The fact that $\{\mathcal{O}_{\mathbb{P}^n}(m),\ m\in\mathbb{Z}\}/\sim \;\xleftarrow{\cong}\; \mathbb{Z}$

follows from $\overset{\Gamma(\mathcal{O}_{\mathbb{P}^n}(m))}{\overset{=}{H^0}}(\mathcal{O}_{\mathbb{P}^n}(m)) \cong S_m \subset S$

and $\mathcal{O}_{\mathbb{P}^n}(m)^{-1} \cong \mathcal{O}_{\mathbb{P}^n}(-m)$:

if $m\neq m'$, then $\mathcal{O}_{\mathbb{P}^n}(m)\not\cong \mathcal{O}_{\mathbb{P}^n}(m')$

if $m,m'>0$, then $S_m\neq S_{m'} \Rightarrow \mathcal{O}_{\mathbb{P}^n}(m)\not\cong\mathcal{O}_{\mathbb{P}^n}(m')$

if $m>0$, $m'<0$, then $H^0(\mathcal{O}_{\mathbb{P}^n}(m))\neq 0$ and $H^0(\mathcal{O}_{\mathbb{P}^n}(m'))=0$

if $m,m'<0$, then $\mathcal{O}_{\mathbb{P}^n}(m)^{-1}\cong \mathcal{O}_{\mathbb{P}^n}(-m)$

$\not\cong \mathcal{O}_{\mathbb{P}^n}(-m') = \mathcal{O}_{\mathbb{P}^n}(m')^{-1}$

now use $\underbrace{(\mathcal{L}^{-1})^{-1}\cong\mathcal{L}}$ to conclude $\mathcal{O}_{\mathbb{P}^n}(m)\not\cong\mathcal{O}_{\mathbb{P}^n}(m')$

More generally, for any locally free sheaf $\mathcal{F}$ of finite rank:

$$(\mathcal{F}^*)^*\cong\mathcal{F}.$$

More operations on sheaves of modules:

Given a morphism of (noetherian) schemes $f: X \to Y$ and an $\mathcal{O}_X$-module $\mathcal{F}$, $f_*\mathcal{F}$ is $(f_*\mathcal{O}_X)$-module.

Using $f^\#: \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$ morphism of sheaves of rings.

this gives a structure of $\mathcal{O}_Y$-algebra on $f_*\mathcal{O}_X$.

So we can think of $f_*\mathcal{F}$ as an $\mathcal{O}_Y$-module, using $f^\#$:

we call this restriction of scalars.

The affine case: If $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$

M an A-module $\mathcal{F} := \tilde{M}$ on X, then

$f_*\mathcal{F} = \widetilde{{}_B M}$ on Y where ${}_B M$ is M regarded as a module over B via $f^\#: B \longrightarrow A$.

A few words about proving this:

Given a basic open set $\operatorname{Spec} B[g^{-1}] \subset Y = \operatorname{Spec} B$,

we saw (last quarter) $f^{-1}(\operatorname{Spec} B[g^{-1}]) = \operatorname{Spec} A[\bar{g}^{-1}]$

where $\bar{g} := f^{\#}(g) \in A$

$$\Rightarrow \quad (f_* \mathcal{F})(\operatorname{Spec} B[g^{-1}]) = \mathcal{F}(f^{-1}(\operatorname{Spec} B[g^{-1}]))$$

$$= \mathcal{F}(\operatorname{Spec} A[\bar{g}^{-1}])$$

$$= M[\bar{g}^{-1}] = {}_B M[g^{-1}]$$

$$\Rightarrow \quad f_* \mathcal{F} \cong \widetilde{{}_B M}$$

Also $\Rightarrow$ $f_*(\text{quasi-coherent}) = \text{quasi-coherent}.$

Pull-backs:

First, general inverse images:

For a continuous map $f: X \to Y$ of topological spaces and a sheaf $\mathcal{G}$ on Y, the inverse image sheaf $f^{-1}\mathcal{G}$ on X is the sheaf associated to the presheaf

$$U \longmapsto \varinjlim_{V \supset f(U)} \mathcal{G}(V) \quad \text{if } f(U) \text{ is open } = \mathcal{G}(f(U))$$

For any subset $Z \overset{i}{\hookrightarrow} Y$, the restriction of $\mathcal{G}$ to Z is

$$\mathcal{G}|_Z := i^{-1}\mathcal{G}.$$

The stalk of $\mathcal{G}|_Z$ at any $P \in Z$ can be naturally identified with $\mathcal{G}_P$.

For a morphism of (noetherian) schemes $f: X \to Y$ and a sheaf $\mathcal{G}$ of $\mathcal{O}_Y$-modules, the sheaf $f^{-1}\mathcal{G}$ is an $(f^{-1}\mathcal{O}_Y)$-module on X.

<u>Note</u>: $f^{\#}: \mathcal{O}_Y \to f_* \mathcal{O}_X$

also induces $f^{\#}: f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ (exercise)

The pull-back of $\mathcal{G}$ to X is

$$f^*\mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

<u>Exercise</u> (see homework): There is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F})$$

for all $\mathcal{O}_X$-modules $\mathcal{F}$ and all $\mathcal{O}_Y$-modules $\mathcal{G}$.

In other words, the functors f^* and f_* are adjoint.

<u>The affine case</u>: $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$

N a B-module , $\mathcal{G} := \tilde{N}$

then (exercise, unwind the definitions):

$$f^* \tilde{N} \cong (N \otimes_B A)^{\sim}$$

extension of scalars.

<u>More about quasi-coherent sheaves</u>: X noetherian scheme.

$\mathcal{F}$ quasi-coherent on X.

By def., $\exists$ open cover of X by open affine schemes $U = \operatorname{Spec} A$ s.t. $\exists$ A-module M with $\mathcal{F}|_U \cong \tilde{M}$.

Theorem: If $\mathcal{F}$ is quasi-coherent on X (noetherian), then $\forall$ open affine $\operatorname{Spec} B \subset X$, $\exists$ B-module N s.t. $\mathcal{F}|_{\operatorname{Spec} B} \cong \tilde{N}$.

The main ingredient of the proof is the following

Lemma (5.3 in Hartshorne): $X = \operatorname{Spec} A$, $\mathcal{F}$ quasi-coherent on X, $f \in A$, then

(a) if $s \in \Gamma(X, \mathcal{F})$ s.t. $s|_{D(f) = \operatorname{Spec} A[f^{-1}]} = 0$

then $\exists\, n > 0$ s.t. $f^n s = 0 \in \Gamma(X, \mathcal{F})$

(b) if $t \in \mathcal{F}(D(f)) = \Gamma(D(f), \mathcal{F})$, then $\exists\, n > 0$ s.t. $f^n t$ is the restriction of a global section of $\mathcal{F}$ over X.

Proof of the lemma: $\exists$ open covering of $X = \operatorname{Spec} A$ by open affine sets $\operatorname{Spec} B$ s.t. $\exists$ B-mod. N with $\mathcal{F}|_{\operatorname{Spec} B} \cong \tilde{N}$.

We can write
$$\begin{array}{c} \operatorname{Spec} B = \bigcup\limits_{\text{some } g \in A} \operatorname{Spec} A[g^{-1}] \\ \cap \\ X = \operatorname{Spec} A \end{array}$$

$$\begin{array}{ccccccc} g \in A & \longrightarrow & B & \longrightarrow & A[g^{-1}] & = & B[\bar{g}^{-1}] \\ g & \longmapsto & \overset{\psi}{\bar{g}} & & & & \end{array}$$

universal prop. of localizations

$$\mathcal{F}|_{\operatorname{Spec} B} \cong \tilde{N} \Rightarrow \mathcal{F}|_{\operatorname{Spec} B[\bar{g}^{-1}]} \cong \widetilde{N[\bar{g}^{-1}]}$$

$$\mathcal{F}|_{\operatorname{Spec} B[\bar{g}^{-1}]} \cong \mathcal{F}|_{\operatorname{Spec} A[g^{-1}]}$$

$$N[\bar{g}^{-1}] = N \otimes_B B[\bar{g}^{-1}] \cong N \otimes_B A[g^{-1}] =: M$$ module over $A[g^{-1}]$

So $\mathcal{F}|_{\operatorname{Spec} A[g^{-1}]} \cong \tilde{M}$

$$\Rightarrow \quad X = \bigcup_{\text{some } g \in A} \operatorname{Spec} A[g^{-1}] \quad \text{s.t. } \forall g \quad \mathcal{F}|_{\operatorname{Spec} A[g^{-1}]} \cong \tilde{M}$$

X affine $\Rightarrow$ X quasi-compact

$$\Rightarrow \quad X = \bigcup_{i=1}^{n} \operatorname{Spec} A[g_i^{-1}]$$

s.t. $\forall i \quad \mathcal{F}|_{\operatorname{Spec} A[g_i^{-1}]} \cong \tilde{M}_i$ $\quad M_i$ module$/A[g_i^{-1}]$

<u>On part (a)</u>: $s \in \mathcal{F}(X)$ s.t. $s|_{\operatorname{Spec} A[f^{-1}]} = 0$

$$\forall i \quad s|_{D(f) \cap D(g_i)} = 0$$

$$D(f) \cap D(g_i) = D(g_i f) = \operatorname{Spec} A[(g_i f)^{-1}]$$

$$\mathcal{F}\big|_{\operatorname{Spec} A[g_i^{-1}]} \cong \widetilde{M_i} \Rightarrow \mathcal{F}(D(g_i f)) = M_i[f^{-1}]$$

$$s_i := s\big|_{D(g_i) = \operatorname{Spec} A[g_i^{-1}]} \in M_i$$

$$s_i\big|_{D(g_i f)} = s\big|_{D(g_i) \cap D(f)} = 0 \qquad s\big|_{D(g_i) \cap D(f)} = s\big|_{D(g_i f)} \in M_i[f^{-1}]$$

$$s_i\big|_{\operatorname{Spec} A[g_i^{-1}][f^{-1}]} = 0$$

$$\Rightarrow \exists\, n_i > 0 \quad \text{s.t.} \quad f^{n_i} s_i = 0$$

put $\quad m := \max_{i=1,\dots,n}\{n_i\} \quad \Rightarrow \quad f^m s\big|_{D(g_i)} = 0 \ \forall i$

$\bigcup_{i=1}^{n} D(g_i) = X \quad \Rightarrow \quad f^m s = 0$ by the sheaf axioms.

<u>For part (b):</u> $\quad X = \bigcup_{i=1}^{n} D(g_i), \quad D(f) = \bigcup_{i=1}^{n} D(g_i f)$

$t \in \Gamma(D(f), \mathcal{F})$

$t\big|_{D(g_i f)} \in \Gamma(D(g_i f), \mathcal{F}) = M_i[f^{-1}]$

$\Rightarrow \quad \exists \begin{cases} m_i \in M_i \\ n_i > 0 \end{cases} \qquad t\big|_{D(g_i f)} = \dfrac{m_i}{f^{n_i}}$

$\Rightarrow \quad f^{n_i}\, t\big|_{D(g_i f)} = m_i\big|_{D(g_i f)}$ after possibly increasing n_i and replacing m_i by $f^{r_i} m_i$ for some $r_i \geq 0$

take $r := \max\limits_{i=1,\dots,n} \{r_i\}$ then $f^r t\big|_{D(g_i f)}$ is the

restriction of some $m_i \in \Gamma(D(g_i), \mathcal{M})$

(might again have to replace m_i with some $f^{r_i} m_i$)

<u>Final step</u>: we need to glue these m_i to a global section of $\mathcal{M}$.

$$m_i\big|_{D(g_i g_j)} \overset{?}{=} m_j\big|_{D(g_i g_j)} \quad \forall\, i,j$$

consider $$\left(m_i\Big|_{D(g_ig_j)} - m_j\Big|_{D(g_ig_j)} \right)\Bigg|_{D(g_ig_jf)} = 0$$

because $$m_i\Big|_{D(g_ig_jf)} = (f^n t)\Big|_{D(g_ig_jf)} = m_j\Big|_{D(g_ig_jf)}$$

we apply part (a): $\exists\ u_{ij} > 0$ s.t.

$$f^{u_{ij}}\left(m_i\Big|_{D(g_ig_j)} - m_j\Big|_{D(g_ig_j)} \right) = 0 \quad \text{in } \Gamma(D(g_ig_j), \mathcal{M})$$

$$u := \max_{i,j}\{u_{ij}\} \quad \Rightarrow \quad \forall\ i,j$$

$$(f^u m_i)\Big|_{D(g_ig_j)} = (f^u m_j)\Big|_{D(g_ig_j)}$$

$\Rightarrow \quad \exists \quad s \in \Gamma(X, \mathcal{F}) \text{ s.t.}$

$$s\big|_{D(g_i)} = f^{u} m_i \quad \forall i$$

$$\forall i \quad (f^{u+r} t)\big|_{D(g_i f)} = (f^{u} m_i)\big|_{D(g_i f)}$$

$$= s\big|_{D(g_i f)}$$

$$\Rightarrow \quad f^{u+r} t = s\big|_{D(f)}. \qquad \square$$