We apply the lemma to compute $Cl(\mathbb{P}^n_k)$:

take $U = U_0 = \operatorname{Spec} k[\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0}]$

$Z_0 = \mathbb{P}^n_k \setminus U_0$ is irreducible

$= Z(X_0)$ $\quad X_0 \in S_1 = \Gamma(\mathcal{O}_X(1))$

$$\mathbb{Z}[Z_0] \longrightarrow Cl(\mathbb{P}^n_k) \longrightarrow Cl(U_0) \longrightarrow 0$$

$U_0 \cong \mathbb{A}^n_k \quad \Rightarrow \quad Cl(U_0) = 0$

$$\Rightarrow \qquad \mathbb{Z}[Z_0] \twoheadrightarrow Cl(\mathbb{P}^n_k)$$

Lemma: The map $\mathbb{Z}[Z_0] \twoheadrightarrow Cl(\mathbb{P}^n_k)$ is injective, i.e.,

$$\mathbb{Z}[Z_0] \xrightarrow{\cong} Cl(\mathbb{P}^n_k)$$

Proof: Injectivity means that there does not exist a rational function $f \in K(\mathbb{P}^n_k)$ s.t. $\mathrm{Div}(f)$ is a multiple of $[Z_0]$.

We study $\mathrm{Div}(f)$ for $f \in K = K(\mathbb{P}^n_k)$.

$$K = K(\mathbb{P}^n) = K(U_0) = \mathrm{Frac}\, k\left[\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0}\right]$$
$$= k\left(\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0}\right)$$

Claim: $k\left(\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0}\right) = k(X_0, \dots, X_n)_0$,

because: we can write any f as $\frac{p}{q}$ where $p, q \in k\left[\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0}\right]$ $\quad \frac{p}{q} = \dfrac{\frac{P}{X_0^r}}{\frac{Q}{X_0^s}}$ where P, Q are homogeneous of degree r and s respectively.

we can multiply P or Q by some power of X_0, hence assume $r = s$.

$$\Rightarrow \quad f = \frac{P}{Q}$$ where P, Q are homogeneous of the same degree (and coprime).

<u>Claim:</u> $\mathrm{Div}(f) = \mathrm{Div}(P) - \mathrm{Div}(Q)$

<u>proof:</u> Choose a point $x \in X = \mathbb{P}^n_k$ of codim. 1

$\exists\ i$ s.t. $x \in U_i = \operatorname{Spec} S[X_i^{-1}]_0 \subset X$

$x \leftrightarrow \mathfrak{p} \subset S[X_i^{-1}]_0$ prime of height 1

$S[X_i^{-1}]_0$ is a UFD $\Rightarrow \mathfrak{p}$ is principal (Prop. I.1.12A)

$\leftarrow$ pol. ring

$\Rightarrow \quad \mathfrak{p} = \langle g \rangle \quad g$ irreducible pol. in $\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}$

$\Rightarrow \quad g = \frac{R}{X_i^d}$ where R is hom. of degree d in $X_0, \dots, X_n$

The local ring of x is $\mathcal{O}_{X,x} \cong k\left[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right]_{\mathfrak{p}}$

$$f = \frac{P}{Q} = \frac{\frac{P}{X_i^m}}{\frac{Q}{X_i^m}}$$

where m = degree of P = degree of Q

(recall $g = \frac{R}{X_i^d}$)

$$v_x(f) = v_{\mathfrak{p}}(f) = v_{\langle g \rangle}\left(\frac{P}{X_i^m}\right) - v_{\langle g \rangle}\left(\frac{Q}{X_i^m}\right)$$

= # times that g occurs as a factor of $\frac{P}{X_i^m}$

— # times that g occurs as a factor of $\frac{Q}{X_i^m}$

= # times that R divides P

— # times that R divides Q

$\Rightarrow \quad \mathrm{Div}(f) = \mathrm{Div}(P) - \mathrm{Div}(Q) \quad \square$

If f is constant, $\mathrm{Div}(f) = 0$

If f is not constant, then $\deg P = \deg Q > 0$

P and Q are coprime $\Rightarrow$ If $[Z_0]$ appears in $\mathrm{Div}(P)$, it will not appear in $\mathrm{Div}(Q)$ and vice-versa.

$\Rightarrow$ $\mathrm{Div}(f)$ cannot be just a multiple of $[Z_0]$. $\square$

Cartier divisors: We do not need $(*)$.

Here X is any noetherian scheme.

Def: The sheaf of total quotient rings $\mathcal{K}_X$:

Since affine open sets form a basis of the topology of X, we only define the sections of $\mathcal{K}_X$ on affine open sets.

Let $U = \operatorname{Spec} A \subset X$ be open affine, we define $\mathcal{K}_X(U)$ to be the total quotient ring of A, i.e., $\mathcal{K}_X(U)$ is the localization of A at all nonzerodivisors. Restriction maps are obtained from localization morphisms.

__Def:__
$$\begin{array}{ccll} \mathcal{K}_X^* & \subset & \mathcal{K}_X & \text{is the subsheaf of invertible elements} \\ \cup & & \cup & \\ \mathcal{O}_X^* & \subset & \mathcal{O}_X & \text{'' '' '' '' '' ''} \end{array}$$

__Note:__ $\mathcal{K}_X$ is a locally constant sheaf (exercise)

If X is integral, $\mathcal{K}_X$ is the constant sheaf with group K.

__Def:__ A Cartier divisor on X is a global section of $\mathcal{K}_X^* / \mathcal{O}_X^*$.

What does the definition mean?

Consider the exact sequence (group law is multiplication)

$$1 \longrightarrow \mathcal{O}_X^* \longrightarrow \mathcal{K}_X^* \longrightarrow \mathcal{K}_X^*/\mathcal{O}_X^* \longrightarrow 1$$

Sections of $\mathcal{K}_X^*/\mathcal{O}_X^*$ can be locally lifted to $\mathcal{K}_X^*$:

$\forall \; f \in \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ $\exists$ covering $X = \bigcup_{i \in I} U_i$

s.t. $\forall i$ $\exists$ $s_i \in \mathcal{K}_X^*(U_i)$ with $s_i \longmapsto f|_{U_i} \in \mathcal{K}_X^*/\mathcal{O}_X^*(U_i)$

and on $U_i \cap U_j$: $\left(s_i/s_j\right)\Big|_{U_i \cap U_j} \in \mathcal{O}_X^*(U_i \cap U_j)$

We call $\{(s_i, U_i)\}$ a representation of f. Note that it is not unique

Def: A Cartier divisor is called principal if it is in the image of the natural map

$$\Gamma(X, \mathcal{K}_X^*) \longrightarrow \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*).$$

In other words it has a representation of the form

$$\{(s, X)\}.$$

Def: Two Cartier divisors are called linearly equivalent if their difference (for the multiplicative group structure) is principal. In other words, they have representations of the form $\{(s_i, U_i)\}$ and $\{(s s_i, U_i)\}$

Relation with invertible sheaves: Given a representation $\{(f_i, U_i)\}$ of a Cartier divisor D, we define an invertible subsheaf $\mathcal{O}_X(D)$ of $\mathcal{K}_X$ as follows.

On each U_i, define $\mathcal{O}_X(D)|_{U_i} := \mathcal{O}_{U_i} \cdot f_i^{-1} \subset \mathcal{K}_{U_i}$

These glue to a subsheaf $\mathcal{O}_X(D)$ of $\mathcal{K}_X$ because on

$U_i \cap U_j$: $\quad \mathcal{O}_{U_i \cap U_j} f_i^{-1}|_{U_i \cap U_j} = \mathcal{O}_{U_i \cap U_j} f_j^{-1}|_{U_i \cap U_j} \subset \mathcal{K}_{U_i \cap U_j}$

because $f_i / f_j \in \mathcal{O}_X^*(U_i \cap U_j)$

Check (exercise) $\mathcal{O}_X(D)$ <u>does not</u> depend on the choice of representation.

Conversely, given an invertible subsheaf $\mathcal{L}$ of $\mathcal{K}_X$, we can associate a Cartier divisor D to it such that $\mathcal{O}_X(D) = \mathcal{L}$ as follows.

Choose an open covering $X = \bigcup_{i \in I} U_i$ s.t. $\forall i$ $\mathcal{L}|_{U_i} \cong \mathcal{O}_{U_i}$.

$\forall i$, define f_i^{-1} as the image of 1:

$$\begin{tikzcd} \mathcal{O}_{U_i} \arrow[r, "\cong", "\varphi_i"'] & \mathcal{L}|_{U_i} \arrow[r, hook] & \mathcal{K}_{U_i} \\ 1 \arrow[rr, mapsto] & & f_i^{-1} \end{tikzcd}$$

On $U_i \cap U_j$, we have $f_i/f_j \in \mathcal{O}_X^*(U_i \cap U_j)$

because $f_i^{-1}|_{U_i \cap U_j}$ and $f_j^{-1}|_{U_i \cap U_j}$ generate the same trivial $\mathcal{O}_{U_i \cap U_j}$-submodule of $\mathcal{K}_{U_i \cap U_j}$.

So $\{(f_i, U_i)\}$ represents a Cartier divisor D, and, by def., $\mathcal{O}_X(D) = \mathcal{L}$.

So: Cartier divisors are in bijection with invertible subsheaves of $\mathcal{K}_X$.