$$\Rightarrow \quad t_{li}\big|_{V_i \cap V_j} = t_{lj}\big|_{V_i\cap V_j} \; t_{ji}\big|_{V_i \cap V_j} \quad \checkmark$$

So the φ_i glue to give $\varphi : X \longrightarrow \mathbb{P}^n$

Note that on U_i : $\quad \mathcal{O}_{\mathbb{P}^n}(1)\big|_{U_i} = \mathcal{O}_{U_i} X_i$

when we pull back $\varphi_i^*\left(\mathcal{O}_{\mathbb{P}^n}(1)\big|_{U_i}\right) = \varphi_i^* \; \mathcal{O}_{U_i} X_i$

$$= \mathcal{O}_{V_i} \varphi_i^* X_i$$

So, on V_i $\quad \varphi^* \mathcal{O}_{\mathbb{P}^n}(1)\big|_{V_i} = \varphi_i^* \left(\mathcal{O}_{\mathbb{P}^n}(1)\big|_{U_i}\right)$

is trivial.

The transition functions of $\varphi^* \mathcal{O}_{\mathbb{P}^n}(1)$ are $\varphi^*\left(\frac{X_i}{X_j}\right)$

we defined $\varphi^*\left(\frac{X_i}{X_j}\right) = \varphi^\#\left(\frac{X_i}{X_j}\right) = \varphi^\#\left(\frac{X_i}{X_j}\right) = t_{ij}$

the t_{ij} are the transition functions of $\mathcal{L}$.

So, if we define isomorphisms:

$$\mathcal{L}|_{V_i} = \mathcal{O}_{V_i}\, s_i|_{V_i} \xrightarrow{\ \sim\ } \varphi^*(\mathcal{O}_{U_i} X_i) = \varphi^*(\mathcal{O}_{\mathbb{P}^n}(1)|_{U_i})$$

$$s_i|_{V_i} \longleftrightarrow \varphi^* X_i|_{U_i}$$

these isomorphisms glue to a global isomorphism

$$\mathcal{L} \cong \varphi^* \mathcal{O}_{\mathbb{P}^n}(1)$$

sending $\quad s_i \longleftrightarrow \varphi^* X_i$

(2) $\quad \varphi : X \longrightarrow \mathbb{P}^n$, put $\mathcal{L} := \varphi^* \mathcal{O}_{\mathbb{P}^n}(1)$

$s_i := \varphi^* X_i$, $\quad V_i := \varphi^{-1}(U_i)$

On U_j, we have $\frac{X_i}{X_j} \in \mathcal{O}_{\mathbb{P}^n}(U_j) = A\left[\frac{X_0}{X_j}, \ldots, \frac{X_n}{X_j}\right]$

put $t_{ij} := \varphi^*\left(\frac{X_i}{X_j}\right)$

we have $X_i\big|_{U_j} = \left(\frac{X_i}{X_j}\right) X_j\big|_{U_j}$

$\Rightarrow$ (pullback) $\boxed{s_i\big|_{V_j} = t_{ij}\, s_j\big|_{V_j}}$

and $\mathcal{L}\big|_{V_j} = \left(\varphi^*\mathcal{O}_{\mathbb{P}^n}(1)\right)\big|_{V_j} = \varphi^*\left(\mathcal{O}_{\mathbb{P}^n}(1)\big|_{U_j}\right)$

$= \varphi^*\left(\mathcal{O}_{U_j} X_j\big|_{U_j}\right) = \mathcal{O}_{V_j}\, \varphi^* X_j\big|_{V_j}$

$\boxed{= \mathcal{O}_{V_j}\, s_j\big|_{V_j}}$

$\Rightarrow$ s_j generates $\mathcal{L}$ on V_j.

Since $\mathbb{P}^n = \bigcup_{i=0}^{n} U_i \Rightarrow X = \varphi^{-1}(\mathbb{P}^n) = \bigcup_{i=0}^{n} V_i$

$\Rightarrow$ $s_0, \dots, s_n$ generate $\mathcal{L}$ on X

Since $\varphi^*\left(\frac{X_i}{X_j}\right) = t_{ij}$, φ is the morphism from part 1.

(determines the morphism on each U_i)

$\square$

<u>Remark:</u> In practice, collections of global sections often do not generate $\mathcal{L}$. However, we can still consider the open set $U \subset X$ where they do generate. So we obtain a morphism $U \longrightarrow \mathbb{P}^n$.

<u>Example</u>: Projections from one projective space to another.

A a ring $\quad S := A[X_0, \dots, X_n] = \bigoplus_{d \geq 0} S_d$

Choose $L_0, \dots, L_n \in S_1$

$P := Z(L_0) \cap \cdots \cap Z(L_n)$ is, by definition, a linear subspace of $\mathbb{P}^n$.

<u>Recall</u>: $I_{Z(L_i)} = \langle L_i \rangle \subset S$

$$\Rightarrow I_P = \langle L_0, \dots, L_n \rangle \subset S$$

Assume now that A is a field k. $\quad S_1 = kX_0 \oplus \cdots \oplus kX_n$

After possibly removing some of the L_i, we can assume they are linearly independent $/k$.

Then (see Chapter 1. Ex. 2.11) P has codim. $r+1$

or dim. $n-1-r$.

After a linear change of coordinates, we can assume

$L_0 = X_0, \dots, L_r = X_r$.

Then $P = Z(X_0, \dots, X_r) \cong \mathbb{P}_k^{n-1-r} \subset \mathbb{P}_k^n$

We project $\mathbb{P}^n$ from P (P is the center of the projection)

Choose $X = \mathbb{P}^n$, $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(1)$

$s_0 = L_0 = X_0, \dots, s_r = L_r = X_r$

The open subset of $\mathbb{P}^n$ where $s_0, \dots, s_r$ generate $\mathcal{O}_{\mathbb{P}^n}(1)$ is

$U := \mathbb{P}^n \setminus P = U_0 \cup \cdots \cup U_r$

By the theorem, we have a morphism

$$\varphi : U \longrightarrow \mathbb{P}^n_k \quad \text{s.t.} \quad \varphi^* \mathcal{O}_{\mathbb{P}^n}(1) \cong \mathcal{O}_{\mathbb{P}^n}(1)\big|_U$$

We often write $\varphi : X \dashrightarrow \mathbb{P}^n$ to indicate that we have a morphism well defined on some open subset of X. We call this a rational map.

<u>Application to quadrics:</u> $A = k$ a field of char. $\neq 2$.

Choose $F \in S_2$

$Q := Z(F) \subset \mathbb{P}^n$ is, by definition, a quadric.

We can use the Gram-Schmitt process to write

$$F = \sum_{i=0}^{n} L_i^2$$ where $L_0, \dots, L_n$ are lin independent linear polynomials.

The integer $r+1$ is the rank of F or $Q := Z(F)$.

The linear space $P = Z(L_0, \dots, L_r)$ is the vertex of Q.

Via the projection of center P, the quadric Q is the inverse image of a quadric $\overline{Q} \subset \mathbb{P}^r$:

$$\begin{tikzcd} \mathbb{P}^n \arrow[r, dashed, "\varphi"] & \mathbb{P}^r \\ Q \arrow[u, phantom, "\cup"] \arrow[r, dashed, "\varphi|_Q"] & \overline{Q} \arrow[u, phantom, "\cup"] \end{tikzcd}$$

i.e., $\varphi^{-1}(\overline{Q}) = Q \setminus P$ $\qquad \varphi : \mathbb{P}^n \setminus P \longrightarrow \mathbb{P}^r$

Let $Y_0, \dots, Y_r$ be the coordinates on $\mathbb{P}^r$: $\varphi^* Y_i = L_i$

define $\overline{Q} := Z\left(\sum_{i=0}^{r} Y_i^2\right)$ $\qquad \varphi^* \overline{Q} = Z\left(\varphi^* \sum_{i=0}^{r} Y_i^2\right) = Q \setminus P$

$\overline{Q}$ has maximal rank (in $\mathbb{P}^{\sim}$), i.e., its vertex is $\emptyset$.

We will see that P is the singular locus of Q.

Applying this to $\overline{Q}$: $\overline{Q}$ is nonsingular, i.e., all the local rings of $\overline{Q}$ are regular local rings.

<u>Ampleness and very ampleness:</u>

<u>Def</u>: (1) An invertible sheaf $\mathcal{L}$ on a scheme $X/_{\operatorname{Spec} A}$ is called very ample if there exists a collection $\{s_0, \dots, s_n\}$ of global sections of $\mathcal{L}$ for which the associated morphism is an embedding.

(2) An invertible sheaf is called ample if, for every

coherent sheaf $\mathcal{F}$ on X, $\exists\ n_0 > 0$ s.t.
$\forall\ n \geq n_0$ $\mathcal{F} \otimes \mathcal{L}^n$ is generated by global sections.

<u>Theorem</u>: Suppose X is of finite type / $\operatorname{Spec} A$, $\mathcal{L}$ invertible.
Then $\mathcal{L}$ is ample iff $\exists\ m > 0$ s.t. $\mathcal{L}^m$ is very ample.
(proof in Hartshorne)

Prop. 7.2 in Hartshorne.

Theorem: $X/_{\operatorname{Spec} A}$, $\mathcal{L}$ invertible sheaf, $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$ generating $\mathcal{L}$. The associated morphism $\varphi : X \longrightarrow \mathbb{P}^n_A$ is a closed embedding iff, $\forall\, i = 0, \dots, n$, the open set X_i where s_i generates $\mathcal{L}$ is affine and the morphism

$$\varphi_i^{\#} : A\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right] \longrightarrow \mathcal{O}_X(X_i)$$

is surjective. Note: there is a typo here in Hartshorne

Proof in Hartshorne, or better yet: do it as an exercise.

Theorem: Suppose $A = k$ is an algebraically closed field, X is projective over k (i.e., $\exists$ a closed embedding $X \hookrightarrow \mathbb{P}^m_k$ for some m). Let $\mathcal{L}$ be an invertible sheaf on X, $s_0, \dots, s_n$ global sections of $\mathcal{L}$ generating $\mathcal{L}$, and let $\varphi : X \longrightarrow \mathbb{P}^n_k$