The embedded tangent space to a projective variety.

$X \subset \mathbb{P}^n_k$ projective.

$C(X) \subset \mathbb{A}^{n+1}$ the affine cone of X

$$\begin{tikzcd}
\mathbb{A}^{n+1}_k \setminus \{0\} \arrow[r, two heads] & \mathbb{P}^n_k \\
D(X_i) = V_i \arrow[u, phantom, sloped, "\subset"] \arrow[r] & U_i = D_+(X_i) \arrow[u, phantom, sloped, "\subset"]
\end{tikzcd}$$

(on closed points: $(x_0, \dots, x_n) \longmapsto$ line through $(0, \dots, 0)$ and $(x_0, \dots, x_n)$)

$$k[X_0, \dots, X_n][X_i^{-1}] \hookleftarrow k\left[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right]$$

subring of degree 0 elements.

$(a_0, \dots, a_n) \longleftrightarrow$ hom. max. ideal $(a_i X_j - a_j X_i)_{0 \le i, j \le n}$

ex. hom. prime ideal $(a_0 X_1 - a_1 X_0)$

$$C(X) = Z(I_X) \qquad I_X \subset k[X_0,\dots,X_n]$$

$\cap$ hom. ideal of X

$\mathbb{A}^{n+1}_k.$

on closed points $C(X) =$ union of the lines through $(0,\dots,0)$ belonging to X.

The Zariski tangent space to $C(X)$ at $y \in C(X)$ closed point mapping to $x \in X$ is the subspace of

$$k^{n+1} = T_y\mathbb{A}^{n+1} = (k\,dX_0 \oplus \dots \oplus k\,dX_n)^*$$

with equations $dP_1(y),\dots,dP_r(y)$ where $(P_1,\dots,P_r) = I_X$ (and P_i are homogeneous.)

$dP_1(y),\dots,dP_r(y)$ are linear forms on $T_y\mathbb{A}^{n+1} \cong k^{n+1}$

$dP_1(y), \dots, dP_r(y)$ are linear polynomials in $X_0, \dots, X_n$.

$$dP_i(y) \iff \sum_{j=0}^{n} \frac{\partial P_i}{\partial X_j}(y)\, X_j$$

$Z(dP_1(y), \dots, dP_r(y))$ is a linear subspace of $\mathbb{P}^n_k$.

Def: $Z(dP_1(y), \dots, dP_r(y)) \subset \mathbb{P}^n_k$ is the embedded projective tangent space to X at x.

ex: It always passes through x.

(Euler's formula: $\forall$ F hom. of degree d

$$\sum_{i=0}^{n} \frac{\partial F}{\partial X_i} X_i = dF$$
)

ex: In some sense, we can think of the projective tangent space as being "generated" by the Zariski tangent space and x itself.

In fact: can write a natural exact sequence.

$$0 \longrightarrow k(y) \longrightarrow T_y C(X) \longrightarrow T_x X \longrightarrow 0$$

$$k(y) = k, \qquad T_y C(X) = Z(dP_1(y), \dots, dP_n(y)) \subset T_y \mathbb{A}^{n+1}$$

Bertini's theorem:

Preliminaries: k alg. closed field

Recall: $\Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n}(d)) = S_d$ the vector space of homogeneous polynomials of degree d in $n+1$ variables $/k$.

Also recall that $\forall\ f \in S_d$, the scheme of zeros $Z(f)$ is the scheme of zeros of f as a global section of $\mathcal{O}_{\mathbb{P}^n}(d)$, $Z(f) \subset \mathbb{P}^n$ is a hypersurface of degree d, $I_{Z(f)} = Sf \subset S$

The sheaf of ideals $\mathcal{I}_{Z(f)}$ is the image of

$$f: \mathcal{O}_{\mathbb{P}^n}(-d) \hookrightarrow \mathcal{O}_{\mathbb{P}^n} \text{ and } \mathcal{I}_{Z(f)} \cong \mathcal{O}_{\mathbb{P}^n}(-d).$$

$Z(f)$ has an associated Weil divisor:

If we write $f = \prod_{i=1}^{r} f_i^{n_i}$ where f_i are irreducible and non proportional.

Then $\mathrm{Div}(f) = \sum_{i=1}^{r} n_i [Z(f_i)]$

($Z(f_i)$ is an integral Weil divisor $\forall i$)

Since $I_{Z(f)} = Sf$, we have that, for two polynomials f, g, $Z(f) = Z(g)$ as subschemes of $\mathbb{P}^n$

iff $\exists\ \lambda \neq 0 \in k$ s.t. $f = \lambda g$.

<u>Fact</u>: More generally, for any nonsingular projective variety $X/_k$ with invertible sheaf $\mathcal{L}$,

$\forall\, s, t \in \Gamma(X, \mathcal{L})$,

$$Z(s) = Z(t) \iff \exists\, \lambda \in k^* \text{ s.t. } t = \lambda s.$$

(Prop. II.7.7)

<u>Def</u>: $|\mathcal{L}| := \mathbb{P}\Gamma(\mathcal{L}) := \{\text{the set of lines in } \Gamma(\mathcal{L})\}$

$= \Gamma(\mathcal{L}) \setminus \{0\} \big/ k^*$ = set of closed points of $\mathbb{P}^n_k$ where $n+1 = \dim_k \Gamma(\mathcal{L})$.

is called the complete linear system of $\mathcal{L}$.

Recall that $\mathrm{Div}(s) = \sum_{i=1}^{r} n_i [D_i]$

where D_i are the irreducible components of $Z(s)$ (with their reduced induced scheme structures) and

$n_i = v_{D_i}(s)$ is the order of vanishing or the valuation of s at the generic point of D_i

note that $\mathrm{Div}(s)$ is effective ($n_i \geq 0 \ \forall i$)

We saw that $\mathcal{I}_{Z(s)} \cong \mathcal{O}_X(-\mathrm{Div}(s)) \subset \mathcal{K}_X$

Theorem (Bertini's): k alg. closed

X a nonsingular closed subvariety of $\mathbb{P}^n_k$.
Then the set of hyperplanes $H \subset \mathbb{P}^n_k$ such that
$H \not\supset X$ and $X \cap H$ is nonsingular is a
non-empty open subset of $|\mathcal{O}_{\mathbb{P}^n}(1)| \cong \mathbb{P}^n_k$.
(closed points of)

Proof: Consider the product $X \times_k |\mathcal{O}_{\mathbb{P}^n}(1)|$.
Recall (Ex II.3.23) that the set of closed points of
$X \times_k |\mathcal{O}_{\mathbb{P}^n}(1)|$ is the product of the sets of closed
points of X and $|\mathcal{O}_{\mathbb{P}^n}(1)|$.

Consider the subset

$$B := \left\{ (x, H) \;\middle|\; \begin{array}{l} x \in H, \text{ either } x \text{ is a singular} \\ \text{point of } X \cap H \text{ or } H \supset X \end{array} \right\}$$

$$\subset X \times_k |\mathcal{O}_{\mathbb{P}^n}(1)|$$

<u>Claim:</u> B has a natural structure of a closed subscheme of $X \times_k |\mathcal{O}_{\mathbb{P}^n}(1)|$.

<u>Proof:</u> If $(x, H) \in B$, let $f \in \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ be an equation for H, i.e., $Z(f) = H$ and let $\{P_1, \ldots, P_r\}$ be a set of generators for I_X. Then, $I_{X \cap H}$ is generated by $(f, P_1, \ldots, P_r)$.