Def. The (Weil) divisor of a rational function:

$f \in K = K(X)$ $\quad X$ satisfies $(*)$, $K(X)=$ field of rational functions

$$\mathrm{Div}(f) := \sum_{\substack{Y \subset X \\ \text{integral Weil divisor}}} v_Y(f)[Y]$$

$v_Y(f) \in \mathbb{Z}$ is the valuation of f for the DVR, $\mathcal{O}_{Y,\eta}$

$\eta =$ generic point of Y.

$K = \mathrm{Frac}(\mathcal{O}_{Y,\eta})$ $\quad \exists\, u \in \mathcal{O}_{Y,\eta}^*$

s.t. $f = u\pi^{v_Y(f)}$

where π is a uniformizer for $\mathcal{O}_{Y,\eta}$

The divisors of rational functions are called principal divisors.

$\forall\ f, g \in K \qquad v_Y(fg^{-1}) = v_Y(f) - v_Y(g)$

$\Rightarrow$ principal divisors form a subgroup of $\mathrm{Div}(X)$.

<u>Def</u>: $Cl(X) := \mathrm{Div}(X) / \mathrm{Prin}(X)$

the class group of X

$\uparrow$ principal divisors

We need to verify that $\sum v_Y(f)[Y]$ is finite to make sure $\mathrm{Div}(f)$ is well-defined.

<u>Lemma</u>: $\forall\ f \in K$, $v_Y(f) = 0$ for all but finitely many integral Weil divisors Y.

<u>Proof</u>: Let $U = \mathrm{Spec}\, A \subset X$ be open and $\neq \emptyset$.

Then $K = \mathrm{Frac}(A) \Rightarrow \exists\ g, h \in A$ s.t. $f = \frac{g}{h}$

After replacing A by $A[h^{-1}]$, we can assume $f \in A$.

$\Rightarrow \forall\, \mathfrak{p} \in \operatorname{Spec} A$ of height 1, $f \in A_{\mathfrak{p}} = \mathcal{O}_{X,\mathfrak{p}}$

$\Rightarrow \quad v_{\mathfrak{p}}(f) \geq 0$

$\Rightarrow$ the set of $\eta \in X$ s.t. $v_\eta(f) < 0$ is contained in $X \setminus U =: Y$ $\qquad Y \subset X$ is closed.

X is noetherian $\Rightarrow Y =$ finite union of irreducible components $= Y_1 \cup \cdots \cup Y_n$

$U \neq \emptyset \Rightarrow \forall\, i,\ Y_i \neq X \Rightarrow \operatorname{codim} Y_i \geq 1$

$\Rightarrow \quad v_\eta(f) < 0 \Rightarrow \eta$ is the generic point of one of the Y_i

$\Rightarrow$ there are at most finitely many η with $v_\eta(f) < 0$.

replace f with $\frac{1}{f}$ $\Rightarrow$ there are at most finitely many η with $v_\eta(f) > 0$. $\square$

<u>Def</u>: We say two Weil divisors $D_1, D_2 \in \mathrm{Div}(X)$ are linearly equivalent if $\exists f \in K$ s.t.

$$D_1 - D_2 = \mathrm{Div}(f),$$

i.e., D_1, D_2 have the same image in $\mathrm{Cl}(X)$.

We also say D_1, D_2 have the same linear equivalence class.

<u>Prop.</u>: If $X = \mathrm{Spec} A$ (in particular A is a noetherian integral domain), then A is a UFD iff X is normal (i.e., A is integrally closed) and $\mathrm{Cl}(X) = 0$.

Proof: (see Hartshorne).

Corollary: If $X = \mathbb{A}^n_k$, then $Cl(X) = 0$

Divisors in projective space $X = \mathbb{P}^n_k = \operatorname{Proj} S$

$$S := k[X_0, \dots, X_n] = \bigoplus_{d \geq 0} S_d$$

recall $S_d = \Gamma(\mathcal{O}_X(d))$

For $s \in S_d$, put $Y := Z(s)$

Lemma: Y is a closed subscheme of pure codimension 1 of X (i.e., all the irreducible components of Y have codim. 1).

The irreducible components of Y are the zeros of the irreducible factors of s. Furthermore, the homogeneous ideal I_Y is generated by s.

Proof: The ideal sheaf $\mathcal{I}_Y$ is the image of

$$s : \mathcal{O}_X(-d) = \mathcal{O}_X(d)^* \longrightarrow \mathcal{O}_X .$$

In each open set $U_i := \operatorname{Spec} S[X_i^{-1}]_0 = \operatorname{Spec} k\left[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right]$

$$\mathcal{O}_X(d) = \mathcal{O}_{U_i} \cdot X_i^d \qquad \mathcal{O}_X(-d) = \mathcal{O}_{U_i} \cdot X_i^{-d}$$

$$\begin{tikzcd} s|_{U_i} : \mathcal{O}_{U_i} X_i^{-d} \arrow[rr, "s"] \arrow[dr, two heads] & & \mathcal{O}_{U_i} . \\ & \mathcal{O}_{U_i} \cdot \frac{s}{X_i^d} \arrow[ur, hook] & \end{tikzcd}$$

$$\frac{s}{X_i^d} \in k\left[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right]$$

$$\Longrightarrow \quad I_{Y \cap U_i} = \Gamma(\mathcal{I}_{Y \cap U_i}) = \left\langle \frac{s}{X_i^d} \right\rangle \subset S[X_i^{-1}]_0$$

The homogeneous ideal $I_Y = \bigoplus_{e \geq 0} I_{Y,e} \subset S$

$$I_{Y,e} = \{ t \in S_d \mid \mathcal{J}_{Z(t)} \subset \mathcal{J}_Y \}$$

on U_i: $\quad \mathcal{J}_{Z(t)\cap U_i} \subset \mathcal{J}_{Y \cap U_i}$

$$\Leftrightarrow \quad I_{Z(t)\cap U_i} \subset I_{Y\cap U_i} = \left\langle \frac{s}{X_i^d} \right\rangle$$

$$\begin{array}{c} \| \\ \left\langle \frac{t}{X_i^e} \right\rangle \end{array}$$

$$\Rightarrow \quad \frac{t}{X_i^e} \in \left\langle \frac{s}{X_i^d} \right\rangle \qquad \forall\, i = 0, \dots, n$$

(exercise) $\Rightarrow$ $\quad t$ is a multiple of s, i.e., $t \in \langle s \rangle \subset S$

$$\Rightarrow I_Y = \langle s \rangle \subset S$$

Now write $\quad s = s_1^{m_1} \cdots s_\ell^{m_\ell}$ where s_i is irreducible (hom.) of degree d_i and s_i, s_j are not proportional for $i \neq j$.

$\Rightarrow \quad \mathcal{I}_{Z(s)} = \text{product of } \mathcal{I}_{Z(s_i)}^{m_i} \text{ in } \mathcal{O}_X$

$$\Rightarrow \quad Z(s) \underset{\text{as a set}}{=} \bigcup_{i=1}^{l} Z(s_i)$$

$\left(\text{recall that } Z(s) \overset{\text{as a set}}{=} \{ x \in X \mid s(x) \in \mathfrak{m}_x \subset \mathcal{O}_{X,x} \} \right.$
$\left. \qquad = \bigcup_{i=1}^{l} \{ x \in X \mid s_i(x) \in \mathfrak{m}_x \} \right)$

$I_{Z(s_i)} = \langle s_i \rangle$

$\Rightarrow Z(s_i)$ is irreducible of codimension 1 in X.
(from last quarter: intersect with U_j s.t. $Z(s_i) \cap U_j \neq \emptyset$) $\square$

Def: The Weil divisor of s is $\mathrm{Div}(s) = \sum_{i=1}^{l} m_i [Z(s_i)]$

Next we wish to determine $Cl(\mathbb{P}^n_k)$.

We first need a general lemma (works for all X satisfying $(*)$).

General lemma: $U \subset X$ open $\neq \phi$, let $Z_1, \dots, Z_n$ be the codimension 1 irreducible components of $X \setminus U$. Then, intersecting divisors of X with U produces the exact sequence

$$0 \to \mathbb{Z}[Z_1] \oplus \cdots \oplus \mathbb{Z}[Z_n] \longrightarrow \mathrm{Div}(X) \longrightarrow \mathrm{Div}(U) \to 0$$
$$D \longmapsto D \cap U$$

which induces the exact sequence

$$\mathbb{Z}[Z_1] \oplus \cdots \oplus \mathbb{Z}[Z_n] \longrightarrow \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(U) \longrightarrow 0.$$

Proof: The first exact sequence is immediate: a divisor of U is the intersection with U of its closure in X.

On the second sequence, surjectivity is true for the same reason.

We prove exactness in the middle:

Note: $K = K(X) = K(U)$

for $f \in K$ $\quad \mathrm{Div}_X(f) \cap U = \mathrm{Div}_U(f)$.

Consider the diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \mathrm{Prin}(X) \arrow[r] \arrow[d] & \mathrm{Prin}(U) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z}[Z_1] \oplus \cdots \oplus \mathbb{Z}[Z_n] \arrow[r] \arrow[d, equal] & \mathrm{Div}(X) \arrow[r] \arrow[d, "D \in \ \mapsto\ \bar{D} \in"'] & \mathrm{Div}(U) \arrow[r] \arrow[d] & 0 \\
 & \mathbb{Z}[Z_1] \oplus \cdots \oplus \mathbb{Z}[Z_n] \arrow[r, "\alpha"] & \mathrm{Cl}(X) \arrow[r, "\beta"] \arrow[d] & \mathrm{Cl}(U) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

$\beta \circ \alpha = 0$ clear.

show $\ker \beta \subset \operatorname{im} \alpha$

If $D \in \mathrm{Div}(X)$ has image in $\ker \beta$, this means

$\exists\ f \in K$ s.t. $D \cap U = \mathrm{Div}_U(f)$.

$$\Rightarrow \quad \big(D - \mathrm{Div}_X(f)\big) \cap U = 0$$

$\Rightarrow \quad D - \mathrm{Div}_X(f)$ supported on $X \setminus U$

$$\Rightarrow \quad D - \mathrm{Div}_X(f) \in \mathbb{Z}[Z_1] \oplus \cdots \oplus \mathbb{Z}[Z_n]$$

$$\Rightarrow \quad \bar{D} = \overline{\big(D - \mathrm{Div}_X(f)\big)} \in \mathrm{Cl}(X)$$

$$\in \text{ image of } \mathbb{Z}[Z_1] \oplus \cdots \oplus \mathbb{Z}[Z_n].$$

$\Rightarrow$ exactness in the middle. $\square$

We apply the lemma to compute $Cl(\mathbb{P}^n_k)$:

take $U = U_0 = \operatorname{Spec} k[\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0}]$

$Z_0 = \mathbb{P}^n_k \setminus U_0$ is irreducible

$= Z(X_0)$ $X_0 \in S_1 = \Gamma(\mathcal{O}_X(1))$

$$\mathbb{Z}[Z_0] \longrightarrow Cl(\mathbb{P}^n_k) \longrightarrow Cl(U_0) \longrightarrow 0$$

$U_0 \cong \mathbb{A}^n_k \Rightarrow Cl(U_0) = 0$

$$\Rightarrow \quad \mathbb{Z}[Z_0] \twoheadrightarrow Cl(\mathbb{P}^n_k)$$

<u>Lemma:</u> The map $\mathbb{Z}[Z_0] \twoheadrightarrow Cl(\mathbb{P}^n_k)$ is injective, i.e.,

$$\mathbb{Z}[Z_0] \xrightarrow{\cong} Cl(\mathbb{P}^n_k)$$