isomorphism.

We say F is represented by (A, φ).

By definition, $A \times B$ represents the functor

$$Hom(\cdot, A) \times Hom(\cdot, B)$$

Dually, the sum $A+B$ represents the functor
$$\mathrm{Hom}(A,\cdot)\times \mathrm{Hom}(B,\cdot).$$

<u>Def:</u> An abelian category is an additive category in which every morphism has a kernel and a cokernel; every monomorphism is the kernel of its cokernel; every epimorphism is the cokernel of its kernel; every morphism can be factored into the composition of an epimorphism followed by a monomorphism.

Examples of abelian categories:

(1) <u>Ab</u> the category of abelian groups.

(2) $\mathcal{M}od(A)$ the category of modules over a ring A.

(3) X topological space.

$\mathcal{Ab}(X)$ the category of sheaves of abelian groups on X.

(4) X ringed space

$\mathcal{Mod}(X)$ the category of sheaves of $\mathcal{O}_X$-modules.

(5) X scheme

$\mathcal{Q}coh(X)$ the category of quasi-coherent sheaves of $\mathcal{O}_X$-modules.

(6) X noetherian scheme

$\mathcal{C}oh(X)$ the category of coherent sheaves of $\mathcal{O}_X$-modules

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There are many examples of additive non-abelian categories such as the category of free abelian groups.

Mitchell's Full Embedding theorem: Let A be a small abelian category. Then $\exists$ a ring R and a fully faithful exact functor $F: \mathcal{A} \longrightarrow Mod(R)$.

Definition: An object I of an abelian category $\mathcal{A}$ is called injective if the functor $Hom(\cdot, I)$ is exact. An object P of $\mathcal{A}$ is called projective if $Hom(P, \cdot)$ is exact.

We say $\mathcal{A}$ has enough injectives if $\forall\ A \in Ob(\mathcal{A})$, $\exists$ a monomorphism $i: A \longrightarrow I$ from A to an injective object I. We say $\mathcal{A}$ has enough projectives if $\forall\ A \in Ob(\mathcal{A})$ $\exists$ an epimorphism $s: P \longrightarrow A$ from a projective object P.

In diagrams: $\quad \mathrm{Hom}(\cdot, I)$

$$
\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r] & C \arrow[r] \arrow[d] & 0 \\
 & & & I &
\end{tikzcd}
$$

exact sequence of $\mathcal{A}$.

$$0 \to \mathrm{Hom}(C, I) \longrightarrow \mathrm{Hom}(B, I) \longrightarrow \mathrm{Hom}(A, I) \to 0$$

always left exact.

exactness on the right means:

$\forall \quad \varphi : A \to I \quad , \quad \exists \quad \psi : B \to I$

s.t. $\qquad \varphi = \psi \circ f$

intuitively: I is "big"

$\mathrm{Hom}(P, -)$

$$0 \to A \to B \xrightarrow{g} C \to 0$$

$$0 \to \mathrm{Hom}(P,A) \to \mathrm{Hom}(P,B) \to \mathrm{Hom}(P,C) \to 0$$

always left exact.

exactness on the right means:

$\forall \quad \varphi : P \to C \quad , \quad \exists \quad \psi : P \to B$

s.t. $\qquad \varphi = g \circ \psi$

intuitively, P is "free"

e.g. in $\mathrm{Mod}(R)$, free modules are projective.

If R is local, free $\Longleftrightarrow$ projective.

In $\mathrm{Vect}(k)$ every vector space is injective and projective.