Prop. 7.2 in Hartshorne.

Theorem: $X/_{\operatorname{Spec} A}$, $\mathcal{L}$ invertible sheaf, $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$ generating $\mathcal{L}$. The associated morphism $\varphi : X \longrightarrow \mathbb{P}^n_A$ is a closed embedding iff, $\forall\, i = 0, \dots, n$, the open set V_i where s_i generates $\mathcal{L}$ is affine and the morphism

$$\varphi_i^\# : A\left[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right] \longrightarrow \mathcal{O}_X(V_i)$$

is surjective. Note: there is a typo here in Hartshorne

Proof in Hartshorne, or better yet: do it as an exercise.

Theorem: Suppose $A = k$ is an algebraically closed field, X is projective over k (i.e., $\exists$ a closed embedding $X \hookrightarrow \mathbb{P}^m_k$ for some m). Let $\mathcal{L}$ be an invertible sheaf on X, $s_0, \dots, s_n$ global sections of $\mathcal{L}$ generating $\mathcal{L}$, and let $\varphi : X \longrightarrow \mathbb{P}^n_k$

be the associated morphism, $V := \langle s_0, \dots, s_n \rangle \subset \Gamma(X, \mathcal{L})$
the k-span of $s_0, \dots, s_n$.

Then φ is a closed embedding iff

(1) V separates points, i.e., $\forall\, x, y \in X$ distinct closed points, $\exists\ s \in V$ s.t. $s(x) = 0$ and $s(y) \neq 0$.

$$\in \mathcal{L}_x / m_x \mathcal{L}_x \cong k(x) \cong k$$

(2) V separates tangent vectors, i.e., $\forall\ x \in X$ closed point, the induced k-linear map

$$\{ s \in V \mid s_x \in m_x \mathcal{L}_x \Leftrightarrow s(x) = 0 \} \longrightarrow m_x \mathcal{L}_x / m_x^2 \mathcal{L}_x \cong m_x / m_x^2$$

as k-vector space

is surjective.

(Proof in Hartshorne)

Remark: Recall from last quarter that the Zariski tangent space to X at x is $(\mathfrak{m}_x/\mathfrak{m}_x^2)^*$ the $k(x)$-dual of $\mathfrak{m}_x/\mathfrak{m}_x^2 = \mathfrak{m}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}/\mathfrak{m}_x = \mathfrak{m}_x \otimes_{\mathcal{O}_{X,x}} k(x)$

X projective$/k \Rightarrow X$ is of finite type$/k$

$\Rightarrow$ the residue field of any closed point of X is a finite extension of k

$\Rightarrow \quad \forall\, x \in X$ closed $\quad k(x) \cong k$ (k alg. closed)

So the Zariski tangent space $T_x X := (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$
$= \mathrm{Hom}_k(\mathfrak{m}_x/\mathfrak{m}_x^2, k)$

and $\mathfrak{m}_x\mathcal{L}_x/\mathfrak{m}_x^2\mathcal{L}_x \cong \mathfrak{m}_x/\mathfrak{m}_x^2 = T_x^* X$ the Zariski cotangent space.

$s \in V$, $s(x)=0$ means $s_x \in m_x \mathcal{L}_x$

so we can send s to the image of s_x in $m_x\mathcal{L}_x / m_x^2 \mathcal{L}_x$.

This gives the natural map

$$V_x := \{ s \in V \mid s(x)=0 \} \longrightarrow m_x\mathcal{L}_x / m_x^2\mathcal{L}_x \cong T_x^* X$$

whose surjectivity is equivalent to the injectivity of the dual map

$$T_x X \longrightarrow \{ s \in V \mid s(x)=0 \}^* = V_x^* \text{ (k-dual)}$$

<u>Geometric set-up:</u> The vector space V_x above is a hyperplane in V because it is the kernel of the evaluation map:

$$V \twoheadrightarrow \mathcal{L}_x / m_x \mathcal{L}_x \cong k$$

In fact, at the level of closed points, the morphism φ is the map
$$\begin{aligned} X &\longrightarrow \mathbb{P}V^* \quad \text{(projective space of hyperplanes in } V) \\ x &\longmapsto V_x \end{aligned}$$

Condition (1) means that for $x \neq y$, $V_x \neq V_y$, hence $\varphi(x) \neq \varphi(y)$

Condition (2) means that the Zariski tangent space at x to X maps injectively to the Zariski tangent space to $\mathbb{P}^n = \mathbb{P}V^*$ at $\varphi(x) = V_x$.

Exercise: $T_{\varphi(x)}\mathbb{P}V^*$ can be canonically identified with V_x^*.

Relative Proj:

In exercise II.5.17 you saw the relative Spec of a quasi-coherent sheaf of $\mathcal{O}_X$-algebras.

For a quasi-coherent sheaf of algebras $\mathcal{A}$, we construct $\mathrm{Spec}_X \mathcal{A} \longrightarrow X$ by glueing the schemes $\mathrm{Spec}\,\mathcal{A}(U) \longrightarrow \mathrm{Spec}\,\mathcal{O}_X(U)$ for all affine open sets $U \subset X$.

We now construct a relative Proj:

Assume that we have a quasi-coherent sheaf $\mathcal{S} = \bigoplus_{d \geq 0} \mathcal{S}_d$ of graded $\mathcal{O}_X$-algebras s.t. $\mathcal{S}_0 = \mathcal{O}_X$, $\mathcal{S}_1$ is coherent and $\mathcal{S}$ is locally generated by $\mathcal{S}_1$ as an $\mathcal{O}_X$-algebra, i.e., $\forall\, d > 0$, the multiplication map $\mathrm{Sym}^d \mathcal{S}_1 \longrightarrow \mathcal{S}_d$ is surjective.

Definition: The scheme $\mathrm{Proj}_X \mathcal{S} \longrightarrow X$ is constructed by glueing the schemes $\mathrm{Proj}\,\mathcal{S}(U) \to U$ for all affine open subschemes $U \subset X$.

On each $\mathrm{Proj}\,\mathcal{S}(U)$, we have the sheaf $\mathcal{O}_U(1)$.
These glue to give the global twisting sheaf $\mathcal{O}(1)$ on $\mathrm{Proj}_X \mathcal{S}$.

The projective bundle of a locally free sheaf of finite rank:

X noetherian scheme. $\mathcal{E}$ a locally free sheaf of finite rank $n+1$ on X. We saw last quarter the symmetric algebra $\mathrm{Sym}\,\mathcal{E} := \bigoplus_{m \geq 0} \mathrm{Sym}^m \mathcal{E}$ (homework)

On $U = \operatorname{Spec} A \subset X$ open s.t. $\mathcal{E}|_U \simeq \mathcal{O}_X^{\oplus(n+1)}$,

choose a basis $X_0, \dots, X_n$ of $\mathcal{E}|_U$, i.e.,

$$\mathcal{E}|_U = \mathcal{O}_U X_0 \oplus \cdots \oplus \mathcal{O}_U X_n$$

then
$$\operatorname{Sym} \mathcal{E}|_U = \bigoplus_{m \geq 0} \operatorname{Sym}^m (\mathcal{O}_U X_0 \oplus \cdots \oplus \mathcal{O}_U X_n)$$

$$= \mathcal{O}_U [X_0, \dots, X_n]$$ the polynomial algebra on $\mathcal{O}_U$ with generators $X_0, \dots, X_n$.

Then $\operatorname{Spec}_X \operatorname{Sym} \mathcal{E}|_U = \operatorname{Spec} \mathcal{O}_U [X_0, \dots, X_n] = \mathbb{A}^{n+1}_U \longrightarrow U$

and $\operatorname{Proj}_X \operatorname{Sym} \mathcal{E}|_U = \operatorname{Proj} \mathcal{O}_U [X_0, \dots, X_n] = \mathbb{P}^n_U \longrightarrow U$

$$\mathcal{O}_X(U)[X_0, \dots, X_n] \longleftarrow \mathcal{O}_X(U)$$

<u>Def:</u> $\mathbb{P}(\mathcal{E}) := \mathrm{Proj}_X \, \mathrm{Sym}\, \mathcal{E}$ is the projective bundle associated to $\mathcal{E}$.

The projections $\mathrm{Proj}\, \mathrm{Sym}\, \mathcal{E}|_U \longrightarrow U$ glue to give the natural projection $\mathbb{P}(\mathcal{E}) \xrightarrow{\pi} X$, and we also have the twisting sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$.

Locally, we saw $\mathrm{Sym}^d \mathcal{E}(U) = (\mathrm{Sym}\, \mathcal{E}(U))_d = \Gamma(\mathbb{P}^n_U, \mathcal{O}(d))$
$= \Gamma(U, \pi_* \mathcal{O}(d))$

$$\Rightarrow \quad \pi_* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(d) \cong \mathrm{Sym}^d \mathcal{E}.$$

$$\Rightarrow \quad \bigoplus_{d \in \mathbb{Z}} \pi_* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(d) \cong \mathrm{Sym}\, \mathcal{E}.$$

Projective morphisms:

Hartshorne defines a projective morphism as a morphism $f: X \longrightarrow Y$ which can be factored into a composition

$$X \xrightarrow{\varphi} \mathbb{P}^n_Y \longrightarrow Y$$ for some n, where φ is a closed embedding.

Grothendieck defines a projective morphism as a morphism which can be factored into $X \xrightarrow{\varphi} P \longrightarrow Y$ where P is a projective bundle over Y and φ is a closed embedding.

<u>Def:</u> A projective n-space bundle over Y is a scheme $\pi: P \longrightarrow Y$ s.t. $\exists$ covering $Y = \bigcup_{i \in I} V_i$ by

open sets $U_i = \operatorname{Spec} A_i$ s.t. $\forall i\ \exists$

$$\varphi_i : \pi^{-1}(U_i) \xrightarrow{\cong} \mathbb{P}^n_{U_i}$$

and $\forall\, i, j$ and all open affine $V = \operatorname{Spec} A \subset U_i \cap U_j$, the isomorphism $\varphi_j \circ \varphi_i^{-1} : \mathbb{P}^n_A \to \mathbb{P}^n_A$ is induced by an A-linear isom. $(\varphi_j \circ \varphi_i^{-1})^{\#} : A[X_0, \dots, X_n] \circlearrowleft$.

<u>Note:</u> (Ex. II.5.18) Every vector bundle is the Spec of the symmetric algebra of some locally free sheaf. However, <u>not</u> every projective bundle is the Proj of the symmetric algebra of some locally free sheaf.

This is the case for "nice" schemes, e.g., regular schemes.

The Hartshorne and Grothendieck definitions coincide for schemes Y that are quasi-projective over an affine scheme.

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open subschemes of projective schemes.

Another nice application of relative Proj: Blow ups.

<u>Blow-ups:</u> We blow up a scheme along a coherent sheaf of ideals or along a closed subscheme.

When we do this, we replace the coherent sheaf of ideals with an invertible sheaf, or we replace the closed subscheme with a locally principal Weil divisor (or closed subscheme).

The ambiant scheme becomes "nicer" (examples later).

<u>Def:</u> The blow up $Bl_{\mathcal{I}} X$, resp., $Bl_Y X$ of X along the coherent sheaf of ideals $\mathcal{I}$, resp., the closed subscheme Y, is the scheme $\mathrm{Proj}_X \, \mathcal{S} \xrightarrow{\pi} X$, where

$$\mathcal{S} := \bigoplus_{d \geq 0} \mathcal{I}^d \quad , \text{ resp.,} \quad \mathcal{S} := \bigoplus_{d \geq 0} \mathcal{I}_Y^d$$

and $\mathcal{I}^d$, resp., $\mathcal{I}_Y^d$, is the d-th power of $\mathcal{I}$, resp., $\mathcal{I}_Y$, in $\mathcal{O}_X$.